NUMERICAL CALCULATION METHOD OF MOLODENSKII'S G_1 , WITH SPECIFIC APPLICATION TO THE GRAVITY DATA OF THE TANZAWA MOUNTAIN AREA

Yukio HAGIWARA

*Earthquake Research Institute,
University of Tokyo,
Tokyo, Japan*

(Received September 20, 1974)

The present paper is devoted to obtaining Molodenskii's G_1 based on the practical data of the Tanzawa Mountain area, southwest of Tokyo. For numerical calculations of the G_1 integral, we adopt 20 km as a truncation distance. It is seen, as a result, that the G_1 distribution has a strong resemblance to short wavelength topographic undulations. Assuming as a rough estimate, the effect of the Molodenskii correction on the deflection of the vertical amounts to about 10 sec. This quantity is significant as compared with the astrogeodetically observed deflections.

1. Introduction

The series solution of the geodetic boundary-value problem, which was derived by MOLODENSKII *et al.* (1962), has over the past decade become of increasing interest to physical geodesists. Some authors (e.g., MORITZ, 1971) have approached through their own proper analytical methods to the Molodenskii solution. The first term of the Molodenskii series corresponds to the conventional Stokes' and Vening Meinesz' integrals for geoidal height and deflection of the vertical. It has been practically indicated that the first-term approximation is accurate enough and the second and higher terms can be neglected in most cases.

However, the topographic effect on deflections of the vertical amounts to a great quantity in mountainous regions. In such a case, the second term should be taken into consideration. The second term is given by G_1 , a kind of terrain correction to gravity anomaly in calculating the Stokes' and Vening Meinesz' integrals. HAGIWARA (1973) has recently evaluated the G_1 distributions over some models of topographic relief, and pointed out that the G_1 -correction to the gravimetric deflections of the vertical is not negligibly small.

G_1 is usually expressed by an integral extended over the whole surface of the earth. When the integration is restricted to a certain distance, the effect of neglecting the distant zones is thought to be very small, because the integrand converges rapidly to zero. It can be assumed, therefore, that G_1 is approximately calculated in the rectangular coordinates. To begin with, the present study aims at practically estimating a truncation distance appropriate for accurate numerical computations of G_1 , in which the truncation error does not exceed 0.5 mgals, on the basis of the gravimetric and topographic data of the Tanzawa Mountain area (JITSUKAWA *et al.*, 1974). Secondly, using the truncation distance thus estimated, the G_1 distribution over the Tanzawa Mountain area will be obtained. The numerical integration method that we make use of here will be given in Appendix.

Furthermore, the relation between G_1 and topographic relief will be discussed. The profile of G_1 , in general, is thought to be similar to that of the topographic relief. Judging from the results obtained from the model calculations (HAGIWARA, 1973), space-variations in G_1 may be strongly affected with short wavelength topographic undulations. According to HEISKANEN and MORITZ (1967), G_1 is divided into two terms: G_{11} and G_{12} . The former corresponds to the vertical gradient of gravity anomaly, which is emphasized in rugged terrain areas. Although the latter is not clear in physical meaning, its profile may be very similar to that of the former across a mountainous region. Consequently, it is supposed that G_1 is approximately proportional to the vertical gradient of gravity anomaly. Such a problem as mentioned above is very interesting to be solved. In the present paper, some discussion will be made about the problem by practically computing G_1 on the basis of the data of the Tanzawa Mountain area.

2. Molodenskii's G_1

MOLODENSKII *et al.* (1962) obtained a correction term G_1 to the Stokes and Vening Meinesz formulas for the geoidal height and the deflection of the vertical as an approximate solution of the geodetic boundary-value problem. Letting ζ , ξ and η be the height anomaly, the meridian and prime-vertical deflections of the vertical at a point P on the earth's surface, these quantities are expressed by using the Molodenskii correction term in the following formulas:

$$\zeta = \frac{R}{4\pi\gamma} \iint_{\sigma} (\Delta g + G_1) S(\cos \phi) d\sigma \quad (1)$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{4\pi\gamma} \iint_{\sigma} (\Delta g + G_1) \frac{dS}{d\phi} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} d\sigma - \frac{\Delta g}{\gamma} \begin{pmatrix} \tan \beta_1 \\ \tan \beta_2 \end{pmatrix} \quad (2)$$

where R is the earth's mean radius, γ the normal gravity, $d\sigma$ the surface element at a moving point P' , Δg the gravity anomaly at P' , S the Stokes' function of ϕ , ϕ the spherical distance between P and P' , α the azimuth of PP' reckoned from north, β_1 and β_2 the inclination angles of north-south and east-west terrain profiles, respectively.

Molodenskii's G_1 is given by

$$G_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_P}{l_0^3} \Delta g d\sigma \quad (3)$$

where h and h_P are the terrain heights at P' and P , respectively, and

$$l_0 = 2R \sin(\phi/2) \quad (4)$$

The integration of (3) should be extended over the whole spherical surface. However, the value of the integrand is very small at greater distances from P , so that we may neglect the effects of the remote zones on the integration. In such a case, we can treat functions in the rectangular coordinates on the horizontal plane, i.e. $\Delta g(x, y)$, $h(x, y)$, etc. The x and y axes are conventionally taken in the north-south and east-west directions, respectively. Then, if the origin of the coordinates is located at P , (3) can be rewritten approximately as

$$G_1 = \frac{1}{2\pi} \int_{-x}^x \int_{-y}^y \frac{f(x, y)}{(x^2 + y^2)^{3/2}} dx dy \quad (5)$$

where

$$f(x, y) = \Delta g(x, y) \{h(x, y) - h(0, 0)\}. \quad (6)$$

The integral is limited to an area of $2X \times 2Y$ centering around the origin P . The numerical integration method adopted here will be given in Appendix.

3. Available Data

The Tanzawa Mountain area (see Fig. 1), southwest of Tokyo City, is predominantly occupied by rugged terrain. The topographic map is shown in Fig. 2. The highest elevation in this area is 1670 meters above sea-level, the central area of which forms a plutonic dome surrounded by volcanic rock. The southern margin of this area is the coastline of Sagami Bay open to the Pacific Ocean. A fault zone called "Sagami Trough" extends oceanwards from the southern foot of the Tanzawa Mountains. According to the current theory of plate tectonics, Sagami Trough is presumed as a boundary between the continental and the Philippine Sea plates. The 1923 Kanto Earthquake, by which more than 100000 people were lost mostly in Tokyo area, was caused by the fault motion of this trough.

JITSUKAWA *et al.* (1974) have recently conducted gravity surveys over

Fig. 1. Location of the Tanzawa Mountain area.

Fig. 2. Simplified topographic map of the Tanzawa Mountains. Shaded areas indicate terrain higher than 1000 m above sea-level (contour interval: 500 m). G_1 is calculated in the central area surrounded by "square".

the Tanzawa Mountain area. Figure 3 shows the terrain-corrected Bouguer anomaly in mgals on the assumption that the mean crustal density is 2.67 g/cm^3 . The open circles in this figure indicate the location of the gravity stations. The Bouguer anomaly is characterized by the topographic and geologic features of this area, that is, the gravity highs coincide with the plutonic dome distribution and the lows with the fault zones in the northern extension of Sagami Trough.

Fig. 3. Simplified map of the terrain-corrected Bouguer anomaly in mgals. Shaded areas indicate the negative anomaly.

In order to obtain Molodenskii's G_1 , we must adopt approximately the free-air anomaly as Δg . It is almost impossible, however, for a free-air anomaly map to be drawn directly from the gravity values observed in a mountainous region. The value of free-air anomaly varies so markedly with height of observation station that the free-air anomaly map has a strong resemblance to the topographic map. No free-air anomaly map can be made up unless the observation net covers all ridges and valleys within the surveying area. On the contrary, as Bouguer anomaly is rather independent of topography, the isoanomaly lines can be readily drawn even in the case that the observation net is not so dense.

Therefore, if we intend to get a free-air anomaly map over a mountainous region, the first thing to do is to draw a Bouguer anomaly map, secondly the values of the Bouguer anomaly Δg_B and the topographic elevation h are read at a grid point of meshes covering the map, and we then finally obtain the value of free-air anomaly as $\Delta g_B + 4\pi G \rho h$ (G : Newton's gravitational constant, ρ : density) at the same point. Terrain correction is not eliminated from the free-air anomaly thus obtained. However, the values of terrain correction, in general, are very small as compared with the absolute values of free-air anomaly, so that we can approximately compute G_1 on the basis of thus obtained terrain-corrected free-air anomaly.

4. Truncation Distance and G_1 Distribution

The truncation error of the G_1 integration should in the strict sense be estimated on the spherical surface of the earth. However, effects of the remote zones on the integration are so small that we can treat the truncation error as a plane approximation. First of all, we have to determine a truncation distance appropriate for the numerical integration of (5). Although it is in general no easy matter to estimate the truncation error, a rough estimation can be carried out by plotting " G_1 vs. truncation distance" based on the practical data.

Fig. 4. G_1 vs. truncation distance.

Figure 4 shows the relation between G_1 and the truncation distance at two points on the ridges along the parallel of latitude $35^{\circ}30'N$. These calculations are based on the data with a grid-spacing of 2 km. It is concluded in the case of $h_p = 1400$ m that the truncation effect beyond 20 km from a station P is estimated to be less than 0.5 mgals and it is not necessary to compute the outside effect. In contrast, in the case of $h_p = 800$ m, 10 km is a truncation distance sufficient for an error of 0.5 mgals or less. Accordingly, the author determines the truncation distance to be 20 km for all the G_1 computations in the present study.

Figure 5 shows the G_1 distribution over the central area of the Tanzawa Mountains, where the negative G_1 area is shaded. It is noticed in the figure

Fig. 5. G_1 distribution over the central area ("square" in Fig. 2) of the Tanzawa Mountains. Negative G_1 areas are shaded. Contour interval: 20 mgals.

Fig. 6. Profiles of G_1 , free-air gravity anomaly and terrain relief along the parallel of latitude $35^{\circ}30'N$.

that the G_1 distribution pattern is very similar to that of topography. The negative G_1 well coincides with the shaded area (higher than 1000 meters above sea-level) in the topographic map (Fig. 2). In other words, the G_1 is negatively correlated to the topographic elevation. Such a tendency was also proved by the model calculations (HAGIWARA, 1973).

For further investigations of the relation between G_1 and topography, Fig. 6 shows a profile of G_1 together with profiles of free-air anomaly and topography along the parallel of latitude $35^\circ 30' \text{ N}$. We see in this figure that a mountain peak corresponds to a minimum of G_1 . It must be noticed here that the positive axis of G_1 is taken in the downward direction. It appears that the higher the elevation is, the less the value of G_1 becomes. Moreover, we see that long wavelength space-variations in free-air anomaly or topographic relief do not affect any space-variations in G_1 . On the contrary, the behavior of G_1 is much emphasized by short wavelength ones. This fact may indicate, in terms of electronics, the G_1 calculation procedure is a kind of high-pass or low-cut filter. Mathematical filtering techniques such as a two-dimensional Fourier analysis can be applied to distributions of free-air anomaly and topographic elevation for approximately obtaining a G_1 distribution.

5. Physical Meaning of G_1

HEISKANEN and MORITZ (1967) described a geometrical interpretation of the Molodenskii approximate solution in terms of spherical harmonics in connection with deflections of the vertical. They divided G_1 into two terms as follows:

$$G_1 = G_{11} + G_{12} \\ = \frac{R^2}{2\pi} \left\{ -h_P \iint_{\sigma} \frac{\Delta g - \Delta g_P}{l_0^3} d\sigma + \iint_{\sigma} \frac{h\Delta g - (h\Delta g)_P}{l_0^3} d\sigma \right\}. \quad (7)$$

Considering Numerov's formula of the vertical gradient of gravity anomaly

$$\left[\frac{\partial \Delta g}{\partial h} \right]_P = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g - \Delta g_P}{l_0^3} d\sigma, \quad (8)$$

we obtain

$$G_{11} = -[h \partial \Delta g / \partial h]_P. \quad (9)$$

This indicates that the term G_{11} corresponds to the reduction of the free-air anomaly from the ground to sea-level by the topographic elevation h . In the customary way of the gravity reduction, we use 0.3086 mgal/m , the normal gradient of gravity, assuming that the earth's total mass is concentrated to the center. Thus, G_{11} is a correction term to the reduction by the deviation from the normal gradient of gravity.

Fig. 7. Profiles of G_{11} , G_{12} and G_1 along the parallel of latitude $35^{\circ}30' N$.

Profiles of G_1 , G_{11} and G_{12} along the parallel of latitude $35^{\circ}30' N$ are shown in Fig. 7. We must pay attention that the positive axes of G_1 and G_{12} are taken in the downward direction in contrast to the case of G_{11} . As seen in this figure, the profiles resemble each other in shape. Although the second term G_{12} is not clear in physical meaning, its behavior is very similar to the vertical gradient of gravity anomaly.

On the basis of the fact that, in the first approximation theory, the Bouguer anomaly is equal to the gravitational attraction of the Bouguer slab, we have

$$\Delta g - \Delta g_P = 2\pi G\rho(h - h_P). \quad (10)$$

Taking the above approximation and Numerov's formula (8) into account, the following formula can be derived (FUJII, 1970).

$$\begin{aligned} G_1 &= G_{11}^* + G_{12}^* \\ &= G\rho R^3 \iint \frac{(h - h_P)^2}{l_0^3} d\sigma + \frac{\Delta g_P}{2\pi G\rho} \left[\frac{\partial \Delta g}{\partial h} \right]_P. \end{aligned} \quad (11)$$

The first term G_{11}^* is equal to twice as much as an ordinary terrain correction and the second G_{12}^* corresponds to the vertical gradient of gravity anomaly. Despite of the assumption of the Bouguer slab, the physical meaning of G_1 is clarified by means of the above derivation. Similarly to G_{11} and G_{12} , the numerical calculations of G_{11}^* and G_{12}^* are made on the basis of the Tanzawa data. The results of the calculations are shown in Fig. 8. We see that G_{11}^* is negligibly small in comparison with G_{12}^* . Accordingly, the procedure of G_1 corresponds

Fig. 8. Profiles of G_{11}^* and G_{12}^* along the parallel of latitude $35^{\circ}30' N$.

approximately to a free-air reduction from the ground to sea-level using the actual gradient of gravity. In other words, G_1 is a correction to the customary free-air reduction with the normal gradient.

MORITZ (1968) derived an approximate solution of Molodenskii's problem by replacing the Molodenskii correction terms by the terrain correction. In the case of the above example, however, the second term concerning with the vertical gradient of gravity anomaly is much larger than the terrain correction. The author thinks that the second term should rather be taken into consideration in an approximation to the Molodenskii correction.

6. Concluding Remarks

The integration of Molodenskii's G_1 is extended over the whole surface of the earth. In numerically calculating G_1 in the rectangular coordinates, we usually restrict the integration to a rectangular area. On the basis of the topographic and gravimetric data of the Tanzawa Mountain area (JITSUKAWA, *et al.*, 1974), the truncation distance of the integration is so determined that the truncation error does not exceed 0.5 mgals. We conclude the truncation distance to be 20 km. In the case that terrain is not so rugged, 10 km is a truncation distance sufficient for an error of 0.5 mgals or less.

The value of G_1 integral is numerically obtained by summing up the effects on G_1 in rectangular segments, which are arranged around the station in a fashion as shown in Fig. 9. The integration area consists of the inner and outer zones, where the digitalized data are read with an interval of 2 km. It is concluded in Fig. 4 that the proportion of the inner effect to the value of G_1 amounts to about 60% in the rugged terrain area. If we desire to obtain the more accurate value of G_1 , we have to provide detailed maps of topography and gravity for data-readings.

Fig. 9. Pattern net for calculating G_1 .

The G_1 distribution over the Tanzawa Mountain area (Fig. 5) indicates that the behavior of G_1 is much emphasized by short wavelength topographic undulations. It seems likely that the space-variations in G_1 do not reflect the topographic relief having wavelengths longer than several kilometers (see Fig. 6). This fact implies that, in terms of electronics, the G_1 calculation procedure is a kind of high-pass or low-cut filter.

Roughly estimating from the G_1 distribution, the effect of G_1 correction on the height anomaly amounts to about 6 cm at maximum. It is not necessary for the height anomaly estimations to take such a quantity into account. In contrast, the difference in the deflection of the vertical amounts to 10 sec., which is not small as compared with the astrogeodetically obtained deflections. It is concluded, therefore, that the G_1 considerations are of course important for the determination of the deflections of the vertical in a mountainous region.

Molodenskii's formulas (2) for the deflection of the vertical have another correction term concerning with the slope of the ground surface. However, the practical determination of a slope tends to be very ambiguous in rugged terrain areas. As a matter of fact, the slope determined from a large-scale map may be quite different from one from a small-scale map. In digital computations, the slope based on terrain data read at grid points depends on the grid interval. How to determine the terrain slope is a serious problem for practical computations of deflections of the vertical by means of the Molodenskii formulas (2).

HEISKANEN and MORITZ (1967) presented a new formula, in which the

term concerning with the slope is cancelled out by the gravity reduction through the vertical gradient of gravity anomaly. It is an approximate solution of the Molodenskii problem, but more convenient for a practical use. Assuming the value of Bouguer anomaly as the gravitational attraction of the Bouguer slab, the author discloses that G_1 corresponds to the vertical gradient of gravity anomaly. This may possibly indicate that the Molodenskii correction to Vening Meinesz' integral can be well approximated by the "gradient method" presented by Heiskanen and Moritz.

APPENDIX

For a numerical integration, functions are digitalized at grid points, the spacings of which are a and b in the x and y directions. Assuming the (m, n) -th grid is located at (x_m, y_n) and

$$\begin{aligned} x_m &= ma \quad (m = -M, -M+1, \dots, M) \\ y_n &= nb \quad (n = -N, -N+1, \dots, N) \end{aligned} \quad (12)$$

we notice that $X=Ma$ and $Y=Nb$. Thus (5) becomes

$$\begin{aligned} G_1 &= \sum_{m=-M}^{M-1} \sum_{n=-N}^{N-1} (G_1)_{m,n} \\ &= \frac{1}{2\pi} \sum_{m=-M}^{M-1} \sum_{n=-N}^{N-1} \int_{x_m}^{x_{m+1}} \int_{y_n}^{y_{n+1}} \frac{f(x, y)}{(x^2 + y^2)^{3/2}} dx dy \end{aligned} \quad (13)$$

Dividing the integration into two parts: the inner and outer zones, we define that the inner zone consists of four grid points $(-1, -1)$, $(-1, 0)$, $(0, -1)$ and $(0, 0)$. Figure 9 shows the pattern net for numerically calculating G_1 . Since $f(0, 0)=0$ in the inner zone, we assume

$$f(x, y) = Bx + Cy + Dxy + Ex^2 + Fy^2. \quad (14)$$

By substituting (14) into (13), for the inner zone we have

$$\begin{aligned} (G_1)_{\text{inner}} &= (G_1)_{-1,-1} + (G_1)_{-1,0} + (G_1)_{0,-1} + (G_1)_{0,0} \\ &= -\frac{bE}{\pi} \log \frac{\sqrt{a^2+b^2}-a}{\sqrt{a^2+b^2}+a} - \frac{aF}{\pi} \log \frac{\sqrt{a^2+b^2}-b}{\sqrt{a^2+b^2}+b} \end{aligned} \quad (15)$$

where, by putting $f_{m,n}=f(x_m, y_n)$ for brevity,

$$\begin{aligned} E &= (f_{1,0} + f_{-1,0})/2a^2 \\ F &= (f_{0,1} + f_{0,-1})/2b^2 \end{aligned} \quad (16)$$

(15) does not include any terms of B , C and D .

For a rectangular area including four grid points (m, n) , $(m+1, n)$, $(m, n+1)$ and $(m+1, n+1)$ in the outer zone, we assume

$$f(x, y) = A + Bx + Cy + Dxy. \quad (17)$$

The coefficients in the above formula are obtained from

$$\left. \begin{aligned} A &= (f_{m,n}x_{m+1}y_{n+1} - f_{m+1,n}x_my_{n+1} - f_{m,n+1}x_{m+1}y_n + f_{m+1,n+1}x_my_n)/ab \\ B &= \{y_{n+1}(f_{m+1,n} - f_{m,n}) - y_n(f_{m+1,n+1} - f_{m,n+1})\}/ab \\ C &= \{x_{m+1}(f_{m,n+1} - f_{m,n}) - x_m(f_{m+1,n+1} - f_{m+1,n})\}/ab \\ D &= (f_{m+1,n+1} - f_{m+1,n} - f_{m,n+1} + f_{m,n})/ab \end{aligned} \right\} \quad (18)$$

By defining

$$\left. \begin{aligned} I(x, y) &= \begin{cases} 1/y & : x=0 \\ 1/x & : y=0 \\ (x+y - \sqrt{x^2+y^2})/xy & : x \neq 0, y \neq 0 \end{cases} \\ J(x, y) &= \log(x + \sqrt{x^2+y^2}) \\ K(x, y) &= \sqrt{x^2+y^2} \end{aligned} \right\} \quad (19)$$

we have the following formula for the outer zone.

$$\begin{aligned} (G_1)_{m,n} &= \frac{1}{2\pi} [A\{I(x_m, y_n) - I(x_{m+1}, y_n) - I(x_m, y_{n+1}) + I(x_{m+1}, y_{n+1})\} \\ &\quad - B\{J(y_n, x_m) - J(y_n, x_{m+1}) - J(y_{n+1}, x_m) + J(y_{n+1}, x_{m+1})\} \\ &\quad - C\{J(x_m, y_n) - J(x_{m+1}, y_n) - J(x_m, y_{n+1}) + J(x_{m+1}, y_{n+1})\} \\ &\quad - D\{K(x_m, y_n) - K(x_{m+1}, y_n) - K(x_m, y_{n+1}) + K(x_{m+1}, y_{n+1})\}] \quad (20) \end{aligned}$$

REFERENCES

- FUJII, Y., Significance and limitation of the gradient method in the geodetic boundary value problem, *J. Geod. Soc. Japan*, **16**, 111-126, 1970 (in Japanese).
 HAGIWARA, Y., G_1 distribution over models of topographic relief, *J. Phys. Earth*, **21**, 305-311, 1973.
 HEISKANEN, W. A. and H. MORITZ, *Physical Geodesy*, Freeman, 1967.
 JITSUKAWA, A., H. TAJIMA, and S. IZUTUYA, Gravity surveys over Tanzawa Mountains and Izu Peninsula, *Spec. Bull. Earthq. Res. Inst.*, **13**, 43-66, 1974 (in Japanese).
 MOLODENSKII, M. S., V. F. EREMEEV, and M. I. YURKINA, Methods for study of the external gravitational field and figure of the earth. Israel Program for Scientific Translations, Jerusalem, 1962.
 MORITZ, H., Series solutions of Molodensky's problem, *Deutsche Geodæt. Komm.*, A, Heft Nr. 63, 1971.
 MORITZ, H., On the use of the terrain correction in solving Molodensky's problem. Report No. 108, Dept. Geodetic Science, Ohio State Univ., 1968.