The Separation of Two Close Hidden Periodicities

By

Michael SHIMSHONI

*Department of Applied Mathematics, The Weizmann Institute of Science
Rehovot, Israel.*

§1. Introduction

Let $f(t)$ be the sum of several periodic components of unknown period, amplitudes and phases. We assume that the periods are generally not in a harmonic ratio to each other and we wish to retrieve the unknown periods, amplitudes and phases.

Lanczos [4, pp. 267-272] gives a detailed account of a method of searching for such hidden periodicities. Jeffreys [2] suggests a somewhat modified approach. In two previous papers [5 and 6] I have suggested methods of solving this problem and estimating the uncertainties in the results.

All these procedures use the idea of Fourier Analysis (or Transform) in one way or another. A problem which remained open is that of the resolution in the case of two periods which are "close" to each other. In the present work a method is proposed to handle close periods.

In order to clarify the ideas, let us consider a function $f(t)$ which is the sum of just two harmonic components, i.e.

$$f(t) = C_1 \cos(2\pi t/T_1 - \varphi_1) + C_2 \cos(2\pi t/T_2 - \varphi_2) \quad (1)$$

and let $f(t)$ be given in the interval

$$-T/2 \leq t \leq T/2. \quad (2)$$

We define as

$$\sigma_1 = T/T_1, \quad \sigma_2 = T/T_2 \quad (3)$$

which give us the number of full cycles the first or second component, respectively, is contained in our interval of interest. Let $T_1 < T_2$ and thus $\sigma_1 > \sigma_2$.

The direct approach for obtaining T_1 , C_1 , φ_1 , T_2 , C_2 and φ_2 would be to compute the two transform functions

$$A(s) = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \frac{2\pi s t}{T} dt, \quad (4.a)$$

$$B(s) = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \frac{2\pi s t}{T} dt \quad (4.b)$$

and from it the functions

$$C(s) = [A^2(s) + B^2(s)]^{1/2};$$

$$\varphi(s) = \tan^{-1}[B(s)/A(s)]. \quad (5)$$

The maxima of $C(s)$ should occur close to the values $s = \sigma_1$, and $s = \sigma_2$, where $\varphi(s) \approx \varphi_1$ and φ_2 respectively.

The methods of the above-mentioned authors use this form of transform in somewhat modified forms, but to the following discussion, these variations are of no consequence.

In order to examine the problems arising when T_1 is close to T_2 , we have to compute the expression (4) and (5) for $f(t)$ given by (1). We get

$$A(s) = \frac{C_1 \cos \varphi_1 \sin[\pi(\sigma_1 - s)]}{\pi(\sigma_1 - s)} + \frac{C_2 \cos \varphi_2 \sin[\pi(\sigma_2 - s)]}{\pi(\sigma_2 - s)} + \frac{C_1 \cos \varphi_1 \sin[\pi(\sigma_1 + s)]}{\pi(\sigma_1 + s)} + \frac{C_2 \cos \varphi_2 \sin[\pi(\sigma_2 + s)]}{\pi(\sigma_2 + s)}, \quad (6.a)$$

$$B(s) = \frac{C_1 \sin \varphi_1 \sin[\pi(\sigma_1 - s)]}{\pi(\sigma_1 - s)} + \frac{C_2 \sin \varphi_2 \sin[\pi(\sigma_2 - s)]}{\pi(\sigma_2 - s)} - \frac{C_1 \sin \varphi_1 \sin[\pi(\sigma_1 + s)]}{\pi(\sigma_1 + s)} - \frac{C_2 \sin \varphi_2 \sin[\pi(\sigma_2 + s)]}{\pi(\sigma_2 + s)}. \quad (6.b)$$

Before computing $C(s)$ we shall look at expressions (6.a) and (6.b).

We take σ_1 being close to σ_2 so that

$$(\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2) \ll 1. \quad (7)$$

The values of s which are of interest are close to σ_1 and σ_2 . Because of that, the two

later terms of both $A(s)$ and $B(s)$ can be ignored in comparison with their respective two earlier terms. Neglecting these terms will still enable us to get a good idea of the nature of $C(s)$. Under these conditions

$$C^2(s) \approx \frac{C_1^2 \sin^2[\pi(\sigma_1 - s)]}{\pi^2(\sigma_1 - s)^2} + \frac{C_2^2 \sin^2[\pi(\sigma_2 - s)]}{\pi^2(\sigma_2 - s)^2} + \frac{2C_1 C_2 \sin[\pi(\sigma_1 - s)] \sin[\pi(\sigma_2 - s)]}{\pi^2(\sigma_1 - s)(\sigma_2 - s)} \times \cos(\varphi_1 - \varphi_2). \quad (8)$$

We see from expression (8) that $C(s)$ is the third side of a triangle, the two other sides of which are

$$y_1 = \frac{C_1 \sin[\pi(\sigma_1 - s)]}{\pi(\sigma_1 - s)} \quad (9.a)$$

and

$$y_2 = \frac{C_2 \sin[\pi(\sigma_2 - s)]}{\pi(\sigma_2 - s)}. \quad (9.b)$$

The angle between y_1 and y_2 is

$$\psi = \pi - (\varphi_1 - \varphi_2). \quad (9.c)$$

Thus, although near $s = \sigma_1$, y_1 tends to a maximum, this tendency might be masked by the relatively big y_2 , especially for unfavourable values of ψ .

In order to estimate the interference of the period T_2 with that of T_1 let us examine the behaviour of the function $S(x) = \frac{\sin \pi x}{\pi x}$ which occurs both in y_1 and y_2 , defining $S(0) = 1$.

As x increases, the maxima of $|S(x)|$ are .217, .128, .091, .071, etc., and they occur for values of $x = 1.43, 2.46, 3.47, 4.48$, etc. The maxima decrease almost proportionately to x^{-1} .

If $\sigma_1 - \sigma_2 > 5$ the interference of the second component with the first is less than 6% (assuming $C_1 \approx C_2$). On the other hand, if $\sigma_1 - \sigma_2 < 2$, we may have interference which exceeds 20%, and for $\sigma_1 - \sigma_2 < 1$ the situation may be so bad that the maxima in $C(s)$ at $s = \sigma_1$ caused by y_1 will be so disturbed by $C_2 S(\sigma_2 - s)$, that in $C(s)$ no maximum will be found near enough to $s = \sigma_1$ to help discover the "hidden" period T_1 .

Lanczos [4, p. 271], when dealing with close

periods, suggests a weighting method which decreases the extreme values of the side lobes faster than in the function $S(x)$, but his method is only useful for cases of $\sigma_1 - \sigma_2 > 1$; if $\sigma_1 - \sigma_2 < 1$ his modification makes the separation even more difficult, because his corresponding $S(x)$ is "somewhat broader" than ours, in particular for small x .

From the above discussion one should not conclude that, from analysing expression (8), we hoped to obtain the exact values of σ_1 and σ_2 , their amplitudes and their phases. All we need is a "good" approximation of either σ_1 , C_1 and φ_1 or σ_2 , C_2 and φ_2 . Such an approximation, say, $\bar{\sigma}_1$, $\bar{C}_1$ and $\bar{\varphi}_1$, is considered "good", if, when we form by subtraction the new function

$$f^1(t) = f(t) - \bar{C}_1 \cos(2\pi \bar{\sigma}_1 t / T - \bar{\varphi}_1), \quad (10)$$

the resulting $f^1(t)$ when analysed will yield σ_2 , C_2 and φ_2 more easily than $f(t)$. In other words we want

$$C_1 \cos(2\pi \sigma_1 t / T - \varphi_1) - \bar{C}_1 \cos(2\pi \bar{\sigma}_1 t / T - \bar{\varphi}_1)$$

to interfere less with

$$C_2 \cos(2\pi \sigma_2 t / T - \varphi_2)$$

than does

$$C_1 \cos(2\pi \sigma_1 t / T - \varphi_1)$$

by itself. If this condition is met, we can then get a "good" approximation $\bar{C}_2$, $\bar{\sigma}_2$, $\bar{\varphi}_2$ of the second periodicity which, when subtracted from $f(t)$ of expression (1), will enable us to get a still better approximation of the first periodicity. Iteration of this process then leads to convergence.

If we fail to obtain a "good" approximation in the above sense, there is little hope to separate the two periodicities.

In order to illustrate the behaviour of $C(s)$ for various pairs of σ_1 , σ_2 and φ_1 , φ_2 we show in Fig. 1 the dependence of $C(s)$ on s for several typical cases. In all examples I took $C_1 = C_2 = 1$ because if $|C_1| > |C_2|$ it is easier to obtain a good approximation to the first periodicity than in the case of equal C_1 and C_2 . As only the difference $\varphi_1 - \varphi_2$ appears in expression (8), I put $\varphi_2 = 0$ and let φ_1 vary between 0 and π . T_2 is taken as 157 seconds

Fig. 1. a-g The Fourier Transform of a sum of two harmonic components.

and T_1 varies. The length of the sample is always 360 seconds, the sampling interval is 1 second.

The continuous line in the figures plots the amplitude $C(T)$, the broken line plots the phase $\varphi(T)$. The two + signs show $C_1(T_1)$ and $C_2(T_2)$ respectively and the two × signs indicate $\varphi_1(T_1)$ and $\varphi_2(T_2)$. In the ideal case the continuous line should have two maxima through the + signs, and the broken line should go through the ×.

We see in Figs. 1a, 1b and 1c that for $T_1 = 67, 87$, and 107 seconds, separation is quite good. For $T_1 = 107$, $\sigma_1 - \sigma_2 = 1.1$ and this seems to suffice for separation. I tried it for several values of φ_1 and separation was possible in most cases.

For $T_1 > 107$ seconds separation becomes increasingly more difficult. Figs. 1d and 1e show $T_1 = 117$ and 127 respectively. In both cases it is impossible to pick out T_1 or T_2 as maxima of the continuous line.

Figs. 1f and 1g treat the case $T_1 = 151$ ($\sigma_1 - \sigma_2 = .09$) for two typical values of φ_2 . It is quite clear from these figures why separation under such circumstances is so difficult.

§ 2. The Method of Separation

Let $f(t)$ be as defined in expression (1). We shall define two other functions, an even function

$$e(t) = \frac{1}{2}[f(t) + f(-t)] \quad (11)$$

and an odd function

$$o(t) = \frac{1}{2}[f(t) - f(-t)]. \quad (11.a)$$

Both $e(t)$ and $o(t)$ are defined for the interval (2), namely

$$-T/2 \leq t \leq T/2.$$

In the following discussion we shall concentrate on $e(t)$, but with minor obvious changes the analysis can be applied to $o(t)$ as well.

Substituting expression (1) into (11), we find

$$e(t) = [C_1 \cos \varphi_1] \cos(2\pi t/T_1) + [C_2 \cos \varphi_2] \cos(2\pi t/T_2). \quad (12)$$

Instead of attempting to determine T_1 and T_2 (or σ_1 and σ_2) from $f(t)$, we shall do it using $e(t)$ as given in expression (12). We see that $e(t)$ is the sum of two pure cosine terms (there is no phase angle) of periods T_1 and T_2 with the amplitudes $A_1 = C_1 \cos \varphi_1$ and $A_2 = C_2 \cos \varphi_2$. Separation of the two terms will depend, as stated before, on the ratio

$$\rho = A_1/A_2 = (C_1/C_2)(\cos \varphi_1/\cos \varphi_2). \quad (13)$$

If $|\rho| \gg 1$, $|A_1| \gg |A_2|$ and it is relatively easy to determine A_1 as A_2 will only disturb in an insignificant way. We should be able to

Fig. 2 The separation of a simple harmonic function into odd and even components.

obtain a "good" estimate of T_1 . Conversely, if $|\rho| \ll 1$, a "good" estimate of T_2 can be obtained.

In order to illustrate how ρ can be altered at will, let us look at Fig. 2 which shows one of the components of $f(t)$, namely

$$\begin{aligned} f_1(t) &= C_1 \cos(2\pi t/T_1 - \varphi_1) \\ &= C_1 \cos[2\pi(t - \varphi'_1)/T_1], \end{aligned} \quad (14)$$

where

$$\varphi'_1 = T_1 \varphi_1 / 2\pi \quad (15)$$

is the rescaled phase angle.

In addition, the two functions

$$e_1(t) = (C_1 \cos \varphi_1) \cos 2\pi t/T_1 \quad (16)$$

and

$$o_1(t) = (C_1 \sin \varphi_1) \sin 2\pi t/T_1 \quad (17)$$

have been drawn in Fig. 2. These two functions are the T_1 components of $e(t)$ and $o(t)$. As stated before, our attention is concentrated at the moment on $e_1(t)$, but we notice that

$$f_1(t) = e_1(t) + o_1(t). \quad (18)$$

We shall now change our time origin by τ and look at the function

$$\begin{aligned} \bar{f}_1(t) &= f_1(t - \tau) = C_1 [\cos 2\pi(t - \tau)/T_1 - \varphi_1] \\ &= C_1 [\cos 2\pi(t - \bar{\varphi}'_1)/T_1], \end{aligned} \quad (19)$$

where

$$\bar{\varphi}'_1 = \varphi'_1 + \tau = T_1 \varphi_1 / 2\pi + \tau. \quad (20)$$

If we now split $\bar{f}_1(t)$ into an even function $\bar{e}_1(t)$ and an odd function $\bar{o}_1(t)$, the amplitude of $\bar{e}_1(t)$ will be

$$\begin{aligned} \bar{A}_1(\tau) &= C_1 \cos(2\pi \bar{\varphi}'_1 / T_1) \\ &= C_1 \cos(\varphi_1 + 2\pi \tau / T_1). \end{aligned} \quad (21)$$

Examining $\bar{A}_1(\tau)$ in expression (21) we see its dependence on τ ; by varying τ , $|\bar{A}_1(\tau)|$ can take any value between C_1 and zero.

In Fig. 3, τ is so chosen that $\bar{A}_1(\tau)$ is rather small; by a slight further shift of τ we could make $\bar{A}_1(\tau)$ disappear altogether.

By choosing τ so that $\bar{A}_1(\tau)$ almost, or completely, vanishes, we can make $|\rho| \ll 1$. Thus for such values of τ it will be easy to determine T_2 with great accuracy. Inspecting Fig. 2 again we see that if the origin is shifted to places where $f_1(t)$ is zero, we have a value of τ which will enable us to reduce the presence of $\bar{e}_1(t)$. If we shift the origin to places where $f_1(t)$ has a maximum or a minimum, we get values of τ where $\bar{o}_1(t)$ has no T_1 component, because in these places $\bar{o}_1(t) = 0$.

It should be realized that by a time shift τ , not only is A_1 changed to $\bar{A}_1(\tau)$, but also A_2 (or the period T_2) changes to $\bar{A}_2(\tau)$. We can write the ratio $\rho(\tau)$ as function of τ :

Fig. 3 The effect of shifting the time origin, on the separation of a simple harmonic function into odd and even components.

$$\rho(\tau) = \bar{A}_1(\tau)/\bar{A}_2(\tau) = (C_1/C_2)[\cos(\varphi_1 + 2\pi\tau/T_1)/\cos(\varphi_2 + 2\pi\tau/T_2)]. \quad (22)$$

Inspecting $\rho(\tau)$ we see that in it we have the ratio of two cosine functions of *different* periods, which will not always vanish at the same moment. So we will have values of τ for which $|\rho(\tau)| \ll 1$ and also some for which $|\rho(\tau)| \gg 1$.

Unfortunately, we have been putting the cart before the horse. We have no direct method of choosing τ so that $\bar{A}_1(\tau)$ of expression (21) is zero, since this would imply knowledge of T_1 and φ_1 , two of the quantities we are trying to find.

In the following, I suggest a procedure to find values of τ which make $\rho(\tau)$ tend either to infinity (yielding T_1) or to zero (yielding T_2).

For any value of τ we try to find the best approximation to $\bar{e}(t)$ in the least squares sense, by a single, even periodic function

$$A \cos(2\pi t/T_0). \quad (23)$$

As $\bar{e}(t)$ is an even function it is defined only in the interval

$$-\bar{T}/2 \leq t \leq \bar{T}/2, \quad (24)$$

where

$$\bar{T}/2 = T/2 - |\tau|. \quad (24a)$$

We must also remember that $\bar{e}(t)$ is a function of τ . We define the positive normalised deviation function

$$R(\tau, T) = \frac{\int_{-\bar{T}/2}^{\bar{T}/2} [\bar{e}(t) - A \cos(2\pi t/T_0)]^2 dt}{\int_{-\bar{T}/2}^{\bar{T}/2} [\bar{e}(t)]^2 dt} \quad (25)$$

and for a fixed τ we find a value of $T_0 = T_\tau$ such that $R(\tau, T_\tau)$ is minimal.

This process is repeated for many values of τ , and we now look for minima among the various $R(\tau, T_\tau)$ for the different τ . It is clear that if τ is a value for which $|\rho(\tau)| \rightarrow \infty$, $R(\tau, T_\tau)$ will be conspicuous by being zero and $T_\tau = T_1$. In analogy, if τ is a value such that $|\rho(\tau)| \rightarrow 0$, we shall find $T_\tau = T_2$.

§3. The Method in Practice

In practical cases we shall have $f(t)$ for discrete values of t only. We shall define,

in analogy with expression (1),

$$f_r = C_1 \cos(2\pi r/T_1 - \varphi_1) + C_2 \cos(2\pi r/T_2 - \varphi_2) \quad (26)$$

for $-n \leq r \leq n$. Without loss of generality, I have re-scaled the time so that the time interval between adjacent points is unity.

In analogy with the expressions for continuous t we can define e_r and $\bar{e}_r$. For simplicity we shall choose integral values for τ , although it is not very difficult to handle non-integral τ as well.

We have now

$$\bar{e}_r = \{C_1 \cos(\varphi_1 + 2\pi\tau/T_1)\} \cos(2\pi r/T_1) + \{C_2 \cos(\varphi_2 + 2\pi\tau/T_2)\} \cos(2\pi r/T_2) \quad (27)$$

defined for values of r in the interval

$$-\bar{n} \leq r \leq \bar{n}, \quad (28)$$

where

$$\bar{n} = n - |\tau|. \quad (28.a)$$

We define the normalised deviation function by the quotient of two sums,

$$R(\tau, T) = \frac{\sum_{r=-\bar{n}}^{\bar{n}} [\bar{e}_r - A \cos(2\pi r/T_0)]^2}{\sum_{r=-\bar{n}}^{\bar{n}} \bar{e}_r^2}. \quad (29)$$

We replaced integration by ordinary summation, and not by an interpolation-summation procedure as suggested by Filon [1] and others. Not only do these interpolation procedures not improve accuracy in cases of harmonic analysis, but they actually give a biased result depending on the period. Jeffreys and Jeffreys [3, p. 456] show this to be so in the case of linear interpolation, but a bias is caused with higher order interpolation formulae as well.

In the Appendix I give explicit formulae for the evaluation of expression (29).

For each value of τ we find $T_0 = T_\tau$ so that $R(\tau, T_\tau)$ is minimal. For all the values of τ we use, we sort the $R(\tau, T_\tau)$ obtained in increasing order of T_τ , retaining the relevant $A(=A_\tau)$ as well. We inspect the list of $R(\tau, T_\tau)$ for small values and these should occur for $T_\tau \simeq T_1$ or T_2 giving

$$A_\tau \simeq C_1 \cos(\varphi_1 + 2\pi\tau/T_1) \quad (30)$$

or

$$A_\tau \simeq C_2 \cos(\varphi_2 + 2\pi\tau/T_2) \quad (31)$$

respectively.

Thus we have an approximation to T_1 and $C_1 \cos(\varphi_1 + 2\pi\tau/T_1)$ (or T_2 and $C_2 \cos(\varphi_2 + 2\pi\tau/T_2)$). Analysis of the odd function would yield approximations to $C_1 \sin(\varphi_1 + 2\pi\tau/T_1)$ or $C_2 \sin(\varphi_2 + 2\pi\tau/T_2)$ as well, enabling us to separate C_1 from φ_1 or C_2 from φ_2 .

§4. Examples

One of the test cases I analysed has the values $T_1=151$ seconds, $T_2=157$ seconds; C_1

Fig. 4a-4b The minima of the normalised deviation function of a sum of two harmonic components.

$=C_2=1$, $\varphi_2=.1$ Radians, and I examined various values of φ_1 . I shall report here on two typical cases: $\varphi_1=.8$ Radians and $\varphi_1=1.6$ Radians. In all the cases, 361 points at 1-second intervals were taken, making the total length of consideration 360 second.

We had $\sigma_1=360/151=2.38$ and $\sigma_2=2.29$; thus the change in phase between the two components over the whole length of the sample was about 33° . In Figs. 4a and 4b the behaviour of the deviation function $R(\tau, T_\tau)$ is given. $R(\tau, T_\tau)$ is the minimum value of $R(\tau, T_0)$ for a fixed τ . The value of T_0 for which $R(\tau, T_0)$ is minimum is called T_τ , and T_τ is the abscissa in Figs. 4a and 4b. In Figs. 4a and 4b the horizontal portions of the lines indicate that there $R(\tau, T_\tau)$ was so great that it was necessary to truncate it so that the scale of the figure remained meaningful. Actually at these truncated sections, the function took values several orders of magnitude bigger than for the minimal points. A comparison of Figs. 4a and 4b with Figs. 1f and 1g indicates the power of the present method.

From these two figures, we see that one can get quite readily very good approximations to T_1 and T_2 . We show only the analysis of $\bar{e}(t)$, but comparable results were obtained using $\bar{o}(t)$. From our results we obtained very good estimates of the amplitudes and phases as well, even without iterating the process in the sense of expression (10). If one iteration is added, convergence was complete in a numerical sense.

§5. Acknowledgement

This research was supported by the U.S. Office of Naval Research under contract N00014-67-C0146.

Appendix

The Approximation of $\bar{e}(t)$ by $A \cos 2\pi t/T_0$.

We shall consider values of $\bar{e}(t)$ for discrete values of t , and use the notation $\bar{e}_r$ for $\bar{e}(r)$. We are trying to minimise $R(\tau, T_0)$ as given in Eq. (29), by choosing the best values of A and T_0 .

To find these A and T_0 we have to solve

$$\frac{\partial R(\tau, T_0)}{\partial T_0} = 0 \quad (A1)$$

and

$$\frac{\partial R(\tau, T_0)}{\partial A} = 0. \quad (A2)$$

From (A2) we get

$$-\sum \bar{e}_r \cos(2\pi r/T_0) + A \sum \cos^2(2\pi r/T_0) = 0 \quad (A3)$$

or

$$A = \frac{\sum \bar{e}_r \cos(2\pi r/T_0)}{\frac{1}{2} \left\{ 2\bar{n} + 1 + \frac{\sin[(2\bar{n}+1)2\pi/T_0]}{\sin(2\pi/T_0)} \right\}} \quad (A4)$$

All the summations are from $r = -\bar{n}$ to $r = \bar{n}$.

We can substitute the value of A from (A4) into (29) and, after some simple manipulations we have

$$R(T_0) = 1$$

$$-\frac{[\sum \bar{e}_r \cos(2\pi r/T_0)]^2}{\frac{1}{2} \left\{ 2\bar{n} + 1 + \frac{\sin[2\pi(2\bar{n}+1)/T_0]}{\sin(2\pi/T_0)} \right\} \sum \bar{e}_r^2} \quad (A5)$$

Unfortunately equation (A1) is unsolvable analytically; thus we have to look for minima in (A5) numerically. This can be done by a modified Newton-Rapson procedure.

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(Received July 31, 1968)

(Not in the Symposium program.)