The static deformation of gravitating elastic earth model is studied in some detail. It is found that the problem becomes well posed if we discard the stress-strain relation in the liquid core. A convenient formulation, in which derivative of density does not appear, is proposed in the liquid core by introducing a new variable. Static solutions obtained by the present theory are compared with dynamic solutions in which the liquid core causes no trouble. The differences between them are within 0.2%. The elastic deformation due to earth's rotation and the deformation of degree 1 modes induced by surface loads are also computed.

1. Introduction

A peculiarity inherent in the equations for the elastic-gravitational equilibrium of a liquid core of the earth has long been noticed and discussed by several authors (Takeuchi, 1950; Longman, 1963; Jeffreys and Vicente, 1966). Recently Smylie and Mansinha reawoke interest in this problem by claiming that radial displacement for static deformation need not be continuous across the core-mantle boundary (Smylie and Mansinha, 1971a, b). As Dahlen (1971) points out, their arguments are self-contradictory. Thus, starting with a statement that equipotential surfaces, isobaric surfaces and surfaces of equal density are parallel in a liquid core, they have ended by denying it. On the other hand, the conventional treatment assumes the condition of Adams-Williamson throughout the core, but according to the present knowledge about the core as revealed by seismic waves it is unlikely that the liquid core of the real earth should strictly be in Adams-Williamson equilibrium.

The present study is intended to reexamine the problem of static deformation of the earth. In the following section the equilibrium equations are studied in some detail for various cases, and integration schemes are proposed for each case. Emphasis is placed upon the case \( \mu(\text{rigidity})=0, n \neq 0 \).
2. The Equations of Equilibrium

We assume the earth to be a gravitating, isotropic, elastic sphere of radius \( a \) whose density and elasticity are variable only with depth. The equations of equilibrium for the deformed earth can be derived immediately from the equations for the free oscillation of the earth by setting the frequency equal to zero. There exist two distinct modes of deformation, corresponding to spheroidal and toroidal oscillations, but since tidal forces, as well as surface mass load, produce only the spheroidal modes of deformation, the toroidal modes will not be considered in this paper.

We strictly follow the notation and formulation by Takeuchi and Saito (1972, subsequently referred to as paper I) which are originally due to Alteman et al. (1959). A modification has been made for the definition of \( y_n \). We define \( y_n \) by

\[
\frac{\partial \psi}{\partial r} - 4\pi G \rho \mu r + \frac{n+1}{r} \psi = y_n(r) Y_n(\theta, \phi).
\]

We will see below that the new definition simplifies boundary conditions slightly.

In the following we discuss several cases separately.

2.1 General case \((\mu \neq 0, n \geq 2)\)

The equations of equilibrium for \( n \geq 2 \) in a solid layer are derived by setting in Eq. I(82) (see Appendix 1, equation numbers preceded by I refer to equations in paper I), \( \omega = 0 \), \( A = C = \lambda + 2\mu \), \( F = \lambda \), and \( L = N = \mu \), where \( \lambda(r) \) and \( \mu(r) \) are Lamé's elastic modulus and rigidity, respectively. Although not stated explicitly, these substitutions are to be understood whenever reference to paper I is made hereafter.

For a moment let us suppose that earth’s core is solid, then Eq. I(82) applies to the entire earth. It has six linearly independent integrations but three of them become infinite at the center. The three regular integrations for a given earth model will be obtained by assigning appropriate initial values to \( y_i \) \((i = 1, 2, \ldots, 6)\) near the center and integrating Eq. I(82) upward. Necessary formulas for initial value calculation will be found in Eqs. I(98)–I(103).

An arbitrary solution is expressed by a linear combination of the three integrations thus obtained. Introducing a second subscript on \( y_i \) in order to distinguish the three independent integrations, we write a solution as

\[
y_i(r) = \sum_{j=1}^{3} Q_j y_{ij}(r) \quad 1 \leq i \leq 6.
\]

Three constants of integration, \( Q_1 \), \( Q_2 \) and \( Q_3 \), are determined by the surface boundary conditions.
Tidal deformation. Let us first consider deformation due to a tidal potential

\[ \phi_t = \left( \frac{r}{a} \right)^n Y_n(\theta, \phi). \]  \hspace{1cm} (2)

The earth yields to the tidal force and undergoes redistribution of masses, which in turn gives rise to the perturbation in the gravitational potential. Let \( \phi_d \) be the perturbation in the gravitational potential due to mass redistribution, then the total perturbation in potential is \( \phi = \phi_t + \phi_d \). Equation I(16), which has been derived for \( \phi_d \), applies also to \( \phi \) since \( \phi_t \) is harmonic everywhere. Note that we define \( \phi = y_0(r) Y_n(\theta, \phi) \) inside the earth. Outside the earth we assume

\[ \phi_d = k_n \left( \frac{a}{r} \right)^{n+1} Y_n(\theta, \phi) \hspace{1cm} r \geq a \]  \hspace{1cm} (3)

where \( k_n \) is some constant to be determined. Actually \( k_n \) denotes one of the tidal Love numbers because Love numbers \( h_n, l_n, \) and \( k_n \) are defined by

\[ u_r(a, \theta, \phi) = \frac{h_n}{g(a)} Y_n(\theta, \phi) \]

\[ u_\theta(a, \theta, \phi) = \frac{l_n}{g(a)} \frac{\partial Y_n(\theta, \phi)}{\partial \theta} \]

\[ \psi_d(a, \theta, \phi) = k_n Y_n(\theta, \phi). \]

The potential should satisfy the following boundary conditions across the surface,

\[ \begin{align*}
(\phi)_i & = (\phi)_e \\
\left( \frac{\partial \phi_d}{\partial r} - 4\pi G \rho u_r \right)_i & = \left( \frac{\partial \phi_d}{\partial r} \right)_e
\end{align*} \]  \hspace{1cm} (4)

where the subscripts \( i \) and \( e \) refer to values just inside and outside the earth's surface, respectively. Remembering \( (\phi)_i = y_0(r) Y_n(\theta, \phi) \), and making use of Eqs. (2) and (3), we find that the surface boundary conditions for tidal deformation are as follows.

\[ y_2(a) = \sum_{j=1}^{n} Q_j y_{2j}(a) = 0 \]

\[ y_4(a) = \sum_{j=1}^{n} Q_j y_{4j}(a) = 0 \]  \hspace{1cm} (5)

\[ y_6(a) = \sum_{j=1}^{n} Q_j y_{6j}(a) = \frac{2n+1}{a}. \]

After solving Eq. (5) for the three constants of integration, we can calculate displacements at arbitrary place by Eq. (1). In particular, we find
Load deformation. Next we consider deformation due to a surface mass load \( \sigma_n Y_n(\theta, \phi) \) per unit area. It exerts a normal stress, \(-g(a)\sigma_n Y_n(\theta, \phi)\), across the surface of the earth, whence we should have \( y_2(a) = -g(a)\sigma_n \). In addition, the load produces a potential

\[
\phi_i = \frac{4\pi G\sigma_n}{2n+1} \left( \frac{r}{a} \right)^n Y_n(\theta, \phi) \quad r \leq a
\]
\[
= \frac{4\pi G\sigma_n}{2n+1} \left( \frac{a}{r} \right)^{n+1} Y_n(\theta, \phi) \quad r \geq a.
\]

The total perturbation in potential is again given in terms of \( \phi_i \) and \( \phi_d \), the latter, in terms of load Love number \( k'_n \), being written as

\[
\phi_d = k'_n \frac{4\pi G\sigma_n}{2n+1} \left( \frac{a}{r} \right)^{n+1} Y_n(\theta, \phi) \quad r \geq a.
\]

Substituting these expressions into Eq. (4) and taking \( \sigma_n = (2n+1)/4\pi Ga \) for the sake of normalization, we find

![Fig. 1. Static tidal deformation of Wang's earth model for \( n=2 \). \( y_1 \) and \( y_3 \) refer to the top scale and are given in unit of \( \text{km}/(\text{km/sec})^2 \). The arrows indicate core-mantle boundary and inner core-outer core boundary.](image-url)
The load Love numbers, $k'_n$, $h'_n$, and $l'_n$, will be found by relations identical to Eq. (6).

For later reference we put Eqs. (5) and (9) into a single equation

$$y_3(a) = -\frac{2n+1}{a} \frac{g(a)}{4\pi G} y_n(a) = 0 \quad y_6(a) = \frac{2n+1}{a}.$$  \hspace{1cm} (9)

where $\sigma'_n = 0$ for tidal deformation and $\sigma'_n = 1$ for load deformation.

Sample solutions are given in Figs. 1 and 2. Since normalization factors have been so chosen that disturbing potentials become nondimensional, radial functions $y_1$ and $y_3$ have dimension of length/potential. Unit of $y_1$ and $y_3$ in these figures is, therefore, km/(km/sec)^2.

Figures 1 and 2 are computed using Wang's earth model which has a liquid outer core (WANG, 1972). Treatment of liquid core will be discussed in Section 2.3.

2.2 Radial deformation ($n=0$)

Although a zero degree tidal potential has no physical meaning, a problem of uniform loading over the earth's surface makes sense. Pertinent
differential equations are given in I(89). One of the integrations \( y_{n}(r) \) will be obtained starting from the following initial values.

\[
\begin{align*}
ry_{1}(r) &= (kr)j_{1}(kr) \\
\ell^{2}y_{2}(r) &= (\lambda + 2\mu)(kr)^{2}j_{0}(kr) - 4\mu(kr)j_{1}(kr) \\
y_{3}(r) &= 3\gamma[1 - j_{0}(kr)]
\end{align*}
\] (11)

\[
k^{2} = \frac{4\gamma}{a^{3}} \quad \gamma = \frac{4\pi G \rho}{3}.
\]

In addition, the third equation in I(89) has a trivial integration, \( y_{3}(r) = \text{constant} \). Thus, an arbitrary solution will be written as

\[
\begin{align*}
y_{1}(r) &= Q_{1}y_{1}(r) \\
y_{2}(r) &= y_{2}(r) \\
y_{3}(r) &= Q_{1}y_{3}(r) + Q_{3}
\end{align*}
\] (12)

Making use of the boundary conditions (9), we find

\[
Q_{1} = \frac{1}{y_{n}(a)} \frac{g(a)}{4\pi Ga} \quad h_{0} = g(a)y_{1}(a) \quad l_{0} = k_{0} = 0.
\] (13)

The vanishing of \( k_{0} \) is expected because \( k_{0} \) is proportional to the change in the earth's mass which should be conserved during deformation (see Eq. (26) below).

2.3 Liquid core case (\( \mu = 0, n \geq 2 \))

A difficulty arises in the system I(82) when \( \mu = 0 \) as in the liquid core of the earth. This has long been noticed and discussed by several authors (LONGMAN, 1963; JEFFREYS and VICENTE, 1966), and it has been customary to assume the condition of neutral equilibrium (Adams-Williamson condition) to prevail within the liquid core to circumvent this difficulty (TAKEUCHI, 1950). The condition holds true only when the core is chemically homogeneous and it has adiabatic temperature gradients. However, such conditions are unlikely to be met in the actual core.

Recently, SMYLIE and MANSINHA (1971a, b) reopened the problem by claiming that the radial displacement need not be continuous across the core-mantle boundary in the case of static deformation. As DAHLEN (1971) points out, their arguments are physically untenable. We see no reasons to assume a jump discontinuity in the radial displacement across the boundary; such a discontinuity would results in cavitation at the boundary.

We would better return to the basic equations of equilibrium to clarify the situation. When \( \mu = 0 \), the equations of equilibrium in the radial and transversal directions and Poisson's equation can be written as
where $X$ is the radial factor of dilatation and the dot stands for $d/dr$. To avoid confusion we deliberately omitted the stress-strain relation from the basic set of equations. We will see, however, that we are able to obtain a unique solution in the liquid core without specifying the stress-strain relation.

Differentiating the second equation in (14) and making use of the first, we obtain

$$
\rho \ddot{y_5} + \rho g X - \rho \frac{d}{dr}(\rho y_1) + \dot{y}_2 = 0
$$

$$
y_3 = \rho (y_1 - y_5)
$$

$$
\ddot{y}_3 + \frac{2}{r} \dot{y}_3 - \frac{n(n+1)}{r^2} y_3 = 4\pi G (\rho y_1 + \rho X)
$$

Thus the potential ($y_5$) is decoupled from displacement when $\mu = 0$. Given a density distribution, we might be able to integrate the above equation, as was done by Takeuchi (1950), provided that $\dot{y}_3$ is continuous everywhere within
the core. However, one should remember that $y_i$ is not necessarily continuous. Instead, $y_i$ should be continuous where $\rho$ has a jump discontinuity. This gives rise to a difficulty unless the density varies continuously, because $y_i$ can not be determined by Eq. (16) alone.

Actually, however, we are able to circumvent this difficulty by introducing a new variable

$$y_7 = y_6 + \frac{4\pi G}{g} y_9 = y_6 + \left( \frac{n+1}{r} - \frac{4\pi G \rho}{g} \right) y_9$$

(17)

where account is taken of the second relation in Eq. (14). The new variable $y_7$, which should not be confused with Longman's $y_7$ (LONGMAN, 1963), is everywhere continuous because $y_6$, $y_9$ and $g$ are continuous even if density may have jump discontinuities. In terms of $y_6$ and $y_7$, Eq. (16) is rewritten in the standard form of first order differential equations

$$\dot{y}_6 = \left( \frac{4\pi G \rho}{g} - \frac{n+1}{r} \right) y_6 + y_7$$

$$\dot{y}_7 = \frac{2(n-1)}{r} \frac{4\pi G \rho}{g} y_6 + \left( \frac{n-1}{r} - \frac{4\pi G \rho}{g} \right) y_7.$$  

(18)

Now we are able to integrate $y_6$ and $y_7$ for any density model without knowing $y_1$, starting from initial values

$$y_6(r) = r^n \quad r y_7(r) = 2(n-1)r^n.$$  

(19)

Boundary conditions at the core-mantle boundary. At the core-mantle boundary, say, $r=b$, all the $y_i$'s except $y_3$, should be continuous, but $y_1$, $y_2$ and $y_6$ can not be determined by Eq. (18). Only one of the three is independent for they are related to each other through the second equation in (14) and Eq. (17). Thus, once $y_1(b)$ is specified, all the remaining variables except $y_3$ will be uniquely determined at the core-mantle boundary. Numerically we proceed as follows. We take $y_3(b)$ and $y_6(b)$ for constants of integration and decompose the boundary conditions into three independent sets:

set 1:

$$y_6^N(b) = 0 \quad y_6^A(b) = -\rho^1(b) y_6^A(b)$$

$$y_9^N(b) = y_9^A(b) \quad y_9^A(b) = y_9^A(b) + \frac{4\pi G \rho^1(b)}{g(b)} y_9^A(b)$$

set 2:

$$y_9^N(b) = 1 \quad y_9^A(b) = \rho^1(b) g(b) y_9^A(b)$$

$$y_9^N(b) = -4\pi G \rho^1(b) y_9^N(b)$$

set 3:

$$y_9^N(b) = 1$$

(20)
Some Problems of Static Deformation of the Earth

where the superscripts $s$ and $l$ refer to values in the mantle and in the liquid core, respectively, $y_s^i(r)$ and $y_l^i(r)$ are integrations of Eq. (18), and where those $y_s^i(b)$ of zero boundary value are not given explicitly. It is evident that any linear combination of the above three sets satisfies the continuity conditions across the boundary. Thus, the boundary values of the three independent solutions within the mantle are completely specified by Eq. (20) and we are able to integrate the solutions upward. Complete solutions in the liquid core and in the mantle are given respectively by

$$y_l^i(r) = Q_l^i y_s^i(r)$$

with coefficients $Q_l^i$ determined by Eq. (10).

**Boundary conditions at the solid inner core-liquid outer core boundary.** If the earth has a solid inner core, Eq. (18) is no more valid in the solid inner core. Instead we have to integrate Eq. I(82) as we have done in the mantle, and hence we have three independent solutions in the solid inner core. Since there is only one in the liquid outer core, we must reduce arbitrariness at the inner core-outer core boundary, say, at $r=c$. Remembering that $y_s^i$ (in this case superscript $s$ refers to solid inner core) vanishes at the boundary, we have

$$y_s^i(c) = \sum_{j=1}^{3} Q_s^j y_s^j(c) = 0$$

$$y_l^i(c) = \sum_{j=1}^{3} Q_l^j y_l^j(c) = y_s^i(c) = \rho^i(c)[g(c)y_s^i(c) - y_l^i(c)]$$

(21)

The above equations determine the ratios of $Q_s^j/Q_l^j$ and $Q_s^j/Q_s^i$, and hence all the $y_i$'s at the boundary are determined except for a free multiplier $Q_l^i$, which is to be determined by the surface boundary condition (10).

**Adams-Williamson condition.** So far the problem is well posed in that the deformation field is determined uniquely. A difficulty arises when we introduce the stress-strain relation in the liquid core. If the liquid core is compressible, the normal stress is related to $X$ by

$$y_3 = \lambda X.$$  

(22)

Substituting this into Eq. (15) we get

$$\left( \frac{\rho g}{\lambda} \alpha + \frac{\beta}{\rho} \right) y_3 = 0.$$

Solution to this equation is either $y_3 = 0$ or $\rho g / \lambda + \beta / \rho = 0$, the latter being the ‘Adams-Williamson condition’ that will prevail in a chemically homogeneous
liquid layer of adiabatic temperature gradients. If such conditions are met in the liquid core, $y_3$ is not constrained and the solution obtained by the scheme just described will remain unchanged. However, they are unlikely to be met in the actual core, and then our problem becomes overdeterminate because we have an additional constraint

$$y_4(b) = \sum_{j=1}^{3} Q_j y_{3j}(b) = 0$$

(23)

at the core-mantle boundary and at the inner core-outer core boundary as well. We would be very fortunate if the four relations, relations in Eqs. (10) and (22), in three unknowns, $Q_1$, $Q_2$, and $Q_3$, hold true simultaneously. That is why the Adams-Williamson condition has been assumed in the ordinary theory of earth tide and load deformation of the earth.

It is not necessary, however, that the condition be met throughout the liquid core. Note that Eq. (23) is called for only at solid-liquid boundaries. Thus, if we assume

$$\frac{\rho g}{\lambda} + \frac{\dot{\rho}}{\rho} = 0 \quad \text{for } c < r < c + \delta \text{ and } b - \varepsilon < r < b$$

where $\delta$ and $\varepsilon$ are some constants, then $y_3$ is not constrained to zero in these layers and we can omit the continuity relation such as Eq. (23). One should note that the solution in the mantle, as well as $y_3$ and $y_5$ in the liquid core, does not depend on the two parameters $\delta$ and $\varepsilon$, because the solution does not depend on $\lambda$ (see Eq. (16)). Therefore, we may take them as small as we like, and in effect we neglect the stress-strain relation within a liquid layer.

As mentioned before, deformation field is not determined uniquely in the liquid core. The situation will remain the same even if we assume the Adams-Williamson condition. In particular, $y_3$ is completely undetermined because a liquid particle can move about force free along an equipotential surface. $y_3$ and $y_5$ are only constrained by the relation $y_3 = \rho(a y_1 - y_5)$. However, necessary informations about the deformation field will be derived from $y_3$ and $y_5$.

2.4 Case $n = 1$

The degree 1 modes of deformation have been considered as 'prohibited' since they represented rigid translation of the earth. This is incorrect in the load deformation problem (FARRELL, 1972). For example, the melting of ice sheets could cause changes in the surface mass loads which might contain degree 1 harmonics when expanded into surface harmonic series, thus producing the degree 1 modes of deformation. It necessarily shifts the center of the earth ($r=0$), but the center of mass of the earth will remain fixed with
Some Problems of Static Deformation of the Earth

respect to space. The shift depends on the choice for the origin of coordinates. We may refer to the center of mass of the earth plus loads, or to the center of mass of the undeformed earth, but either choice will put a constraint on the solution.

An immediate integration of the fundamental differential equations will be obtained for the case $n=1$. We find that

$$y_1 = y_3 = 1 \quad y_5 = g \quad y_2 = y_4 = y_6 = 0$$

(constitute a solution for $n=1$). Evidently this solution represents a rigid translation of the earth. From the reciprocity theorem (Appendix 1) as applied between the solution (24) and an arbitrary solution, or directly from Eq. I(82), we can prove an identity

$$\frac{d}{dr} \left[ r^2 \left( y_3 + 2y_4 + \frac{g}{4\pi G} y_6 \right) \right] = 0$$

where $y_i (1 \leq i \leq 6)$ represents an arbitrary solution of Eq. I(82) for $n=1$. Assuming, for the moment, that the earth's core is entirely solid, and integrating the above equation from $r=0$ to $r$, we obtain

$$y_3 + 2y_4 + \frac{g}{4\pi G} y_6 = 2y_4 + \frac{g}{4\pi G} y_7 = 0. \tag{25}$$

Thus we have proved the consistency relation which Farrell (1972) suggested numerically (contrary to Farrell's inference the relation does not hold for dynamic case, see Appendix 2). The relation implies that among three integrations only two are linearly independent. Therefore, a $3 \times 3$ linear equation to determine three constants of integration becomes singular when $n=1$. It is evident, however, by comparing Eq. (25) and the surface boundary conditions, Eq. (10), that if any two of the three boundary conditions are satisfied the remaining one is also satisfied.

The above consistency relation is valid even if the earth's core is liquid. To show this we first mention that

$$y_3 = Q_3^1 g + Q_3^2 \frac{dr}{r^2 g}$$

$$y_i = \frac{Q_i^1}{r^2 g}$$

is a solution of Eq. (18) when $n=1$, where $Q_i^1$ and $Q_i^2$ are arbitrary constants. When the core is liquid down to the center the above solution applies to the center, and then $Q_i^2$ should vanish because of the regularity condition. On the other hand, when the earth has a solid inner core, $y_i$ vanishes at the top of the inner core because of Eq. (25), and hence $Q_i^2 = 0$. This proves that $y_i$ is identically zero in a liquid layer and that Eq. (25) is valid throughout the earth.
Numerically we proceed as follows. We propagate first two solutions in Eq. (20) through the mantle to the surface using \( y_n(b) = g(b) \) and \( y_n^l(b) = 0 \) as initial values, and determine \( Q_1 = Q_i \) and \( Q_1^l \) using any two of the three boundary conditions. The remaining condition will be satisfied automatically because of the consistency relation. It remains the shift of the earth’s center to be determined.

According to the well-known formula due to MacCullaugh (e.g., Jeffreys and Jeffreys, 1956, p. 543), the perturbation in the gravitational potential outside the earth can be written as

\[
(\psi_d) = \frac{G}{r} \left\{ M + \frac{M}{r} \left[ 2P_1(\cos \theta) + (x \cos \phi + y \sin \phi)P_1'(\cos \theta) \right] \right\} + O(r^{-4}) \tag{26}
\]

where
- \( M \): total mass of the earth
- \( \Delta M \): perturbation in the mass of the earth
- \( x, y, z \): coordinates of the center of gravitation of the earth.

We see that \( \psi_d(a) \) of a \( n=1 \) deformation is proportional to the coordinates of the center of gravitation. If we refer to the center of gravitation of the undeformed earth, we should have \( \psi_d(a) = 0 \). Remembering Eq. (7) and \( \phi(a) = \phi_1(a) + \psi_d(a) \), we determine the remaining constant \( Q_3 \) by

Fig. 4. Static load deformation for \( n=1 \).
Some Problems of Static Deformation of the Earth

\[ y_5(a) = Q_1 y_3^5(a) + Q_2 y_4^5(a) + Q_3 y_6^5(a) = 1 \]  

(27)

where \( Q_1 \) and \( Q_2 \) are already determined. A complete solution in the mantle is given by \( y_6 = \sum Q_3 y_4^6 \), \( y_6^a \) being solution (24). In this formulation the displacement at the center of the earth \((r = 0)\) is not always zero but its amount is negligible; in the case shown in Fig. 4 the shift of earth’s center is about \( 2 \times 10^{-3} \) of the surface displacement.

2.5 Rotational deformation

The potential for the centrifugal force due to earth’s rotation is expressed by

\[ \phi_r = \frac{1}{3} r^2 \Omega^2 + \frac{1}{3} r^2 \left( \frac{\Omega_x^2 + \Omega_y^2}{2} - \Omega_z^2 \right) P_2(\cos \theta) \]

\[ - \frac{1}{3} r^2 \Omega_z (\Omega_x \cos \phi + \Omega_y \sin \phi) P_2(\cos \theta) \]

\[ - \frac{1}{6} r^3 \left( \frac{\Omega_x - \Omega_y}{2} \cos 2\phi + \Omega_z \Omega_z \sin 2\phi \right) P_2(\cos \theta) . \]  

(28)

where \( \Omega = (\Omega_x, \Omega_y, \Omega_z) \) is the rotation vector. Each term except for the first represents a harmonic function and the deformation due to such a potential can be computed by using the scheme described before.

The first term is exceptional in that it is not harmonic. Noting \( \phi = \phi_r + \phi_d \), we should solve

\[ \mathcal{P}^2 \psi = 4\pi G \text{div} (\rho u) + 2\Omega^2 \]

instead of the ordinary Poisson’s equation. The equations become inhomogeneous. For the sake of normalization we take \( \Omega^2 = 3/a^2 \), then the equations to be solved are

\[ y_1 = \frac{1}{\lambda + 2\mu} \left( y_2 - \frac{2\lambda}{r^2} y_1 \right) \]

\[ y_2 = \frac{2}{r} (\lambda y_1 - y_2) + \frac{4}{r} \left( \frac{\lambda + \mu}{r} - \rho g \right) y_1 - \frac{2\rho r}{a^3} \]

\[ y_3 = 4\pi G \rho y_3 + \frac{2r}{a^3} \]

\[ y_4 = \frac{1}{r} y_5 + \frac{2r}{a^3} . \]

(29)

Naturally the equations are identical to Eq. I(89) except for the terms containing \( 2r/a^3 \). A solution of these equations will be expressed in terms of a solution of homogeneous equations, i.e., a solution of Eq. I(89), and a particular solution of Eq. (29). The former, \( y_6 \), will be obtained by the method.
discussed in Sec. 2.2, and the latter, \( y_{i2} \), by integrating Eq. (29) using initial values

\[
\frac{y_1}{y_2} = -\frac{r}{2\gamma a^2} \quad \frac{y_2}{y_2} = -\frac{3\lambda + 2\mu}{3\gamma a^2}.
\]  

A general solution is

\[
y_i = Q_i y_{i1} + y_{i2} \quad i = 1, 2
\]

\[
y_5 = 4\pi G \int \rho y_1 \, dr + \left( \frac{r}{a} \right)^2 + Q_5
\]

\[
y_6 = \frac{1}{r} y_6 + \frac{2r}{a^2}.
\]

Surface boundary condition for \( y_6 \) is a little different from the one in the general case. We assume

\[
\phi_6 = k_6^* \frac{a}{r} \quad r \geq a
\]

and call \( k_6^* \) 'tidal Love number' for the rotational deformation. The remaining two Love numbers, \( h_6^* \) and \( l_6^* \), will also be defined, corresponding to \( h_6' \) and \( l_6' \). By definition we have

\[
y_6(a) = 1 + k_6^*
\]

\[
y_6(a) = \left[ \frac{d}{dr} \left( \frac{r^2}{a^2} + k_6^* \frac{a}{r} \right) + \frac{1}{r} \left( \frac{r^2}{a^2} + k_6^* \frac{a}{r} \right) \right]_{r=a} = \frac{3}{a}.
\]  

This, together with \( y_6(a) = 0 \), determines the two constants, \( Q_6 \) and \( Q_5 \), in Eq. (31). Evidently, we get \( l_6^* = k_6^* = 0 \), but we get a non-trivial value for \( h_6^* = g(a)y_1(a) \). The vanishing of \( k_6^* \) is also evident from Eq. (26) because the mass of the earth is conserved during deformation.

3. Discussion and Conclusion

Results of calculation for Wang's earth model (Wang, 1972) based on the present theory are shown in Figs. 1-4, and summarized in Table 1. Wang's model was chosen because it was intended to represent an average earth and because its parameter values are given in tabular form. All the computations were performed in double precision and using following units:

- length: km
- density: g/cm³
- seismic velocity: km/sec.

Radial functions, \( y_i \), are normalized so that disturbing potentials become
Some Problems of Static Deformation of the Earth

$Y_n(\theta, \phi)$ at the surface. The present results compares well with previous results (e.g., Longman, 1966; Farrell, 1972).

An objection that might be raised against the present theory will be that we have discarded the stress-strain relation in the liquid core, although we are not claiming that the relation breaks down in a liquid layer (elasticity of the core is taken into account for radial deformation). The equations of equilibrium, Eq. (14), are of course of primary importance in equilibrium theory; any deformation fields must satisfy those equations in the first place whatever the constitutive law may be. With the relative importance of Eq. (14) and the stress-strain relation (22) in mind, we have tried to solve Eq. (14) first, and succeeded. It turned out in consequence that Eq. (22) was just not required to construct relevant solution in the liquid core. We must emphasize, however, that the present solution is identical to the conventional one if the liquid core actually follows the Adams-Williamson equation. Compare Eq. (20) with Longman's equation (20) (Longman, 1963).

Another reasoning for the present theory is based on a belief that an arbitrary earth model should be in equilibrium state in response to external force. This belief is partly supported by the fact that deformation field is always determined uniquely in dynamic cases. To see if the dynamic solution approaches to the static solution as frequency decreases, semidiurnal tide and load problems are solved. In terms of Love numbers the differences between dynamic and static solutions are within 0.2% (Table 1). Figure 5 shows radial solutions for semidiurnal tide. These curves are almost identical to the solutions shown in Fig. 1 in the mantle. This agreement is not a priori evident because if we had assumed the Adams-Williamson condition in the liquid outer core, density structure would be different for the static case and the agreement would not be justified. Rugged feature of $y_3$ in the liquid core is not unexpected because $y_3$ need not be continuous there.

<table>
<thead>
<tr>
<th>Static solution</th>
<th>n</th>
<th>$h_n$</th>
<th>$l_n$</th>
<th>$k_n$</th>
<th>$h'_n$</th>
<th>$l'_n$</th>
<th>$k'_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.09835*</td>
<td>0</td>
<td>0</td>
<td>-0.13437</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-0.29472</td>
<td>0.10132</td>
<td>0</td>
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<td>0.08586</td>
<td>0.30028</td>
<td>-1.09955</td>
<td>0.02205</td>
<td>-0.30819</td>
</tr>
<tr>
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<td>3</td>
<td>0.29145</td>
<td>0.01548</td>
<td>0.09310</td>
<td>-1.07371</td>
<td>0.06967</td>
<td>-0.19835</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dynamic solution (12 hr)</th>
<th>n</th>
<th>$h_n$</th>
<th>$l_n$</th>
<th>$k_n$</th>
<th>$h'_n$</th>
<th>$l'_n$</th>
<th>$k'_n$</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>0.61150</td>
<td>0.08605</td>
<td>0.30185</td>
<td>-1.00479</td>
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<td>-0.30986</td>
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<tr>
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<td>3</td>
<td>0.29201</td>
<td>0.01545</td>
<td>0.09328</td>
<td>-1.07434</td>
<td>0.06967</td>
<td>-0.19876</td>
</tr>
</tbody>
</table>

* This value refers to $h'_n$, rotational Love number.
Rotational deformation is computed for the first time (Fig. 4). It consists of an almost uniform extension in the mantle and a linearly increasing extension in the core. Munk and MacDonald referred to the term $\frac{Q^2r^2}{3}$ in Eq. (28) as a potential leading to a purely radial deformation that consists of a contraction near the center and an extension in the outer part of the earth (Munk and MacDonald, 1960, p. 25). But their reference to Love’s solution seems inappropriate; the effect of self-gravitation was not included in Love’s solution (Love, 1934, p. 143). Displacement due to the diurnal rotation amounts to 0.7 km at the surface. In other words, the mean radius of the earth would be shorter by 0.7 km than the present length if the earth were not rotating.

All the computations in this study were performed on HITAC 8700/8800 at the Computer Centre, University of Tokyo.

Appendix 1. Reciprocity Theorem

Let $y_{\alpha}$ and $y_{\beta}$ be two arbitrary integrations of Eq. I(82)

$$\frac{dy_\alpha}{dr} = \frac{1}{\lambda + 2\mu} \left( y_\beta - \frac{\lambda}{r} \left[ 2y_\beta - n(n+1)y_\beta \right] \right)$$
Some Problems of Static Deformation of the Earth

\[
\frac{dy_3}{dr} = -\omega^2 \rho y_1 + \frac{2}{r} \left( \frac{\lambda}{\mu} \frac{dy_1}{dr} - y_3 \right) + \frac{1}{r} \left( \frac{2(\lambda + \mu)}{r} - \rho g \right) \left[ 2y_1 - n(n+1)y_3 \right]
\]
\[
+ \frac{n(n+1)}{r} y_4 - \rho \left( y_6 - \frac{n+1}{r} y_5 + \frac{2g}{r} y_1 \right)
\]

\[
\frac{dy_4}{dr} = \frac{1}{\mu} \frac{dy_1}{dr} + \frac{1}{r} \left( y_9 - y_3 \right)
\]

\[
\frac{dy_5}{dr} = -\omega^2 \rho y_9 - \frac{\lambda}{r} \frac{dy_1}{dr} - \frac{\lambda + 2\mu}{r^3} \left[ 2y_1 - n(n+1)y_3 \right] + \frac{2\mu}{r^3} (y_1 - y_3)
\]
\[
- \frac{3}{r} y_4 - \rho \left( y_5 - gy_1 \right)
\]

\[
\frac{dy_6}{dr} = y_6 + 4\pi G \rho y_1 - \frac{n+1}{r} y_5
\]

\[
\frac{dy_5}{dr} = \frac{n-1}{r} \left( y_6 + 4\pi G \rho y_1 \right) + \frac{4\pi G \rho}{r} \left[ 2y_1 - n(n+1)y_3 \right]
\]

then an identity

\[
\frac{d}{dr} \left[ \mu \left( y_{11}y_{22} - y_{12}y_{21} \right) + n(n+1) \left( y_{31}y_{42} - y_{32}y_{41} \right) + (4\pi G)^{-1} \left( y_{51}y_{62} - y_{52}y_{61} \right) \right] = 0
\]

holds. This is another expression of the well-known Betti's reciprocity theorem.

**Appendix 2. Consistency Relation for Dynamic Case**

A solution that represents rigid translation in dynamic case \((\omega \neq 0, n = 1)\) is given by

\[
y_1 = y_3 = 1 \quad y_2 = y_4 = 0 \quad y_5 = g - \omega^2 r \quad y_6 = -3\omega^2.
\]

If we apply the reciprocity theorem to this particular solution and an arbitrary solution for \(n = 1\), we obtain

\[
y_2 + 2y_4 + (4\pi G)^{-1} [3\omega^2 y_5 + (g - \omega^2 r) y_6] = 0.
\]

Strictly speaking, the \(3 \times 3\) matrix that results from Eq. (9) is not singular for finite \(\omega\), since any linear combination of \(y_3, y_4, \) and \(y_6\) does not vanish identically. But numerically the matrix may become singular for small \(\omega\). Incidentally, if the boundary condition (9) is substituted the above relation is reduced to \(y_6(a) = 1\), which is consistent with Eq. (27) as it should be.

**REFERENCES**


JEFFREYS, H. and R. O. VICENTE, Comparison of forms of the elastic equations for the earth, Mémoires de l'académie Royale de Belgique, 37, 5-31, 1966.


