THE ASYMPTOTIC DISTRIBUTION OF TORSIONAL EIGENFREQUENCIES OF A SPHERICAL SHELL. III

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In the two previous papers of this series (SATO and LAPWOOD, 1977a, b) we examined approximate methods for calculating eigenfrequencies of radial overtones of torsional oscillations of spherically symmetrical shells. For shells composed of uniform layers we were able to obtain an exact frequency equation, in terms of spherical Bessel functions, for which roots could be computed with any desired precision. They thus supplied a standard for the measurement of the accuracy of approximate methods.

In applications to shells of two and three uniform layers, which were simple representations of an Earth with inner surfaces of discontinuity, we noted the presence of the solotone effect, which is the existence of recurring patterns of eigenfrequencies owing to internal reflection.

In this paper we take up the analysis of the solotone effect, showing how it may be predicted from knowledge of the shell structure, and how it may be interpreted in terms of ray theory. Applications to the same Earth-models as used before show that for them the theory of the solotone gives an excellent fit to the precisely computed eigenfrequencies. The pattern of eigenfrequencies proves to be very sensitive to changes in layer thickness, and thus offers the possibility of future use in determining the positions of surfaces of discontinuity within the mantle of the Earth.

1. Solotone Effect Due to a Single Discontinuity in Parameters

In Paper I of this series we found the distribution of eigenfrequencies of radial overtones in torsional oscillations of a spherical shell made up of two uniform layers. The approximation measured there by \( m_n \), which allows for Earth-curvature but not internal reflections, deviates from \( m \), for \( m = 1, 2, \ldots, 60 \), in a pattern of values determined by the solotone effect, i.e. by the modifications of frequency due to internal reflection and refraction. Figure 1 shows \( m_1 - m \) (Earth-curvature neglected) and \( m_2 - m \), graphed against \( m \) for (Legendre order) \( n = 2 \), for the Earth model (used in I) specified by:
Fig. 1. For two-layer shell, \( m_2-m, m_1-m \) and \( S=\left(-m^{1/2}/\rho\right)\sin(\theta-(\theta+\phi)m\pi) \) graphed against \( m \) for \( n=2 \) and \( m=0, 1, 2, \ldots, 60 \).

Inner radius: \( r=a=3,470 \text{ km} \).

Surface of discontinuity: \( r=c=5,960 \text{ km} \).

Outer radius: \( r=b=6,370 \text{ km} \).

In region

(1), \( a<r<c, \beta_1=6.58 \text{ km/sec}, \rho_1=5.35 \text{ gm/cm}^3 \) where \( \beta \) is shear wave velocity and \( \rho \) is density.

(2), \( c<r<b, \beta_2=4.46 \text{ km/sec}, \rho_2=3.77 \text{ gm/cm}^3 \).

Thus the coefficient of reflection for a ray going from (1) to impinge normally on the surface of discontinuity is

\[
R = \frac{\rho_1\beta_1 - \rho_2\beta_2}{\rho_1\beta_1 + \rho_2\beta_2} = 0.3535 .
\]

In Fig. 1 the solotone effect shows clearly in an oscillation of period 5 units of \( m \). We now show that this oscillation could have been predicted from theory—not merely its existence but even its period and amplitude.

We start from our formula (I (9.9))

\[
\sin(\chi_3+\chi_1) + R' \sin(\chi_3-\chi_1) = 0 ,
\]

where \( R' \), given by (I (9.5)), is the coefficient of reflection for oblique incidence from region (2) on the (1, 2) interface and \( \chi_1, \chi_3 \) are as defined in (I, § 9). A zero-order approximation to solutions of (1.2) is

\[
\chi_1+\chi_3=m\pi ,
\]

where \( m \) is an integer; the corresponding first-order solution is obtained by
writing \( \chi_1 = \omega_n \Theta, \chi_2 = \omega_n \Phi \), and

\[
\chi_1 + \chi_2 = m\pi + \delta_m.
\]

We obtain

\[
\omega_n \approx \frac{1}{\Theta + \Phi} m\pi + (-)^m \frac{R'}{\Theta + \Phi} \sin \frac{\Theta - \Phi}{\Theta + \Phi} m\pi.
\]

(1.5)

If we now neglect Earth-curvature, and approximate to \( \Theta \) by \( \Theta = (c-a)/\delta_1 \), to \( \Phi \) by \( \Phi = (b-c)/\delta_1 \), and to \( R' \) by \(-R\) (since they are defined in opposite senses), we get, from I (12.7) and (12.8),

\[
m_1 \pi = \omega_n (\Theta + \Phi),
\]

so that

\[
m_1 - m \approx S,
\]

where

\[
S = (-)^{n+1} \frac{R}{\pi} \sin \frac{\Theta - \Phi}{\Theta + \Phi} m\pi.
\]

(1.7)

This is Anderssen's formula (ANDERSSEN, 1977), quoted in I (11.4), now shown to be a special case of (1.5).

In Fig. 1 \( S \) has been graphed against \( m \) for \( n = 2 \). We see that except for values of \( m \) less than 10 the fit of \( S \) to \( m_1 - m \) is remarkably good. The amplitude and period of the solotone pattern, and even the actual values of \( \omega_n \), are given very closely by (1.6), in spite of the approximations made in deriving (1.6).

There is, however, a divergence of the values of \( S \) from those of \( m_1 - m \) as \( m \) decreases from about 10 towards zero. In fact, it is clear from Fig. 1 that \( S \) conforms even closer to \( m_2 - m \) than to \( m_1 - m \). We can substantiate this observation as follows. \( m_2 \) was defined in I (12.4) as

\[
m_2 = \frac{1}{2\pi} \omega_n \left( T_n - p_n J_n \right)
\]

\[
= \frac{1}{\pi} (\chi_{1a} - \chi_{1n}) + \frac{1}{\pi} (\chi_{2b} - \chi_{2n}),
\]

(1.8)

i.e.,

\[
\pi m_2 = \chi_1 + \chi_2.
\]

Hence

\[
(m_2 - m)\pi = \delta_m
\]

\[
= (-)^{n+1} R' \sin (\chi_n - \chi_1).
\]

If we now approximate to \( R' \) by \(-R\), and to \( \chi_1, \chi_2 \) by \((\Theta/(\Theta + \Phi))m\pi, (\Phi/(\Theta + \Phi))m\pi\), respectively, we get, in place of (1.6),

\[
m_2 - m \approx S,
\]

(1.9)
Fig. 2. For two-layer shell, \( m_1 - m \), \( m_2 - m \) and \( S = (-1)^{m+1} n \sin \theta \alpha \) graphed against \( m \) for \( n = 15, m = 1, 2, 3, \ldots, 60 \).

in an approximation which has taken Earth-curvature into account. In other words, \( m_1 \) differs from \( m \) because of Earth-curvature and internal reflections, whereas \( m_2 \) differs from \( m \) because of internal reflections only. Hence the measure \( S \) of the solotone effect is closer to \( m_2 - m \) than to \( m_1 - m \), as Fig. 1 illustrates. This relationship is much more clearly in evidence for \( n = 15 \), where Earth-curvature plays a larger part. Figure 2 shows \( m_1 - m \), \( m_2 - m \) and \( S \) for \( n = 15, m = 1, 2, 3, \ldots, 60 \). Note that the approximations which we have used for \( S \) are independent of \( n \), and that the graph of \( m_i \) against \( m \) varies little with \( n \).

2. Solotone Effect Due to Two Surfaces of Discontinuity

In Paper II (see Figs. 6, 7, 8) we showed the existence of a solotone effect in a 3-layer Earth model. We now examine the detail of the patterns of eigen-frequencies for the same model—an averaged PEM-A (Dziewonski et al., 1975)—with three uniform layers specified by:

- Inner radius: \( r = r_0 = a = 3,485.7 \) km.
- Lower discontinuity: \( r = r_1 = 5,701 \) km.
- Upper discontinuity: \( r = r_2 = 5,951 \) km.
- Outer radius: \( r = r_3 = b = 6,368 \) km.
In region

1. \( a < r < r_1 \), \( \beta_1 = 6.769 \text{ km/sec}, \ \rho_1 = 5.011 \text{ gm/cm}^3 \),
2. \( r_1 < r < r_2 \), \( \beta_2 = 5.301 \text{ km/sec}, \ \rho_2 = 3.953 \text{ gm/cm}^3 \),
3. \( r_2 < r < b \), \( \beta_3 = 4.505 \text{ km/sec}, \ \rho_3 = 3.421 \text{ gm/cm}^3 \)

Fig. 3. For three-layer shell,
(a) \( m_n - m \) is graphed against \( m \) for \( n = 5, m = 1, 2, \ldots, 60 \).
(b) \( S \) is graphed against \( m \).

Note the close resemblance between Figs. (a) and (b).
Here

\[ R_1 = \frac{\rho_2 \beta_2 - \rho_1 \beta_1}{\rho_2 \beta_2 + \rho_1 \beta_1} = -0.236 , \]

\[ R_2 = \frac{\rho_2 \beta_2 - \rho_2 \beta_2}{\rho_2 \beta_2 + \rho_2 \beta_2} = -0.152 . \]

In Fig. 3(a) \( m_2 - m \) is plotted against \( m \) for \( n = 5, m = 0, 1, 2, \ldots, 60 \). We see that the main solotone period in \( m \) is 10 units, but the complex structure of the pattern of eigenfrequencies suggests interference between oscillations of different periods. The patterns for \( n = 2, 10 \) and 15 (not reproduced here) are almost identical with that for \( n = 5 \). We wish to obtain this pattern from the theory set out in I and II. It would be simple to start from \( \delta \) given in II (6.8), but in order to tie our work in with previous results we start from the main theorem of McNabb et al. (1976).

This theorem states that, for a Sturm-Liouville system with two discontinuities, the asymptotic distribution of eigenvalues is given by

\[
\begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\sin \chi_1 & \cos \chi_1 & 0 & \gamma_1 & 0 & 0 \\
\cos \chi_1 & -\sin \chi_1 & 1/\gamma_1 & 0 & 0 & 0 \\
0 & 0 & \sin \chi_2 & \cos \chi_2 & 0 & \gamma_2 \\
0 & 0 & \cos \chi_2 & -\sin \chi_2 & 1/\gamma_2 & 0 \\
0 & 0 & 0 & 0 & \cos \chi_3 & -\sin \chi_3 \\
\end{vmatrix} = 0, \quad (2.1)
\]

where, with the notation of (I) and (II), we have neglected \( q_1 \) and \( q_2 \) (and hence Earth-curvature) and used

\[
\chi_p = \int_{r_{p-1}}^{r_p} \frac{dr}{\beta}, \quad \gamma_p = (\beta_p \rho_p)^{1/2} (\beta_p + \rho_{p+1})^{-1/2}, \quad p = 1, 2. \quad (2.2)
\]

Expanding the determinant in (2.1) we get

\[
(1 + \gamma_1^2)(1 + \gamma_2^2) \sin (\chi_1 + \chi_s + \chi_b) + (1 - \gamma_1^2)(1 - \gamma_2^2) \sin (\chi_1 - \chi_b + \chi_s) \\
- (1 + \gamma_1^2)(1 - \gamma_2^2) \sin (\chi_1 + \chi_b - \chi_s) - (1 - \gamma_1^2)(1 + \gamma_2^2) \sin (\chi_1 - \chi_b - \chi_s) = 0 ,
\]

i.e.

\[
\sin (\chi_1 + \chi_b + \chi_s) = R_s \sin (\chi_1 - \chi_b - \chi_s) + R_3 \sin (\chi_1 + \chi_b - \chi_s) \\
- R_1 R_4 \sin (\chi_1 - \chi_b + \chi_s) . \quad (2.3)
\]

A zero order approximation \( \chi_1 + \chi_b + \chi_s = m \pi \), where \( m \) is an integer, is found by assuming \( R_1 = 0, R_4 = 0 \) in (2.4). A first order approximation, obtained by putting

\[
\chi_1 + \chi_b + \chi_s = m \pi + \delta \quad (2.5)
\]

in (2.4), has

\[
\tan \delta = \frac{R_4 \sin 2\chi_b + R_1 R_4 \sin 2\chi_s}{1 + R_4 \cos 2\chi_b} \quad (2.6)
\]
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as in II (6.8). Thus for small $R_1, R_3$ the solotone effect should be given approximately by

$$S = \frac{R_1}{\pi} \sin 2\chi_1 - \frac{R_3}{\pi} \sin 2\chi_3.$$  \hspace{1cm} (2.7)

In an approximation which neglects Earth-curvature,

$$\chi_1 = \frac{r_1-a}{\beta_1} \frac{m\pi}{\gamma}, \quad \chi_3 = \frac{b-r_3}{\beta_3} \frac{m\pi}{\gamma},$$  \hspace{1cm} (2.8)

where

$$\gamma = \frac{r_1-a}{\beta_1} + \frac{r_3-r_1}{\beta_2} + \frac{b-r_3}{\beta_3}.$$  \hspace{1cm} (2.9)

Having evaluated these for the 3-layer Earth-model defined at the beginning of this section, we plot $S$ in Fig. 3(b). Our graph is very close to that of Fig. 3(a), and shows that the solotone effect is here the superposition of two sinusoidal oscillations of periods (in m) $\gamma/T_1$ and $\gamma/T_3$, where $T_1$ and $T_3$ are the times of normal transit of a ray across regions (1) and (3) respectively, and $\gamma$ is the time of complete normal transit through the three layers of the shell.

3. Pattern of Frequencies for a Two-Layer Shell

In Fig. 1 the points corresponding to frequencies were joined by a broken line: the roughly sinusoidal character of this broken line has been shown to

\[(m_n - m) \times 10^1\]

![Graph showing pattern of frequencies for two-layer shell for n=2, m=1, 2, 3, ..., 60. The points are the same as in Fig. 1, but are here shown to lie on smooth sinusoidal lines.](image)
match Anderssen's expression for the solotone effect due to a single surface of discontinuity. If we take the same set of points and join up in a new way we discover a remarkable and significant pattern. In Fig. 4, taking advantage of the almost periodic structure of the distribution of eigenvalues, we have drawn five continuous curves through the sets of points corresponding to:

\[
\begin{align*}
m &= 3, 8, 13, \ldots, 58, \\
m &= 4, 9, 14, \ldots, 59, \\
m &= 5, 10, 15, \ldots, 60, \\
m &= 6, 11, 16, \ldots, 56, \\
m &= 7, 12, 17, \ldots, 57.
\end{align*}
\]  

(3.1)

The points of one set are seen to lie on a smooth curve which changes slowly in direction. This can be understood by reference to the expression (1.5) for \( S \). For this model (§ 1) \( \theta - \phi = 286.497, \theta + \phi = 378.420 \), so that

\[
\frac{\theta - \phi}{\theta + \phi} = 0.609125 = \frac{3}{5} + \epsilon,
\]  

(3.2)

where \( \epsilon = 0.009125 \). If we now write

\[
m = 5p + q, \tag{3.3}
\]

where \( q = 0, 1, 2, 3, 4 \) and \( p = 0, 1, 2, \ldots, 9 \), then

\[
S = (-)^{p+q-1} \frac{R}{\pi} \left\{ \sin \left( \frac{3}{5} + \epsilon \right) (5p + q)\pi \right\}
\]

\[
= (-)^{q+1} \frac{R}{\pi} \sin (5q\pi + \delta)
\]  

(3.4)

where

\[
\delta = \left( \frac{3}{5} + \epsilon \right) q \pi.
\]  

(3.5)

The curves drawn in Fig. 4 are given by (3.4) for \( q = 0, 1, 2, 3, 4 \): that beginning from \( m = 3 \) corresponds to \( q = 3 \) and that from \( m = 4 \) to \( q = 4 \). The amplitudes \( (R/\pi) \) and periods \( (2/5\epsilon) \) of all five curves are the same, and the details of phase can be obtained from (3.4) and (3.5).

The pattern seen in Fig. 4 is for \( n = 2 \). Similar computations for \( n = 5, 10, 15 \) show that the same pattern arises, with small variations, whatever the value of \( n \). Thus the approximations made in deriving (3.4), which make the result independent of \( n \), hold for the ranges of \( m, n \) here considered. This pattern is not sensitive to the Legendre order-number of the oscillation.

The amplitude of the solotone oscillation depends on \( R \), and this is determined by the contrast in impedance at the surface of discontinuity of pa-
rameters. But the period of the solotone oscillation depends on the ratio of times \( \theta \) and \( \phi \) taken in radial travel through the layers. The pattern of eigenfrequencies depends delicately on the ratio of \( \theta - \phi \) to \( \theta + \phi \). If this ratio can be expressed as \( \pm (h/k) \), where \( h, k \) are small integers such that \( h + k \) is even, there will be \( k \) sine curves on which the points lie in the diagram corresponding to Fig. 4.

4. Shift of Frequency Pattern When Layer Thicknesses Change

In Fig. 5(a), (b), (c) we compare the eigenfrequency patterns for three shells which differ only in the relative thicknesses of the two layers. In each shell the shear velocity in the upper layer is 4.46 km/sec and in the lower layer is 6.58 km/sec. In model A the thicknesses of upper and lower layers are 700 km and 2,200 km respectively, in B 1,400 km and 1,500 km, and in C 2,100 km and 800 km. These thicknesses are shown in Fig. 5.

Figures 5(a), (b) and (c) show the patterns of eigenfrequencies computed for \( n=10, \ m=1, 2, 3, \ldots, 60 \) for the three models. These patterns differ greatly—there being 3, 7 and 5 curves, respectively, on which the points lie. We now show how these could have been predicted from knowledge of layer thicknesses and velocities.

The values of \( (\theta - \phi)/(\theta + \phi) \) for the three shells are as follows:

\[
\begin{align*}
(A) & \quad 0.361079 \\
(B) & \quad -0.158595 \\
(C) & \quad -0.589554
\end{align*}
\]

We now tabulate \( k |(\theta - \phi)/(\theta + \phi)| \) for \( k=1, 2, \ldots, 9, 10 \) to see whether we get a result near to an integer for any \( k \):

\[
\begin{array}{ccc}
 k & A & B & C \\
1 & 0.361 & 0.159 & 0.590 \\
2 & 0.722 & 0.317 & 1.179 \\
3 & \textbf{1.083} & 0.476 & 1.769 \\
4 & 1.444 & 0.634 & 2.358 \\
5 & 1.805 & 0.793 & \textbf{2.948} \\
6 & 2.166 & 0.952 & 3.537 \\
7 & 2.528 & \textbf{1.110} & 4.127 \\
8 & 2.889 & 1.269 & 4.716 \\
9 & 3.250 & 1.427 & 5.306 \\
10 & 3.611 & 1.586 & 5.896
\end{array}
\]
Thus, in

\[ A: \frac{\theta - \phi}{\theta + \phi} = \frac{1}{3} + \varepsilon_a, \]

\[ B: \frac{\theta - \phi}{\theta + \phi} = -\frac{1}{7} + \varepsilon_b, \tag{4.2} \]

\[ C: \frac{\theta - \phi}{\theta + \phi} = -\frac{3}{5} + \varepsilon_c, \]
Fig. 5. Patterns of eigenfrequencies for \( n = 10, m = 1, 2, \ldots, 60 \) for three two-layer shells, which differ only in position of inner surface of discontinuity.

(a) \( \text{In (b) this lies 1,400 km below the upper surface, 1,500 km above the lower.} \)

(b) \( \text{In (b) this lies 700 km below the upper surface, 2,200 km above the lower.} \)

(c) \( \text{In (c) this lies 1,400 km below the upper surface, 1,500 km above the lower.} \)

The eigenfrequencies lie on 3 smooth curves in (a), 7 in (b) and 5 in (c).

where \( \varepsilon_a, \varepsilon_b, \varepsilon_c \) are small. Thus the patterns of eigenfrequencies for \( A, B \) and \( C \) will contain 3, 7 and 5 curves respectively. Further discussion of the patterns can be carried out as in §3. We see from this example that small changes in \( \theta \) and \( \phi \) are likely to produce large changes in the diagram of eigenfrequency distribution, as suggested at the end of §3. In another paper (Lapwood and Satô, 1977) we show how the pattern changes dramatically as the thicknesses of layers are changed.

5. Pattern of Frequencies for a Three-Layer Shell

Turning now to the three-layer Earth model (averaged PEM-A) introduced in §2, we draw, for \( n = 2 \), both the solotone graph with an approximate main period of 10 in \( m \), and the ten lines in which points \( q + 10p \) (\( q \) and \( p \) integers) lie on curves which deviate in direction only slightly, being almost parallel to the axis of \( m \). These are shown in Fig. 6. We found Anderssen's expression \( S \) in (2.7) and showed in Fig. 3(b) that \( S \) is very close to \( m_2 - m \): we now use (2.7) to show why the period \( m = 10 \) arises in this solotone oscillation. Let a certain \( m = m_0 \) give, from (2.7)

\[
\pi S = R_1 \sin \alpha_1 m_0 - R_2 m_0 \sin \alpha_2 m_0,
\]

(5.1)
where

\[ \alpha_1 = 1.401544 \pi, \quad \alpha_2 = 0.396460 \pi. \quad (5.2) \]

we now increase \( m \) by 10 to \( m_0 + 10 \), and find the corresponding \( S_2 \):

\[ \pi S_2 = R_1 \sin \alpha_1 (m_0 + 10) - R_2 \sin \alpha_2 (m_0 + 10). \quad (5.3) \]

The condition that \( S_2 \) is almost equal to \( S_1 \) for any \( m \) is that \( E \) should be small, where

\[ E = R_1 \{ \sin \alpha_1 (m + 10) \sin \alpha_1 m \} - R_2 \{ \sin \alpha_2 (m + 10) \sin \alpha_2 m \} = R_1 \{ \sin \alpha_1 m (\cos 10 \alpha_1 - 1) + \cos \alpha_1 m \sin 10 \alpha_1 \} - R_2 \{ \sin \alpha_2 m (\cos 10 \alpha_2 - 1) + \cos \alpha_2 m \sin 10 \alpha_2 \}. \quad (5.4) \]

But

\[ 10 \alpha_1 = 14.0154 \pi, \quad 10 \alpha_2 = 3.9646 \pi \]
\[ \sin 10 \alpha_1 = 0.0438, \quad \sin 10 \alpha_2 = 0.1110 \]
\[ \cos 10 \alpha_1 = 0.9988, \quad \cos 10 \alpha_2 = 0.9938 \]

so that

\[ E = 0.236 (\sin \alpha_1 m (0.0012) - \cos \alpha_1 m (0.0438)) - 0.152 (\sin \alpha_2 m (0.0062) + \cos \alpha_2 m (0.1110)) \quad (5.5) \]

which is very small for all \( m \) between 10 and 60. Thus points of one set in Fig. 6 lie on an almost straight line almost parallel to the axis of \( m \). Note that

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**Fig. 6.** For three-layer shell, \( n=2, m=0, 1, \ldots, 60 \). This figure corresponds to Fig. 3 in solotone shape (zigzag graph with period 10 in \( m \)), and has superposed the ten (almost straight) lines defining the pattern of eigenfrequencies.
this flatness of the pattern needs both $10\alpha_1$ and $10\alpha_2$ to be very nearly integral multiples of $2\pi$ at the same time.

It is clear that the pattern of eigenfrequencies could have been predicted by examination of $k\alpha_1$ and $k\alpha_2$ for integral $k$.

6. Interpretation of Frequency Equation in Terms of Ray Theory

In I, §9 we showed that the frequency equation (9.9), namely

$$\sin(\chi_3 + \chi_1) + R' \sin(\chi_3 - \chi_1) = 0,$$

(6.1)
could be expanded into the relation

$$R'e^{-2i\chi_2} + Te^{-2i(\chi_2 + \chi_1)} + TR'e^{-2i(\chi_2 + 2\chi_1)} + TR''e^{-2i(\chi_2 + 3\chi_1)} + \ldots = 1,$$

(6.2)

where $R'' = -R'$ is the coefficient of reflection for a ray impinging on the $(1, 2)$ interface from the region $(1)$ and $T = 1 - R'^2$ is the transmission coefficient for a double transmission across the $(1, 2)$ boundary. (6.2) is then interpreted as stating that in any normal mode the total disturbance at any point is the vector sum of all contributions from rays multiply reflected in one layer (see Fig. 6 of Paper I). We note that in this problem the disturbance in one normal mode needs an infinite sum of rays for its representation, just as the disturbance in one pulse needs an infinite sum of normal modes for its representation.

If we turn to a three-layer shell we can obtain from the frequency equation (2.4) the relation

$$(1 + R_1 e^{-2i\chi_1})(1 - R_2 e^{-2i\chi_3}) - e^{-2i\chi_2}(e^{-2i\chi_1} + R_1)(e^{-2i\chi_3} - R_2) = 0.$$

(6.3)

Here $R_1$ is coefficient of reflection for a ray impinging on the $(1, 2)$ boundary from medium $(2)$, and $R_2$ for a ray impinging on the $(2, 3)$ boundary from medium $(3)$. We write $R_1'(=-R_1)$ for reflection at the $(1, 2)$ boundary of a ray from $(1)$, and $R_2'(=-R_2)$ for reflection at the $(2, 3)$ boundary of a ray from $(2)$. Then

$$e^{-2i\chi_2} R_1 + e^{-2i\chi_3} R_2' + e^{-2i\chi_3} = 1.$$

(6.4)

As in (I), we employ the identity

$$\frac{A+B}{1+AB} = A + B(1 - A^j) \sum_0^\infty (-AB)^j$$

(6.5)

to obtain

$$1 = e^{-2i\chi_2}[R_1 + e^{-2i\chi_1}T_1(1 - R_2 e^{-2i\chi_3} + R_2' e^{-4i\chi_1} - R_2'' e^{-6i\chi_1} \ldots)]$$

$$\times [R_2' + e^{-2i\chi_2}T_2(1 - R_1 e^{-2i\chi_3} + R_1' e^{-4i\chi_1} - R_1'' e^{-6i\chi_1} \ldots)]$$

$$= e^{-2i\chi_2} R_1 + e^{-2i\chi_2} T_1(1 - R_2 e^{-2i\chi_3} + R_2'' e^{-4i\chi_1} - R_2'' e^{-6i\chi_1} \ldots)$$

$$\times [R_2' + e^{-2i\chi_2} T_2(1 - R_1 e^{-2i\chi_3} + R_1'' e^{-4i\chi_1} - R_1'' e^{-6i\chi_1} \ldots)]$$

(6.6)
In (6.6) the term contributed by the product of the \((p+2)\)th term in the first curled bracket and the \((q+2)\)th term in the second curled bracket is
\[
e^{-2i\theta_{1}2R_{1}R_{1}'2}\cdot (6.7)
\]
This arises from a ray which set out from a certain point of the middle layer and came back to the same point after \(p\) reflections in the first layer and \(q\) reflections in the third layer. The first term of the product in (6.6) is \(e^{-2iz_{3}R_{3}'R_{3}}\), which comes from a ray which returns to the same point of the middle layer after one reflection at each interface. Every term of (6.6) can be associated in this way with a ray which undergoes multiple reflections in the two outer layers. The frequency equation states that the vector sum of displacements in all these rays is unity. This is the condition for constructive interference.

We have examined the form of the frequency equation in which the curvature of the Earth has been neglected, \(R_{1}\) and \(R_{3}\) are coefficients of normal reflection, and \(\chi_{1}, \chi_{3}, \chi_{3}\) have simple form. The same analysis holds for the more complicated parameters which occur where Earth-curvature is allowed for, as it was in I § 6.

Turning to our expressions for the solotone frequency pattern, we can interpret them as resonance effects. For a two-layer shell we have (1.5)
\[
S = (-)^{m+1} \frac{R}{\pi} \sin \frac{T_{1} - T_{3}}{T} m\pi, \tag{6.8}
\]
where \(T_{1}\) is the time taken for a ray to cross layer (1) and \(T_{3}\) to cross (2), and \(T = T_{1} + T_{3}\). Then if \(T_{1} - T_{3}\) divides into \(T\) almost an integral number of times, say \(k\), the pattern of frequencies recurs (approximately) with period \(k\) in \(m\). In other words, the pattern shows resonance between (a) the time of full radial transit of a ray, and (b) the time-lapse between arrivals of rays through the separate layers. This is shown schematically in Fig. 7(a).
For a three-layer shell (from (2.7)),
\[
S = \frac{R_1}{\pi} \sin \frac{T_1}{T} m \pi - \frac{R_2}{\pi} \sin \frac{T_3}{T} m \pi \\
= \left( - \right)^{m+1} \frac{R_1}{\pi} \sin \frac{T_1+T_2}{T} m \pi - \left( - \right)^{m+1} \frac{R_2}{\pi} \sin \frac{T_1+T_3}{T} m \pi .
\]
(6.9)

The first term on the right of (6.9) arises from resonance between \( T \) and the timelapse between rays through (2) and (3) and through (1), while the second arises from resonance between \( T \) and the timelapse between rays through (1) and (2) and through (3). These paths are shown in Fig. 7(b).

7. Conclusions

In this paper we have utilised computed solutions and analytic approximations obtained in papers (I) and (II) to examine the solotone effect in spherical shells with two or three uniform layers, as seen in the pattern of frequencies of radial overtones in torsional oscillations. We have shown how amplitude and period of solotone oscillation are determined by the elastic parameters and layer thicknesses, and how the detailed pattern of the diagram of frequencies depends critically on resonance effects.

Similar analysis could be performed for other sequences of overtones. We expect that the solotone effect and frequency diagram will prove to depend delicately on the internal structure of the shell. This should be very useful in judging the relative merits of different proposed models, when data become sufficiently refined.

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