Head and Reflected Waves from an SH Line Source in a Dipping Layer Overlying an Elastic Medium

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Abstract

The displacements due to a periodic and impulsive line source of SH waves in a dipping layer overlying an elastic solid are theoretically investigated using a reflected wave formulation. A formal solution is found by which the contributions due to head and reflected waves are determined by evaluation of the integrals by the method of steepest descent. Although the individual waveforms are similar for the wedge and the horizontally layered case, the synthetic seismograms are markedly different due to the difference in travel times.

Using ray paths, the contributions of the integrals have been interpreted. The range of existence of head waves has been examined and the discontinuities in the wedge associated with diffracted waves studied. In the case of a free or rigid lower boundary of the wedge, the dispersion relation has been determined.

§1. Introduction

A number of workers (e.g. LAPWOOD (1961), HUDSON (1963), and HUDSON and KNOPOFF (1964)) have investigated the propagation of surface and diffracted waves in wedged-shaped media. HUDSON (1963) pointed out that the solution in the case of a rigid lower boundary could be divided into two parts—the multiply reflected and diffracted wave solutions. However, the early part of the seismogram consisting of head and multiply reflected waves which are often used in interpretation has not been well studied for a dipping layer overlying an elastic media. In the present paper the authors will establish one method of solution and will theoretically investigate the problem for an SH line source in an elastic wedge overlying an elastic medium.

In an earlier paper (ISHII and ELLIS, 1970), a solution for the problem of multiple reflection of plane SH waves by a dipping layer has been found. By integration of a solution of this type, the disturbance due to a line source is sought which does not include the diffracted wave term. However, for a transient input, an observation point

distant from the vertex receives the reflected and refracted waves earlier than the diffracted waves which result from collisions with the vertex. Therefore the present solution should apply to the composition of the initial section of the seismogram. The formal solution is evaluated by the steepest descent technique as recommended by HONDA and NAKAMURA (1954) for evaluation of branch line integrals and as applied by EMURA (1960) and others. In this way the wave forms and the ranges of existence of the head waves are determined for various dip angles for comparison with the case of a horizontal layer. For all computations the following elastic parameters corresponding to those of HASKELL (1960) are used with S wave velocity in the upper layer $C_{b1}=3.64$ km/sec, the velocity ratio $J=C_{b2}/C_{b1}=1.27$, and the rigidity ratio $\bar{\sigma}=\mu_2/\mu_1=1.88$.

As surface waves are obtained from the contributions of poles, the surface wave problem reduces to finding poles of the finite series expression of our solution. Further, the discontinuities in displacement associated with the diffracted waves in the wedge have been found and hence this problem is separated from the determination of the

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solution due to other waves. The solution of the present problem is therefore an important step for the consideration of surface and diffracted waves.

§ 2. Equation of motion and boundary conditions

The propagation of SH waves through a system consisting of an elastic medium of rigidity μ_1 , density ρ_1 , and a dip angle $\theta_1 + \theta_2$, overlying an elastic medium of rigidity μ_2 and density ρ_2 (Fig. 1) will be considered. The free surface is $\theta = -\theta_1$ and the boundary between the elastic media is $\theta = \theta_2$. A cartesian system (x, y, z) is related to the cylindrical coordinates (r, θ, z) by the standard relationships $x = r \cos \theta$, $y = r \sin \theta$, and $z = z$. The motion is generated by a line source (S) of SH waves located at $(d, 0)$ in the cylindrical coordinate system.

Fig. 1. Geometry of the problem: the line source (S) is located at $(d, 0)$ and the receiver (R) at (r, θ) in the wedge bounded by the free surface ($\theta = -\theta_1$) and the boundary ($\theta = \theta_2$) between the two media.

For the above problem, the motion is independent of z , and the displacement has only a z -component. Then, assuming a time variation of the form $e^{i\omega t}$, the equation of motion

$$\nabla^2 u_i = \frac{1}{C_{bi}^2} \frac{\partial^2 u_i}{\partial t^2}, \quad i=1, 2, \quad (1)$$

becomes in cylindrical coordinates

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + k_{bi}^2 \right) u_i = 0, \quad i=1, 2, \quad (2)$$

where $C_{bi} = \sqrt{\mu_i/\rho_i}$ is the velocity of the S waves and $k_{bi} = \omega/C_{bi}$. The only non-zero

component of stress is

$$(P_{z\theta})_i = \mu_i \frac{\partial u_i}{\partial \theta}. \quad (3)$$

The boundary conditions then become

$$(P_{z\theta})_i = 0 \quad \text{at} \quad \theta = -\theta_1 \quad (4)$$

and

$$\left. \begin{aligned} u_1 &= u_2 \\ (P_{z\theta})_1 &= (P_{z\theta})_2 \end{aligned} \right\} \quad \text{at} \quad \theta = \theta_2. \quad (5)$$

To solve the line source problem, a plane wave solution satisfying the boundary conditions will first be obtained. The line source solution can then be obtained by integration of this solution with respect to the cylindrical angle.

§ 3. Steady state plane wave solution

The initial displacement due to a plane wave is expressed in the form

$$\begin{aligned} u'_0 &= A_i \cdot e^{-ik_{b1}\{(d-x)\cos\alpha_i + |y|\sin\alpha_i\}} \\ &= A_i \cdot e^{ik_{b1}r\cos(\alpha_i + |\theta|) - ik_{b1}d\cos\alpha_i}, \end{aligned} \quad (6)$$

where α_i is the angle between the x -axis and wave normal of the SH waves and may take on complex values in the evaluation of the effect due to a line source. For $\theta > 0$, u'_0 represents waves downgoing from the x -axis, while for $\theta < 0$, u'_0 represents waves upgoing from the x -axis. In those cases where the waves interact with the boundaries, the former collides with the dipping boundary first and the latter with the free surface first. In a previous paper (ISHII and ELLIS, 1970), two expressions were found to express the motion, with the choice of expression depending on whether the final reflection is from the boundary or the free surface. These essentially were

$$N_i = A_i \sum_{m=1}^L \left(\prod_{n=1}^m A_n \right) e^{ik_{b1}r\cos\{\theta - 2\theta_d(m-1) - \beta_1\}}$$

$$N'_i = A_i \sum_{m=1}^L \left(\prod_{n=1}^m A_n \right) e^{ik_{b1}r\cos\{\theta + 2\theta_d(m-1) + \beta_1\}},$$

where

$$\beta_1 = \theta_d + \frac{\pi}{2} - \sin^{-1} \left(\frac{1}{d} \cos(\theta_d - \alpha) \right),$$

α —initial propagation direction.

For the geometry under consideration in that case, the free surface was located at $\theta=0$ and the boundary between media at $\theta=\theta_d$.

In the present case the same procedures are used to determine the four expressions that are required to express the motion depending on whether the initial direction of the waves is positive or negative and whether the final reflection is from the boundary (1) or the free surface (2). These expressions are

$$S_1^+(N) = A_i \sum_{n=1}^N \left(\prod_{k=1}^n A_k^+ \right) \times e^{ik_{b1}r \cos(\alpha_i + 2(n-1)\theta_1 + 2n\theta_2 - \theta) - ik_{b1}d \cos \alpha_i}, \quad (7)$$

$$S_2^+(N) = A_i \sum_{n=1}^N \left(\prod_{k=1}^n A_k^+ \right) \times e^{ik_{b1}r \cos(\alpha_i + 2n\theta_1 + 2n\theta_2 + \theta) - ik_{b1}d \cos \alpha_i}, \quad (8)$$

$$S_1^-(N') = A_i \sum_{n=1}^{N'} \left(\prod_{k=1}^n A_k^- \right) \times e^{ik_{b1}r \cos(\alpha_i + 2n\theta_1 + 2n\theta_2 - \theta) - ik_{b1}d \cos \alpha_i}, \quad (9)$$

$$S_2^-(N') = A_i \sum_{n=1}^{N'} \left(\prod_{k=1}^n A_k^- \right)$$

$$A_n = \frac{\Delta \sin(\theta_d + 2\theta_d(n-2) + \beta_1) - \delta \sqrt{1 - \Delta^2 \cos^2(\theta_d + 2\theta_d(n-2) + \beta_1)}}{\Delta \sin(\theta_d + 2\theta_d(n-2) + \beta_1) + \delta \sqrt{1 - \Delta^2 \cos^2(\theta_d + 2\theta_d(n-2) + \beta_1)}}, \quad n \geq 2$$

(equation (20) in the previous paper).

In the present problem, however, we have two different reflection coefficients because of the existence of a source within the wedge. The reflection coefficients in the present geometry for a wave which started as a downgoing wave with respect to the x -axis are

$$A_k^+ = \frac{\Delta \sin\{\theta_2 + 2(k-1)(\theta_1 + \theta_2) + \alpha_i\} - \delta \sqrt{1 - \Delta^2 \cos^2\{\theta_2 + 2(k-1)(\theta_1 + \theta_2) + \alpha_i\}}}{\Delta \sin\{\theta_2 + 2(k-1)(\theta_1 + \theta_2) + \alpha_i\} + \delta \sqrt{1 - \Delta^2 \cos^2\{\theta_2 + 2(k-1)(\theta_1 + \theta_2) + \alpha_i\}}}, \quad (17)$$

and for a wave which started as an upgoing wave with respect to x -axis are

$$A_k^- = \frac{\Delta \sin\{\theta_2 + 2k\theta_1 + 2(k-1)\theta_2 + \alpha_i\} - \delta \sqrt{1 - \Delta^2 \cos^2\{\theta_2 + 2k\theta_1 + 2(k-1)\theta_2 + \alpha_i\}}}{\Delta \sin\{\theta_2 + 2k\theta_1 + 2(k-1)\theta_2 + \alpha_i\} + \delta \sqrt{1 - \Delta^2 \cos^2\{\theta_2 + 2k\theta_1 + 2(k-1)\theta_2 + \alpha_i\}}}, \quad (18)$$

Δ and δ are defined by $\Delta = C_{b2}/C_{b1}$ and $\delta = \mu_2/\mu_1$.

The solution u'_1 on and close to the surface due to the initial disturbance and satisfying the boundary conditions (4) and (5) is:

$$u'_1 = S_0 + S_0^- + S_1^+(N-1) + S_2^+(N-1) + S_1^-(N'-1) + S_2^-(N'-1) \quad (19)$$

for conditions (11) and (13)

$$u'_1 = S_0 + S_0^- + S_1^+(N-1)$$

$$\times e^{ik_{b1}r \cos(\alpha_i + 2(n+1)\theta_1 + 2n\theta_2 + \theta) - ik_{b1}d \cos \alpha_i}, \quad (10)$$

where the maximum number of reflections from the boundary N and N' are seen by examination of the phase in (7) to (10) to be determined by

$$\pi + \theta_2 \geq 2(N-1)\theta_1 + 2N\theta_2 \geq \pi - \theta_1 \quad (11)$$

or

$$\pi + \theta_1 \geq 2N\theta_1 + 2N\theta_2 \geq \pi - \theta_2, \quad (12)$$

and by

$$\pi + \theta_2 \geq 2N'\theta_1 + 2N'\theta_2 \geq \pi - \theta_1 \quad (13)$$

or

$$\pi + \theta_1 \geq 2(N'+1)\theta_1 + 2N'\theta_2 \geq \pi - \theta_2. \quad (14)$$

The expressions for the direct wave and the wave once reflected from the free surface are

$$S_0 = A_i e^{ik_{b1}r \cos(\alpha_i + |\theta|) - ik_{b1}d \cos \alpha_i} \quad (15)$$

and

$$S_0^- = A_i e^{ik_{b1}r \cos(\alpha_i + 2\theta_1 + \theta) - ik_{b1}d \cos \alpha_i}. \quad (16)$$

In the problem investigated in the previous paper, we obtained the reflection coefficient as

$$+ S_2^+(N-1) + S_1^-(N') + S_2^-(N') \quad (20)$$

for conditions (11) and (14)

$$u'_1 = S_0 + S_0^- + S_1^+(N) + S_2^+(N) + S_1^-(N'-1) + S_2^-(N'-1) \quad (21)$$

for conditions (12) and (13)

$$u'_1 = S_0 + S_0^- + S_1^+(N) + S_2^+(N) + S_1^-(N') + S_2^-(N') \quad (22)$$

for conditions (12) and (14). It should be pointed out that these formal solutions are not a physical solution for a particular real angle α_i but are the plane wave forms from which the line source solution will be obtained.

The last terms of the series expressions (7)–(10) give rise to discontinuities in displacement and stress which serve as boundary conditions for the diffracted waves. These will be investigated in detail in a later section. Neglecting these last terms of the series,

$$u'_1 = S_0 + S_0^- + S_1^+(N-1) + S_2^+(N-1) + S_1^-(N'-1) + S_2^-(N'-1)$$

is valid everywhere in medium 1.

For evaluation of the displacement due to a line source, it is convenient to express formulae (7)–(10) and (15) and (16) as

$$S_1^+(N) = A_i \sum_{n=1}^N \left(\prod_{k=1}^n A_k^+ \right) e^{-ik_{b1} R_{n1}^+ \cos(\alpha_i - \theta_{n1}^+)}, \quad (23)$$

$$S_2^+(N) = A_i \sum_{n=1}^N \left(\prod_{k=1}^n A_k^+ \right) e^{-ik_{b1} R_{n2}^+ \cos(\alpha_i - \theta_{n2}^+)}, \quad (24)$$

$$S_1^-(N') = A_i \sum_{n=1}^{N'} \left(\prod_{k=1}^n A_k^- \right) e^{-ik_{b1} R_{n1}^- \cos(\alpha_i - \theta_{n1}^-)}, \quad (25)$$

$$S_2^-(N') = A_i \sum_{n=1}^{N'} \left(\prod_{k=1}^n A_k^- \right) e^{-ik_{b1} R_{n2}^- \cos(\alpha_i - \theta_{n2}^-)}, \quad (26)$$

$$S_0 = A_i e^{-ik_{b1} R_0 \cos(\alpha_i - \theta_0)}, \quad (27)$$

$$S_0^- = A_i e^{-ik_{b1} R_0^- \cos(\alpha_i - \theta_0^-)}, \quad (28)$$

where

$$R_{n1}^+ = \sqrt{\{d - r \cos(2(n-1)\theta_1 + 2n\theta_2 - \theta)\}^2 + r^2 \sin^2(2(n-1)\theta_1 + 2n\theta_2 - \theta)},$$

$$\tan \theta_{n1}^+ = \frac{r \sin(2(n-1)\theta_1 + 2n\theta_2 - \theta)}{\{d - r \cos(2(n-1)\theta_1 + 2n\theta_2 - \theta)\}}, \quad (29)$$

$$R_{n2}^+ = \sqrt{\{d - r \cos(2n\theta_1 + 2n\theta_2 + \theta)\}^2 + r^2 \sin^2(2n\theta_1 + 2n\theta_2 + \theta)},$$

$$\tan \theta_{n2}^+ = \frac{r \sin(2n\theta_1 + 2n\theta_2 + \theta)}{\{d - r \cos(2n\theta_1 + 2n\theta_2 + \theta)\}}, \quad (30)$$

$$R_{n1}^- = \sqrt{\{d - r \cos(2n\theta_1 + 2n\theta_2 - \theta)\}^2 + r^2 \sin^2(2n\theta_1 + 2n\theta_2 - \theta)},$$

$$\tan \theta_{n1}^- = \frac{r \sin(2n\theta_1 + 2n\theta_2 - \theta)}{\{d - r \cos(2n\theta_1 + 2n\theta_2 - \theta)\}}, \quad (31)$$

$$R_{n2}^- = \sqrt{\{d - r \cos(2(n+1)\theta_1 + 2n\theta_2 + \theta)\}^2 + r^2 \sin^2(2(n+1)\theta_1 + 2n\theta_2 + \theta)},$$

$$\tan \theta_{n2}^- = \frac{r \sin(2(n+1)\theta_1 + 2n\theta_2 + \theta)}{\{d - r \cos(2(n+1)\theta_1 + 2n\theta_2 + \theta)\}^2}, \quad (32)$$

$$R_0 = \sqrt{\{d - r \cos|\theta|\}^2 + r^2 \sin^2|\theta|}, \quad \tan \theta_0 = \frac{r \sin|\theta|}{\{d - r \cos|\theta|\}}, \quad (33)$$

$$R_0^- = \sqrt{\{d - r \cos(2\theta_1 + \theta)\}^2 + r^2 \sin^2(2\theta_1 + \theta)}, \quad \tan \theta_0^- = \frac{r \sin(2\theta_1 + \theta)}{\{d - r \cos(2\theta_1 + \theta)\}}. \quad (34)$$

§ 4. Formal steady state solution for a line source

In order to generalize the results to the case of a line source, the operator

$$k_{b1} \int_{-i\infty}^{\pi+i\infty} d\alpha_i, \quad (35)$$

is applied to the plane wave solution. In particular, the displacement u_0 due to the initial disturbance can be written using (6) as

$$u_0 = k_{b1} \int_{-i\infty}^{\pi+i\infty} u'_0 d\alpha_i$$

$$= A_i k_{b1} \int_{-i\infty}^{\pi+i\infty} e^{-ik_{b1}((d-x) \cos \alpha_i + |y| \sin \alpha_i)} d\alpha_i$$

$$= A_i k_{b1} \pi H_0^{(2)}(k_{b1} R_0). \quad (36)$$

Equation (36) can easily be derived from the results of NAKAMURA (1960). When $k_{b1} R_0$ is large, (36) can be approximated by the asymptotic formula,

$$u_0 = A_i \sqrt{\frac{2\pi k_{b1}}{R_0}} e^{-ik_{b1}R_0 + (\pi/4)i}, \quad (37)$$

which is an outgoing wave from the line source.

Using (19) to (22) and (23) to (28), the reflected wave solution in the wedge corresponding to a line source (36) can be obtained as

$$u_1 = k_{b1} \int_{-i\infty}^{\pi+i\infty} u'_1 d\alpha_i. \quad (38)$$

§ 5. Evaluation of the first series term of the integrals

In this section the integration of the terms which are produced by waves which are reflected once by the boundary between the media are evaluated. (As a guide to the evaluation of the higher order series terms, the contributions due to waves twice reflected from the boundary between the media are calculated in the Appendix). From (23)-(26) and (38), we see that they have the following forms:

$$I_{1l}^m = k_{b1} \int_{-i\infty}^{\pi+i\infty} A_1^m \cdot e^{-ik_{b1} R_{1l}^m \cos(\alpha_i - \theta_{1l}^m)} d\alpha_i, \quad (39)$$

where

$$A_1^m = \frac{\Delta \sin(\phi_1^m + \alpha_i) - \delta \sqrt{1 - \Delta^2 \cos^2(\phi_1^m + \alpha_i)}}{\Delta \sin(\phi_1^m + \alpha_i) + \delta \sqrt{1 - \Delta^2 \cos^2(\phi_1^m + \alpha_i)}}, \quad (40)$$

and $l=1, 2$ and $m=+, -$. From the equations (17) and (18), $\phi_1^+=\theta_2$ and $\phi_1^-=2\theta_1+\theta_2$ and R_{il}^m and θ_{il}^m are given by the equations (29) to (32).

For this integration, the original path L is taken in the plane for which $\text{Re}(\lambda_s) > 0$. The integrand of (39) contains the two-valued function $\lambda_s = \sqrt{1 - \Delta^2 \cos^2(\phi_1^m + \alpha_i)}$ and its branch points are given by the relation $\cos(\phi_1^m + \alpha_i) = \pm 1/\Delta$ and are therefore located on the real axis of the α_i -plane at the points $B(\theta_B = \theta_0 - \phi_1^m)$, $B'(\theta_{B'} = \pi - \theta_0 - \phi_1^m)$, $\dots$ (Figure 2) where $\cos \theta_0 = 1/\Delta = n$, the refractive index. To facilitate evaluation, it is assumed that the medium is very slightly absorptive by setting $n = n_0 - i\varepsilon$ where ε is a very small positive quantity. The branch point B is then displaced by $\varepsilon/\sqrt{1 - n_0^2}$ parallel to the positive imaginary

Fig. 2. The α_i -plane ($\alpha_i = x + iy$) on which $Re(\lambda_s) > 0$ and the regions of positive and negative $Im(\lambda_s)$, separated by the curves L_B and $L_{B'}$, indicated. Notation: S -saddle-point; B, B' -branch points; L -original path of integration; L_s -path of steepest descent through the saddle point; L_1, L_2 -paths of branch line integral; L_B -branch cut $Re(\lambda_s) = 0$; and $L_{B'}$ -curve along which $Im(\lambda_s) = 0$.

axis on the α_i -plane.

We choose the branch cut given by $\text{Re}(\lambda_s)=0$ which is defined by:

$$\cos (x+\phi_1^m) \sin (x+\phi_1^m) \cosh y \sinh y=n_0 \varepsilon \quad (41)$$

$$\cos^2(x + \phi_1^m) \cosh^2 y - \sin^2(x + \phi_1^m) \sinh y > n_0^2 \quad (42)$$

$$\alpha_i = x + iy. \quad (43)$$

In Figure 2, L_B is $\text{Re}(\lambda_s)=0$, $L_{B'}$ is $\text{Im}(\lambda_s)=0$ and the signs of $\text{Im}(\lambda_s)$ in the α_t -plane are indicated by plus and minus. When for large distances from the origin,

$$e^{-ik_{1l} R_{1l}^m} \cos(\alpha_i - \theta_{1l}^m)$$

vanishes, the path of integration can therefore be shifted on the Riemann surfaces. The region where

$$\text{Im} [\cos (\alpha_i - \theta_{1l}^m)] < 0 \quad \text{or} \quad e^{-ik_{0l} R_{1l}^m \cos (\alpha_i - \theta_{1l}^m)}$$

vanishes at a large distance from the origin is shown by hatching. The original path L , along which the relations $\text{Im}(\sin \alpha_i) < 0$ and $\text{Im}(\lambda_i) < 0$ hold, can be replaced by L_n and

(L_1, L_2) , where L_s passes through the saddle point S and (L_1, L_2) goes around the branch point B . The dotted lines denote that they are on the second Riemann sheet where $\text{Re}(\lambda_s) < 0$. Each of them is drawn along the path of steepest descent,

$$\cos(x - \theta_{1l}^m) \cosh y = 1 \quad (44)$$

and

$$\cos(x - \theta_{1l}^m) \cosh y = \cos(\theta_B - \theta_{1l}^m) \quad (45)$$

respectively.

Contribution from the saddle point (Reflected waves)

From (44), L_s makes an angle $\pi/4$ with the x -axis. In the neighbourhood of the saddle point the following approximations are valid:

$$\alpha_i - \theta_{1l}^m = \rho e^{i\pi/4} \quad (46)$$

and

$$\cos(\alpha_i - \theta_{1l}^m) = 1 - \rho^2/2. \quad (47)$$

The contour integral along L_s for (39) is then

$$u_s = A_i k_{b1} \int_{-\infty}^{\infty} A_1^m(\alpha_i) \times e^{(\pi/4)i - ik_{b1} R_{1l}^m - (1/2)k_{b1} R_{1l}^m \rho^2} d\rho. \quad (48)$$

Expanding the term $A_1^m(\alpha_i)$ near the saddle point and using Watson's Lemma (JEFFREYS, 1956) we obtain

$$u_s = A_i \sqrt{\frac{2\pi k_{b1}}{R_{1l}^m}} A_1^m(\theta_{1l}^m) e^{-ik_{b1} R_{1l}^m + (\pi/4)i} \quad (49)$$

where, if $\theta_{1l}^m > \theta_B$,

$$A_1^m(\theta_{1l}^m) = \frac{\sin(\phi_1^m + \theta_{1l}^m) - \delta \sqrt{1/D^2 - \cos^2(\phi_1^m + \theta_{1l}^m)}}{\sin(\phi_1^m + \theta_{1l}^m) + \delta \sqrt{1/D^2 - \cos^2(\phi_1^m + \theta_{1l}^m)}} \quad (50)$$

and, if $\theta_{1l}^m < \theta_B$,

$$A_1^m(\theta_{1l}^m) = \frac{\sin(\phi_1^m + \theta_{1l}^m) + i\delta \sqrt{\cos^2(\phi_1^m + \theta_{1l}^m) - 1/D^2}}{\sin(\phi_1^m + \theta_{1l}^m) - i\delta \sqrt{\cos^2(\phi_1^m + \theta_{1l}^m) - 1/D^2}} = e^{2i\varphi} \quad (51)$$

where

$$\tan \varphi = \frac{\delta \sqrt{\cos^2(\phi_1^m + \theta_{1l}^m) - 1/D^2}}{\sin(\phi_1^m + \theta_{1l}^m)}.$$

Contribution from the branch point (Head waves)

Next the contribution from the integral along $L_1 L_2$ is considered. L_1 and L_2 are taken along the path of steepest descent around B and tend to $\pi/2 + \theta_{1l}^m + i\infty$. Setting $\gamma = \sin(\phi_1^m + \alpha_i)$, we have from (39)

$$u_{L_1 L_2} = k_{b1} A_i \int_{L_2} \left\{ \frac{\gamma - \delta \sqrt{1/D^2 - 1 + \gamma^2}}{\gamma + \delta \sqrt{1/D^2 - 1 + \gamma^2}} \right\} e^{-ik_{b1} R_{1l}^m \cos(\alpha_i - \theta_{1l}^m)} d\alpha_i \\ = -4\delta k_{b1} A_i \int_{L_2} \frac{\gamma \sqrt{1/D^2 - 1 + \gamma^2}}{\gamma^2(1 - \delta^2) + \delta^2(1 - 1/D^2)} \\ \times e^{-ik_{b1} R_{1l}^m \cos(\alpha_i - \theta_{1l}^m)} d\alpha_i. \quad (52)$$

It should be noted that along L_2 near B

$$\text{Re}(\lambda_s) > 0 \text{ and } \text{Im}(\lambda_s) > 0. \quad (53)$$

From (45) along L_2 we can write

$$\cos(\alpha_i - \theta_{1l}^m) = \cos(\theta_B - \theta_{1l}^m) - i\tau, \quad \tau > 0, \quad (54)$$

therefore

$$d\alpha_i = i d\tau / \sin(\alpha_i - \theta_{1l}^m), \quad (55)$$

along L_2 near B . Putting

$$\alpha_i = \theta_B + u + iv, \quad (56)$$

for very small u and v , we have approximately

$$(u + iv) \sin(\theta_B - \theta_{1l}^m) = i\tau. \quad (57)$$

On the other hand, along L_2 we have

$$\sin(\alpha_i + \phi_1^m) = \sin(\theta_0 + u + iv) \\ = \sqrt{1 - 1/D^2} + \frac{1}{D}(u + iv), \quad (58)$$

and from (57)

$$\sin(\alpha_i + \phi_1^m) - \sqrt{1 - 1/D^2} = \frac{i\tau}{D \sin(\theta_B - \theta_{1l}^m)}. \quad (59)$$

Hence, in the limit when

$$\sin(\alpha_i + \phi_1^m) \rightarrow \sqrt{1 - 1/D^2},$$

along L_2 , we have using (53)

$$\sqrt{\sin^2(\alpha_i + \phi_1^m) + 1/D^2 - 1} \\ = \frac{\sqrt{2}(1 - 1/D^2)^{1/4}}{\sqrt{D} \cdot \sqrt{\sin(\theta_B - \theta_{1l}^m)}} \sqrt{\tau} \cdot e^{i\pi/4}, \quad (60)$$

As

$$\int_0^\infty \sqrt{\tau} \cdot e^{-ik_{b1} R_{11}^m \tau} d\tau = \frac{\sqrt{\pi}}{2} \frac{1}{(k_{b1} R_{11}^m)^{3/2}}, \quad (61)$$

we have

$$\begin{aligned} u_{L_1 L_2} &\approx -A_i \frac{2\sqrt{2\pi}\delta}{\sqrt{A}} \\ &\times \frac{1}{(k_{b1})^{1/2} (R_{11}^m)^{3/2}} \frac{1}{(1-1/A^2)^{1/4} [\sin(\theta_B - \theta_{11}^m)]^{3/2}} \\ &\times e^{-ik_{b1} R_{11}^m \cos(\theta_B - \theta_{11}^m) + (3/4)\pi i}. \end{aligned} \quad (62)$$

§ 6. Aperiodic solution

For computation of synthetic seismograms, it is convenient to choose a displacement of the form

$$\phi(t) = -\frac{A}{t^2 + C^2}, \quad A > 0, C > 0.$$

Performing the operation

$$\frac{1}{\pi} \operatorname{Re} \int_0^\infty d\omega \int_{-\infty}^\infty \phi(\sigma) e^{-i\omega\sigma} d\sigma, \quad (63)$$

on (37), (49) and (62) we obtain the following solutions

(1) Direct waves

$$\begin{aligned} \bar{u}_1 &= -A_i \frac{\pi}{\sqrt{2C_{b1}R_0}} \frac{A}{C^{5/2}} \frac{1}{\{1 + ((t-t_D)/C)^2\}^{3/4}} \\ &\times \cos \left\{ \frac{3}{2} \tan^{-1} \frac{t-t_D}{C} + \frac{\pi}{4} \right\}, \end{aligned} \quad (64)$$

where

$$t_D = R_0/C_{b1}.$$

(2) Waves reflected once from the interface

$$\begin{aligned} \bar{u}_s &= -A_i \frac{\pi}{\sqrt{2C_{b1}R_{11}^m}} \frac{A}{C^{5/2}} \frac{A_1^m(\theta_{11}^m)}{\{1 + ((t-R_{11}^m)/C)^2\}^{3/4}} \\ &\times \cos \left\{ \frac{3}{2} \tan^{-1} \frac{t-R_{11}^m}{C} + \frac{\pi}{4} \right\} \end{aligned} \quad (65)$$

for $\theta_{11}^m > \theta_B$

$$\begin{aligned} \bar{u}_s &= -A_i \frac{\pi}{\sqrt{2C_{b1}R_{11}^m}} \frac{A}{C^{5/2}} \frac{1}{\{1 + ((t-R_{11}^m)/C)^2\}^{3/4}} \\ &\times \cos \left\{ \frac{3}{2} \tan^{-1} \frac{t-R_{11}^m}{C} + \frac{\pi}{4} + 2\varphi \right\} \end{aligned}$$

$$\begin{aligned} t_{SAR} &= \frac{\overline{SA} + \overline{AR}}{C_{b1}} = \frac{\overline{S'R}}{C_{b1}} = \frac{\sqrt{\{d \cos \theta_2 - r \cos(\theta_2 - \theta)\}^2 + \{d \sin \theta_2 + r \sin(\theta_2 - \theta)\}^2}}{C_{b1}} \\ &= \frac{\sqrt{\{d - r \cos(2\theta_2 - \theta)\}^2 + r^2 \sin^2(2\theta_2 - \theta)}}{C_{b1}} = \frac{R_{11}^+}{C_{b1}} = R_{11}^+. \end{aligned} \quad (67)$$

Also, the travel time along the path $\overline{SBCR}$ is given by

for $\theta_{11}^m < \theta_B$, where $R_{11}^m = R_{11}^m/C_{b1}$.

(3) Head waves

$$\begin{aligned} \bar{u}_{L_1 L_2} &= A_i \frac{2\sqrt{2\pi}\delta}{\sqrt{A(1-1/A^2)^{1/4}}} \frac{1}{(R_{11}^m)^{3/2}} \\ &\times \frac{1}{[\sin(\theta_B - \theta_{11}^m)]^{3/2}} \frac{A}{C^{3/2}} \frac{1}{\{1 + ((t-H_{11}^m)/C)^2\}^{1/4}} \\ &\times \cos \left\{ \frac{1}{2} \tan^{-1} \frac{t-H_{11}^m}{C} + \frac{3}{4}\pi \right\}, \end{aligned} \quad (66)$$

where

$$H_{11}^m = \frac{R_{11}^m \cos(\theta_B - \theta_{11}^m)}{C_{b1}}.$$

§ 7. Interpretation of the travel time

In a study of wave propagation from a point source in a horizontal layer, HONDA and NAKAMURA (1954) evaluated the contributions from branch points and a saddle point and associated the time factors of these contributions with head waves and reflected waves. Similarly, in this section we will determine the reflected and head wave travel times and find that they are given by R_{11}^m and H_{11}^m , the time factors of the saddle point and branch point contributions as previously implied. Consider first the path $\overline{SAR}$ in Figure 3. Then

Fig. 3. Basic ray paths used in physical interpretation of contributions from branch and saddle points.

$$\begin{aligned}
t_{SBCR} &= \frac{\overline{SB}}{C_{b1}} + \frac{\overline{BC}}{C_{b2}} + \frac{\overline{CR}}{C_{b1}} \\
&= \frac{d \sin \theta_2}{C_{b1} \sin (\theta_B + \theta_2)} + \frac{1}{C_{b2}} \left\{ d \cos \theta_2 - r \cos (\theta_2 - \theta) - \frac{d \sin \theta_2 + r \sin (\theta_2 - \theta)}{\tan (\theta_B + \theta_2)} \right\} + \frac{r \sin (\theta_2 - \theta)}{C_{b1} \sin (\theta_B + \theta_2)} \\
&= \frac{1}{C_{b1}} \{ (d \cos \theta_2 - r \cos (\theta_2 - \theta)) \cos (\theta_B + \theta_2) + (d \sin \theta_2 + r \sin (\theta_2 - \theta)) \sin (\theta_B + \theta_2) \} \\
&= \frac{R_{11}^+}{C_{b1}} \cos (\theta_B - \theta_{11}^+) = H t_{11}^+,
\end{aligned} \tag{68}$$

since

$$\cos (\theta_B + \theta_2) = C_{b1} / C_{b2}.$$

Hence we have verified the interpretation of R_{11}^m and $H t_{11}^m$ as the reflected and head wave respectively for $m=+$ and $l=1$. In a similar manner, R_{11}^m and $H t_{11}^m$ can be interpreted for different values of m and l as shown in Figure 4. Obviously, for the observation point distant from the vertex, these waves arrive earlier than the diffracted waves which are produced by collisions of waves with the vertex. Hence this solution should adequately describe the early section of the seismogram.

Fig. 4. Ray paths of the head and reflected waves expressed by the first series term of the integrals.

§ 8. Range of existence of head waves

The range of existence of head waves can be determined by considering the process by which the integral (39) is evaluated. In Figure 2, when $\theta_B > \theta_{11}^m$, we can form a closed contour which connects with the original path and which includes contributions of the saddle point and the branch point on application of Cauchy's theorem. On the other hand when $\theta_B < \theta_{11}^m$, a closed contour cannot be made without excluding the integral

around the branch point. Therefore in the first case we have contributions from both the branch point (head waves) and saddle point (reflected waves) while in the second case there is only a saddle point contribution (reflected waves). It is clear that the critical condition to decide the existence of head waves is $\theta_B = \theta_{11}^m$. Obtaining θ_B and θ_{11}^m by the use of (17), (18) and (29), (30), (31), (32) respectively and using the critical condition we obtain the following transcendental equations:

$$\tan^{-1} \frac{r \sin (2\theta_2 - \theta)}{d - r \cos (2\theta_2 - \theta)} = \cos^{-1} \frac{1}{A} - \theta_2, \tag{69}$$

$$\tan^{-1} \frac{r \sin (2\theta_1 + 2\theta_2 + \theta)}{d - r \cos (2\theta_1 + 2\theta_2 + \theta)} = \cos^{-1} \frac{1}{A} - \theta_2, \tag{70}$$

$$\tan^{-1} \frac{r \sin (2\theta_1 + 2\theta_2 - \theta)}{d - r \cos (2\theta_1 + 2\theta_2 - \theta)} = \cos^{-1} \frac{1}{A} - \theta_2 - 2\theta_1, \tag{71}$$

$$\tan^{-1} \frac{r \sin (4\theta_1 + 2\theta_2 + \theta)}{d - r \cos (4\theta_1 + 2\theta_2 + \theta)} = \cos^{-1} \frac{1}{A} - \theta_2 - 2\theta_1. \tag{72}$$

In Figure 5, the range of existence of head waves whose paths are illustrated in Figure 4 is shown for $\theta_1 = 5^\circ$ and the observation point on the line $\theta = 0$. The abscissa is the ratio of source to observation point distances from the apex while the ordinate is the maximum θ_2 for which head waves exist. As expected from the small path difference, the range of existence of (a) and (b) are close as are (c) and (d). The range of existence also decreases with increasing number of reflections and decreasing ratio d/r .

For an observation point at 5° from the free surface and a source-vertex to observa-

Fig. 5. Maximum value of θ_2 for which the head waves shown in Fig. 4 exist versus the ratio of source to observation distances. The observation and source points are 5° from the free surface.

Fig. 6. Maximum value of the wedge angle ($\theta_1 + \theta_2$) for which the head waves of the types shown in Fig. 4 exist for an observation point at 5° from the free surface and $d/r=10.0$.

tion-vertex distance ratio of 10.0, Figure 6 shows the range of existence of head waves with changing θ_1 , which corresponds to a change of depth of the line source. It is noted that with increasing θ_1 , the dip angle for which the head waves (a) and (b) exist linearly increases while for head waves of type (c) and (d) it linearly decreases.

§ 9. Discontinuities

Discontinuities in displacement and stress in medium (1) arise because the first collision of the wave with a boundary changes from the free surface to the boundary between the media as α_i passes through zero. This

Fig. 7. Discontinuities in medium (1) due to interaction of the wave with the vertex. The lined areas indicate the regions for which the geometric wave from the last reflection exists with the term from which it arises indicated in brackets.

collision with the vertex results in a diffracted wave which is not considered in this solution. When both initially up-going and down-going waves and the interface of the last reflection are considered, we have four cases of the combination of the discontinuity as shown in Figure 7. The cross-hatched areas indicate the regions for which the geometric wave from the last reflection exists. From equations (11)–(14) the equations of these lines of discontinuity are

$$(1) \quad S_1^+: \theta = 2(N-1)\theta_1 + 2N\theta_2 - \pi \quad (73)$$

$$(2) \quad S_2^+: \theta = \pi - 2N\theta_1 - 2N\theta_2 \quad (74)$$

$$(3) \quad S_1^-: \theta = 2N'\theta_1 + 2N'\theta_2 - \pi \quad (75)$$

$$(4) \quad S_2^-: \theta = \pi - 2(N'+1)\theta_1 - 2N'\theta_2 \quad (76)$$

The discontinuities indicate a discrepancy of our solution from the complete solution of the physical problem. In order to obtain a quantitative estimation of the displacement discontinuities, plane waves incident toward the vertex and propagating at very small angles upward ($m=-$) and downward ($m=+$) have been examined. The resulting discontinuities are shown in Figure 8. For this geometry the reflected wave solution is a good approximation to the complete solution for $\theta_2 < 13^\circ$. However for $\theta_2 > 21^\circ$ the diffracted wave plays an important role. However, this formulation should adequately

Fig. 8. Relative amplitudes of the displacement discontinuities due to a plane initial wave close to the x -axis for propagation upward ($m=-$) and downward ($m=+$).

describe the early part of the seismogram as the diffracted waves from the vertex will arrive later than the initial phases.

It is seen that coincidence of the discontinuities in Figures 7b and 7c leads to at least partial cancellation of the discontinuities. Two special cases are of interest as total cancellation results.

(1) Lower boundary free or rigid

When the lower boundary of the wedge is either free or rigid then A_k^+ and A_k^- are +1 or -1 respectively. The condition

$$\theta_1 + \theta_2 = \frac{\pi}{N+N'},$$

leads to (73)=(76) or (74)=(75) so that the two lines of the discontinuities are coincident and no discontinuities exist. Hence the solution is complete and no diffracted waves exist.

(2) Surface source

If the line source is placed in the surface ($\theta_1=0$) then from (17) and (18) $A_k^+ = A_k^-$. For the particular situation

$$\theta_2 = \frac{\pi}{N+N'},$$

we have (73)=(76) or (74)=(75) and the two discontinuities coincide, hence in this case no discontinuities exist in medium (1).

As discussed in the paper (ISHII and ELLIS, 1970), in addition to the discontinuity within the wedge, discontinuities are generated by the vertex on each refraction into medium (2). The effect of these will not normally

be large on the surface of the wedge except close to the vertex as the amplitude decreases rather rapidly with distance and the wave will be partially reflected at the lower boundary of the wedge. Hence, in this discussion the discontinuities in medium (2) are expected to be less important than those in medium (1).

§ 10. Dispersion equation for the lower boundary free or rigid

In this section the dispersion equation is derived for a dipping structure in the simple case where medium (2) is either air or rigid. When medium (2) is air or rigid, A_n^+ and A_n^- become +1 or -1 respectively.

For $2m(\theta_1 + \theta_2) \ll 1$ we can write $\cos(\alpha_i + 2m(\theta_1 + \theta_2)) = \cos \alpha_i - 2m(\theta_1 + \theta_2) \sin \alpha_i$. In this case

$$\begin{aligned} S_1^+(N) = & A_i e^{-ik_{b1}d \cos \alpha_i} \\ & \times \frac{1 - (\pm 1)^m e^{-ik_{b1}r(2(m-1)\theta_1 + 2m\theta_2) \sin \alpha_i}}{1 \mp e^{-ik_{b1}r(2\theta_1 + 2\theta_2) \sin \alpha_i}} \\ & \times \{e^{ik_{b1}r \cos(\alpha_i + 2\theta_2 - \theta)} \\ & + e^{ik_{b1}r \cos(\alpha_i + 2m\theta_1 + 2(m+1)\theta_2 - \theta)} + \dots\}, \quad (77) \end{aligned}$$

$$\begin{aligned} S_2^+(N) = & A_i e^{-ik_{b1}d \cos \alpha_i} \\ & \times \frac{1 - (\pm 1)^m e^{-ik_{b1}r(2m\theta_1 + 2m\theta_2) \sin \alpha_i}}{1 \mp e^{-ik_{b1}r(2\theta_1 + 2\theta_2) \sin \alpha_i}} \\ & \times \{e^{ik_{b1}r \cos(\alpha_i + 2\theta_1 + 2\theta_2 + \theta)} \\ & + e^{ik_{b1}r \cos(\alpha_i + 2(m+1)\theta_1 + 2(m+1)\theta_2 + \theta)} + \dots\}. \quad (78) \end{aligned}$$

Similarly the expressions for $S_1^-(N)$ and $S_2^-(N)$ can be obtained. Operating with

$k_{b1} \int_{-i\infty}^{\pi+i\infty} d\alpha_i$, poles appear from the relation

$$1 \mp e^{-ik_{b1}r(2\theta_1 + 2\theta_2) \sin \alpha_i} = 0,$$

which yields

$$\begin{aligned} \sin(k_{b1}r(\theta_1 + \theta_2) \sin \alpha_i) &= 0 \\ \cos(k_{b1}r(\theta_1 + \theta_2) \sin \alpha_i) &= 0. \end{aligned}$$

For real α_i , we can then write

$$\frac{\omega}{C_n} r(\theta_1 + \theta_2) \sqrt{\left(\frac{C_n}{C_{b1}}\right)^2 - 1} = \begin{cases} n\pi \\ (n-1/2)\pi \end{cases}, \quad (79)$$

where C_n is the phase velocity as $\cos \alpha_i = C_{b1}/C_n$ ($0 < \alpha_i < \pi$).

For the case of a rigid bottom, the ex-

pression is the same as that obtained by NAGUMO (1961). Further, if we put $r(\theta_1 + \theta_2) = H$ (H is the depth in the case of a horizontal layer), the dispersion relation (79) coincides with that of the horizontally layered case. NAGUMO (1961) has called C_n and $V_n = \partial\omega/\partial k_n$ the formal phase and group velocity to differentiate from the observed surface velocities.

§ 11. The horizontal layer solution

It is interesting to derive the solution for a horizontal layer using our method as the transition of the solution to the horizontal layer case may suggest a method for obtaining the surface wave solutions for the dipping layer. For the diffracted wave problem, it is useful to study this transition as the quantitative and qualitative behaviour of the discontinuities as they approach zero for zero dip angle may indicate the nature of the diffracted solution.

Fig. 9. Coordinate system for the horizontal layer case with the source (S) at $(d, 0)$ and the receiver (R) at (x, y) .

The same (x, y) coordinate system is used with the x -axis now being horizontal (Figure 9). The source is placed at $(d, 0)$ in the layer of thickness $H = H_1 + H_2$. Employing the same procedure as for the dipping layer, we obtain the displacement for the time variations $e^{i\omega t}$,

$$\begin{aligned}
 u_1 = & k_{b1} \int_{-\infty}^{\pi+i\infty} A_i [e^{-ik_{b1}\{(d-x)\cos\alpha_i + |y|\sin\alpha_i\}} + e^{-ik_{b1}\{(d-x)\cos\alpha_i + (2H_1+y)\sin\alpha_i\}} \\
 & + \sum_{n=1}^{\infty} A_1^n \{e^{-ik_{b1}\{(d-x)\cos\alpha_i + (2((n-1)H_1+nH_2)-y)\sin\alpha_i\}} + e^{-ik_{b1}\{(d-x)\cos\alpha_i + (2(nH_1+nH_2)+y)\sin\alpha_i\}} \\
 & + e^{-ik_{b1}\{(d-x)\cos\alpha_i + (2(nH_1+nH_2)-y)\sin\alpha_i\}} + e^{-ik_{b1}\{(d-x)\cos\alpha_i + (2((n+1)H_1+nH_2)+y)\sin\alpha_i\}}\}] d\alpha_i \\
 = & k_{b1} \int_{-\infty}^{\pi+i\infty} A_i [e^{-ik_{b1}R_0\cos(\alpha_i-\theta_0^-)} + e^{-ik_{b1}R_0^-\cos(\alpha_i-\theta_0^-)} + \sum_{n=1}^{\infty} A_1^n \{e^{-ik_{b1}R_{n1}^+\cos(\alpha_i-\theta_{n1}^+)} \\
 & + e^{-ik_{b1}R_{n2}^+\cos(\alpha_i-\theta_{n2}^+)} + e^{-ik_{b1}R_{n1}^-\cos(\alpha_i-\theta_{n1}^-)} + e^{-ik_{b1}R_{n2}^-\cos(\alpha_i-\theta_{n2}^-)}\}] d\alpha_i, \quad (80)
 \end{aligned}$$

where

$$A_1 = \frac{d \sin \alpha_i - \delta \sqrt{1 - d^2 \cos^2 \alpha_i}}{d \sin \alpha_i + \delta \sqrt{1 - d^2 \cos^2 \alpha_i}}, \quad (81)$$

$$\left. \begin{aligned}
 R_0 &= \sqrt{(d-x)^2 + y^2} & \tan \theta_0 &= |y|/(d-x) \\
 R_0^- &= \sqrt{(d-x)^2 + (2H_1+y)^2} & \tan \theta_0^- &= (2H_1+y)/(d-x) \\
 R_{n1}^+ &= \sqrt{(d-x)^2 + \{2(nH-H_1)-y\}^2} & \tan \theta_{n1}^+ &= \{2(nH-H_1)-y\}/(d-x) \\
 R_{n2}^+ &= \sqrt{(d-x)^2 + (2nH+y)^2} & \tan \theta_{n2}^+ &= (2nH+y)/(d-x) \\
 R_{n1}^- &= \sqrt{(d-x)^2 + (2nH-y)^2} & \tan \theta_{n1}^- &= (2nH-y)/(d-x) \\
 R_{n2}^- &= \sqrt{(d-x)^2 + \{2(nH+H_1)+y\}^2} & \tan \theta_{n2}^- &= \{2(nH+H_1)+y\}/(d-x)
 \end{aligned} \right\}. \quad (82)$$

As equation (80) is of the same form as (38), formulae (65) and (66) can be applied for the reflected and head waves respectively. Therefore the variations of the waveforms depend only on the values R_{ni}^m , θ_{ni}^m and θ_B . Equation (80) can also be derived from (38),

the formal dipping layer solution, if we take the limit

$$r \rightarrow \infty, \quad \theta_1 \rightarrow 0, \quad \theta_2 \rightarrow 0$$

as

$$r \sin \theta_1 = H_1 \quad \text{and} \quad r \sin \theta_2 = H_2. \quad (83)$$

In the case of a horizontal layer, surface waves appear from contributions of poles. As our solution transforms to the horizontal layer solution, we could investigate surface waves in the case of a dipping layer if the finite series expression of our solution can be changed into a compact form which corresponds to a normal mode expression.

§ 12. Computation of displacement seismograms

For the direct wave, head waves, and waves once reflected from the boundary, displacements have been calculated for the three cases shown in Figure 10. Elastic constants are again those employed by HASKELL (1960). Ray paths of waves which contribute to the seismogram are shown in Figure 11.

The component waves are shown on the time-displacement plot of Figure 12 and the arrival times corresponding to the ray paths indicated by lettered arrows. A detailed feature is the small amplitude of the head waves compared to the direct and reflected waves. This is expected from inspection of equations (65) and (66) which show that the head waves decrease as $1/(R_{11})^{3/2}$ and the reflected waves as $1/(R_{11})^{1/2}$. Although the

Fig. 10. Three cases for which theoretical seismograms were calculated. The parameters used were: $H_1=9.59$ km, $H_2=3.00$ km, $D=99.6$ km, $d=10.0$ km, and the displacement parameter $c=0.05$ sec.

Fig. 11. Ray paths which contribute to the theoretical seismograms.

travel times changed significantly, the wave forms of the refracted and reflected waves do not undergo large changes for the three cases illustrated.

Figure 13 represents the seismograms synthesized from the components of Figure 12. The seismograms look very different. However, the different arrivals are all recognizable except for the head wave (b) is embedded in the wave forms of the direct wave (c) and the reflected wave (d) in the case of the horizontal layer. A very noticeable feature is the late arrival of the reflected wave (e) in the case of the horizontal layer. More multiply reflected waves will appear as later phases. As the distance between the observation point and the vertex is 10.0 km and the velocities of medium (1) and medium (2) are 3.64 km/sec and 4.62 km/sec respectively, the diffracted waves hardly contribute to the section of the seismogram

Fig. 12. Displacements of the component waves for the geometries given in Figs. 10(a), 10(b) and 10(c).

Fig. 13. Synthesized seismograms resulting from the displacements of Fig. 12.

shown here as the diffracted wave arriving 4 to 5 sec after the first arrival is due to a head wave of small amplitude interacting with the vertex.

§13. Summary

The propagation of SH waves from a line source in a dipping layer overlying an elastic medium has been investigated using multiple reflection formulation. A formal solution which does not include diffracted waves has been obtained. The first two series

terms of the integral have been evaluated using the method of steepest descent to obtain displacements for both a harmonic and an aperiodic time variation and contributions have been interpreted using ray paths in terms of head and reflected waves. If in the integral the branch points are smaller than the saddle points, head waves do not appear. Hence the range of existence of the various types of head waves may be determined.

Using the same technique, the solution in

the case of a horizontal layer has also been found and comparison made with the dipping layer through numerical examples. The wave forms of the arrivals do not differ greatly; however, the character of the synthetic seismogram markedly changes due to changes in arrival times.

Discontinuities in displacement which are associated with the diffracted wave have been studied. For the special cases in which the lower boundary is free or rigid it is found that our solution is the complete solution. In other cases, our solution can be applied to the initial section of the seismogram.

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Appendix

Evaluation of the second series terms of the integral

As a guide to computation of higher order terms in the series, a summary of the evaluation procedures and results for the contributions by waves twice reflected from the boundary between the elastic media are evaluated here. The second terms of the series have the form

$$I_{2l}^m = k_{b1} \int_{-i\infty}^{\pi+i\infty} A_1^m A_2^m e^{-ik_{b1} R_{2l}^m \cos(\alpha_i - \theta_{2l}^m)} d\alpha_i, \quad (A.1)$$

where

$$\left. \begin{aligned} A_1^m &= \frac{d \sin(\phi_1^m + \alpha_i) - \delta \sqrt{1 - d^2 \cos^2(\phi_1^m + \alpha_i)}}{d \sin(\phi_1^m + \alpha_i) + \delta \sqrt{1 - d^2 \cos^2(\phi_1^m + \alpha_i)}} \\ A_2^m &= \frac{d \sin(\phi_2^m + \alpha_i) - \delta \sqrt{1 - d^2 \cos^2(\phi_2^m + \alpha_i)}}{d \sin(\phi_2^m + \alpha_i) + \delta \sqrt{1 - d^2 \cos^2(\phi_2^m + \alpha_i)}} \end{aligned} \right\} \quad (A.2)$$

and $l=1, 2$ and $m=+, -$.

From equations (17) and (18)

$$\left. \begin{aligned} \phi_1^+ &= \theta_2 & \phi_1^- &= 2\theta_1 + \theta_2 \\ \phi_2^+ &= 2\theta_1 + 3\theta_2 & \phi_2^- &= 4\theta_1 + 3\theta_2 \end{aligned} \right\}, \quad (A.3)$$

R_{2l}^m and θ_{2l}^m have been given by equations (29) to (32).

As the integrands of (A.1) contain the expressions

$$\left. \begin{aligned} \lambda_{s1} &= \sqrt{1 - d^2 \cos^2(\phi_1^m + \alpha_i)} \\ \lambda_{s2} &= \sqrt{1 - d^2 \cos^2(\phi_2^m + \alpha_i)} \end{aligned} \right\}, \quad (A.4)$$

which are both two-valued, a four-sheeted Riemann surface is required for their representation. The branch cuts, along which the four sheets coalesce are defined by $\text{Re}(\lambda_{s1})=0$ and $\text{Re}(\lambda_{s2})=0$. For evaluation purposes the medium is assumed to be very slightly absorptive as before. The sheets I, II, III and IV are defined corresponding to the combinations $(\text{Re}(\lambda_{s1}) < 0, \text{Re}(\lambda_{s2}) > 0)$,

$$\left. \begin{aligned} (\text{Re}(\lambda_{s1}) > 0, \text{Re}(\lambda_{s2}) > 0) \\ (\text{Re}(\lambda_{s1}) > 0, \text{Re}(\lambda_{s2}) < 0) \end{aligned} \right\}, \quad (A.5)$$

and $(\text{Re}(\lambda_{s1}) < 0, \text{Re}(\lambda_{s2}) < 0)$ respectively. The original path of integration can be shifted on any sheet of the Riemann surface for the factor $e^{-ik_{b1} R_{2l}^m \cos(\alpha_i - \theta_{2l}^m)}$ vanishing along the path at a large distance from the origin. The original path L is taken on sheet II where the relations $\text{Im}(\sin \alpha_i) < 0$, $\text{Im}(\lambda_{s1}) < 0$ and $\text{Im}(\lambda_{s2}) < 0$ hold along L .

As an example, when

$$\theta_B > \theta_C > \theta_s,$$

where

$$\left. \begin{aligned} \theta_B &= \theta_0 - \phi_1^m \\ \theta_C &= \theta_0 - \phi_2^m \\ \theta_0 &= \text{Arc} \cos(1/d) \\ \theta_s &= \theta_{2l}^m \end{aligned} \right\}, \quad (A.6)$$

the original path can be shifted to L_s , (L_3, L_4) and (L_1, L_2) as shown in Figure A1. L_s passes through the saddle point S , and the contours (L_3, L_4) and (L_1, L_2) go around the branch points C and B respectively, each one of them being drawn along the path of steepest descent given by

$$\cos(x - \theta_{2l}^m) \cosh y = 1, \quad (A.7)$$

$$\cos(x - \theta_{2l}^m) \cosh y = \cos(\theta_C - \theta_{2l}^m) \quad (A.8)$$

and

Fig. A1. The α_i -plane showing branch cuts and integral paths for evaluation of the second series term of the integrals. Notation: B, C-branch points; S-saddle point; L-original path of integration; L_s -path of steepest descent through saddle point; and L_i ($i=1, 2 \dots$)-paths of branch line integral.

$$\cos(x - \theta_{2l}^m) \cosh y = \cos(\theta_B - \theta_{2l}^m), \quad (\text{A.9})$$

where

$$\alpha_i = x + iy,$$

Integral around C

The contour integrals along (L_3, L_4) can be evaluated by the same procedure as before. By noting the relations

$$\left. \begin{array}{l} \text{Re}(\lambda_{s1}) < 0 \quad \text{Im}(\lambda_{s1}) < 0 \\ \text{Re}(\lambda_{s2}) > 0 \quad \text{Im}(\lambda_{s2}) > 0 \end{array} \right\}, \quad (\text{A.10})$$

for the path L_4 near C on sheet I, for a harmonic time variation the contribution to the displacement is found to be

$$u_{L_4} = -A_i \frac{2\sqrt{2\pi} \cdot \delta}{\sqrt{A}}$$

$$\times \frac{1}{(k_{b1})^{1/2} (R_{2l}^m)^{3/2}} \frac{1}{(1 - 1/A^2)^{1/4} \{\sin(\theta_C - \theta_{2l}^m)\}^{3/2}} \times e^{-ik_{b1} R_{2l}^m \cos(\theta_C - \theta_{2l}^m) + 2i\varphi_4 + (3\pi/4)i}, \quad (\text{A.11})$$

where

$$\tan \varphi_4 = \frac{\delta \sqrt{A^2 \cos^2(\phi_1^m + \theta_C) - 1}}{A \sin(\phi_1^m + \theta_C)}. \quad (\text{A.12})$$

Integral around B

The contour integrals along (L_1, L_2) can also be evaluated by the same procedure as before. By noting the relations

$$\left. \begin{array}{l} \text{Re}(\lambda_{s1}) > 0 \quad \text{Im}(\lambda_{s1}) > 0 \\ \text{Re}(\lambda_{s2}) > 0 \quad \text{Im}(\lambda_{s2}) > 0 \end{array} \right\}, \quad (\text{A.13})$$

for the path L_2 near B lying on sheet II, for a harmonic time variation the contribution to the displacement is found to be

$$u_{L_1 L_2} = -A_i \frac{2\sqrt{2\pi} \cdot \delta}{\sqrt{A}} \times \frac{1}{(k_{b1})^{1/2} (R_{2l}^m)^{3/2}} \frac{1}{(1 - 1/A^2)^{1/4} \{\sin(\theta_B - \theta_{2l}^m)\}^{3/2}} \times \frac{A \sin(\phi_2^m + \theta_B) - \delta \sqrt{1 - A^2 \cos^2(\phi_2^m + \theta_B)}}{A \sin(\phi_2^m + \theta_B) + \delta \sqrt{1 - A^2 \cos^2(\phi_2^m + \theta_B)}} \times e^{-ik_{b1} R_{2l}^m \cos(\theta_B - \theta_{2l}^m) + (3\pi/4)i}. \quad (\text{A.14})$$

Integral through S

The contour integral along L_s can also be evaluated by the same procedure as before. For a harmonic time variation the contribution to the displacement is found to be

$$u_s = A_i \sqrt{\frac{2\pi k_{b1}}{R_{2l}^m}} A_1^m(\theta_{2l}^m) A_2^m(\theta_{2l}^m) e^{-ik_{b1} R_{2l}^m + (\pi/4)i}, \quad (\text{A.15})$$

where for $\theta_{2l}^m > \theta_B > \theta_C$

$$\left. \begin{array}{l} A_1^m(\theta_{2l}^m) = \frac{A \sin(\phi_1^m + \theta_{2l}^m) - \delta \sqrt{1 - A^2 \cos^2(\phi_1^m + \theta_{2l}^m)}}{A \sin(\phi_1^m + \theta_{2l}^m) + \delta \sqrt{1 - A^2 \cos^2(\phi_1^m + \theta_{2l}^m)}} \\ A_2^m(\theta_{2l}^m) = \frac{A \sin(\phi_2^m + \theta_{2l}^m) - \delta \sqrt{1 - A^2 \cos^2(\phi_2^m + \theta_{2l}^m)}}{A \sin(\phi_2^m + \theta_{2l}^m) + \delta \sqrt{1 - A^2 \cos^2(\phi_2^m + \theta_{2l}^m)}} \end{array} \right\} \quad (\text{A.16})$$

if $\theta_B > \theta_{2l}^m > \theta_C$

$$\left. \begin{array}{l} A_1^m(\theta_{2l}^m) = e^{2i\varphi_1} \\ A_2^m(\theta_{2l}^m) = \text{same as (A.16)} \end{array} \right\}, \quad (\text{A.17})$$

$$\tan \varphi_1 = \frac{\delta \sqrt{A^2 \cos^2(\phi_1^m + \theta_{2l}^m) - 1}}{A \sin(\phi_1^m + \theta_{2l}^m)}, \quad (\text{A.18})$$

if $\theta_B > \theta_C > \theta_{2l}^m$

as

$$\left. \begin{aligned} A_1^m(\theta_{2l}^m) &= e^{2i\varphi_1} \\ A_2^m(\theta_{2l}^m) &= e^{2i\varphi_2} \end{aligned} \right\}, \quad (A.19) \quad \phi(t) = -\frac{A}{t^2 + C^2}, \quad A > 0, \quad C > 0,$$

the operation

$$\tan \varphi_2 = \frac{\delta \sqrt{\Delta^2 \cos^2(\phi_2^m + \theta_{2l}^m) - 1}}{\Delta \sin(\phi_2^m + \theta_{2l}^m)}. \quad (A.20) \quad \frac{1}{\pi} \operatorname{Re} \int_0^\infty d\omega \int_{-\infty}^\infty \phi(\sigma) e^{-i\omega\sigma} d\sigma,$$

Aperiodic solution

applied to (A.11), (A.14) and (A.15) yields the

When the motions are aperiodic and vary following solutions: *Head waves*,

$$\begin{aligned} \bar{u}_{L_3 L_4} &= A_i \frac{2\sqrt{2}\pi\delta}{\sqrt{\Delta}(1-\Delta^2)^{1/4}} \frac{\sqrt{C_{b1}}}{(R_{2l}^m)^{3/2}} \frac{1}{\{\sin(\theta_C - \theta_{2l}^m)\}^{3/2}} \\ &\times \frac{A}{C^{3/2}} \frac{1}{\{1 + ((t - H_C t_{2l}^m)/C)^2\}^{1/4}} \cos \left\{ \frac{1}{2} \tan^{-1} \frac{t - H_C t_{2l}^m}{C} + 2\varphi_4 + \frac{3}{4}\pi \right\}, \end{aligned} \quad (A.21)$$

where

$$H_C t_{2l}^m = \frac{R_{2l}^m \cos(\theta_C - \theta_{2l}^m)}{C_{b1}}, \quad (A.22)$$

$$\begin{aligned} \bar{u}_{L_1 L_2} &= A_i \frac{2\sqrt{2}\pi\delta}{\sqrt{\Delta}(1-\Delta^2)^{1/4}} \frac{\sqrt{C_{b1}}}{(R_{2l}^m)^{3/2}} \frac{1}{\{\sin(\theta_B - \theta_{2l}^m)\}^{3/2}} \frac{A}{C^{3/2}} \frac{1}{\{1 + ((t - H_B t_{2l}^m)/C)^2\}^{1/4}} \\ &\times \frac{\Delta \sin(\phi_2^m + \theta_B) - \delta \sqrt{1 - \Delta^2 \cos^2(\phi_2^m + \theta_B)}}{\Delta \sin(\phi_2^m + \theta_B) + \delta \sqrt{1 - \Delta^2 \cos^2(\phi_2^m + \theta_B)}} \cos \left\{ \frac{1}{2} \tan^{-1} \frac{t - H_B t_{2l}^m}{C} + \frac{3}{4}\pi \right\}, \end{aligned} \quad (A.23)$$

where

$$H_B t_{2l}^m = \frac{R_{2l}^m \cos(\theta_B - \theta_{2l}^m)}{C_{b1}}. \quad (A.24)$$

Reflected waves,

$$\bar{u}_s = -A_i \frac{\pi}{\sqrt{2C_{b1}R_{2l}^m}} \frac{A}{C^{5/2}} \frac{A_1^m(\theta_{2l}^m)A_2^m(\theta_{2l}^m)}{\{1 + ((t - R t_{2l}^m)/C)^2\}^{3/4}} \cos \left\{ \frac{3}{2} \tan^{-1} \frac{t - R t_{2l}^m}{C} + \frac{\pi}{4} \right\} \quad \text{for } \theta_{2l}^m > \theta_B > \theta_C, \quad (A.25)$$

$$\bar{u}_s = -A_i \frac{\pi}{\sqrt{2C_{b1}R_{2l}^m}} \frac{A}{C^{5/2}} \frac{A_2^m(\theta_{2l}^m)}{\{1 + ((t - R t_{2l}^m)/C)^2\}^{3/4}} \cos \left\{ \frac{3}{2} \tan^{-1} \frac{t - R t_{2l}^m}{C} + 2\varphi_1 + \frac{\pi}{4} \right\} \quad \text{for } \theta_B > \theta_{2l}^m > \theta_C, \quad (A.26)$$

$$\bar{u}_s = -A_i \frac{\pi}{\sqrt{2C_{b1}R_{2l}^m}} \frac{A}{C^{5/2}} \frac{1}{\{1 + ((t - R t_{2l}^m)/C)^2\}^{3/4}} \cos \left\{ \frac{3}{2} \tan^{-1} \frac{t - R t_{2l}^m}{C} + 2\varphi_1 + 2\varphi_2 + \frac{\pi}{4} \right\} \quad \text{for } \theta_B > \theta_C > \theta_{2l}^m, \quad (A.27)$$

where

$$R t_{2l}^m = R_{2l}^m / C_{b1}. \quad (A.28)$$

If the branch points are smaller than the saddle points head waves do not appear. The ray paths for arrivals which travel along part of the path as head waves are

shown in Figure A 2.

Hence, except for diffracted waves, we can formally obtain a complete synthetic seismogram in the case of a dipping layer

Fig. A 2. Ray paths of the head waves expressed by the second series term of the integrals.

by applying this procedure to the third and higher order series terms of the formal integral solution.

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