Comment on “On the Derivation of the Elastic Equation of Motion” by R. J. Geller

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Geller (1988) concludes that the variational principle for continua does not in general lead to Newton's second law. Also he asserts that the generalized form of Hooke's law with 36 independent elastic constants always leads to the existence of a strain energy function. And he recommends that the equation derived from variational principle ought to be used when the equation is different from the equation based on Newton's second law.

In this comment, only the first case (Hookean material in classical linear elasticity theory) treated by Geller (1988) is considered. Therefore my conclusion does not apply to his second case (a linearly elastic, self-gravitating medium). The same notations for physical quantities as Geller (1988) are used in this comment. His equation is cited as (Gn), when n is the equation number.

The above-mentioned conclusions derived by Geller (1988) are summarized in Table 1. Table 2 shows the traditional conclusions established in the classical linear elasticity theory (Love, 1944). According to them, the strain energy function (potential energy per unit volume) can be defined for the Hookean material with 21 independent elastic constants ($C_{klij} = C_{ijkl}$). On the contrary, the strain energy function cannot be defined for the Hookean material with 36 independent elastic constants ($C_{klij} \neq C_{ijkl}$). So the variational principle for the conservative system with kinetic and potential energies is valid only for the $C_{klij} = C_{ijkl}$ material. In this case, the equation derived from the variational principle is the same as the well-known (Newton-Euler-)Cauchy equation. Here the generalized Hooke's law is the same as Eqs. (G1) and (G5).

$$\sigma_{ij} = C_{ijkl} e_{kl} = C_{ijkl} u_{k,l}, \quad (\text{G1, G5})$$

where σ_{ij} and e_{kl} are stress and strain components. Definition of e_{ij} is given by

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}). \quad (\text{G2})$$

The definition of e_{ij} and properties of C_{ijkl} expressed in Eqs. (G3) and (G4) are used to derive Eq. (G5) from Eq. (G1):

$$C_{ijkl} = C_{jikl} = C_{ijlk}. \quad (\text{G3, G4})$$

Equation set No. 1 in Tables 1 and 2 means the combination of the (Newton-Euler)-

Table 1. Summary of Geller's (1988) conclusions about the strain energy function and applicability of equation sets. Equation set No. 1 means the combination of Eqs. (1) and (2) in the text. Equation set No. 2 means Eqs. (3) and (4).

Number of independent elastic constants	Strain energy function	Equation set No. 1	Equation set No. 2
21($C_{kl ij} = C_{ij kl}$)	$\frac{1}{2} u_{i,j} C_{ijkl} u_{k,l}$	Applicable	Applicable
36($C_{kl ij} \neq C_{ij kl}$)	$\frac{1}{2} u_{i,j} C_{ijkl} u_{k,l}$	Not applicable*	Applicable

* Geller (1988) claims, without any proof, the exact equivalency of the two equation sets in Section 4. But later he discusses the difference in hermiticity of bilinear operators corresponding to the two equation sets in Sections 5, 6, 7, 8, 11, and 12. He attributes this difference to flaws in the "standard derivation" on the equation set No. 1. To my opinion, this difference is that between the properties of equation sets, and shows the non-equivalency of the two equation sets. So his evaluation of equation set No. 1 is not consistent.

Table 2. Summary of the traditional conclusions about the strain energy function and applicability of equation sets (Love, 1944). Equation sets are the same as in Table 1.

Thermal process	Number of independent elastic constants	Strain energy function	Equation set No. 1	Equation set No. 2
Adiabatic or isothermal	21($C_{kl ij} = C_{ij kl}$)	$\frac{1}{2} e_{ij} C_{ijkl} e_{kl}$	Applicable	Applicable
Otherwise	36($C_{kl ij} \neq C_{ij kl}$)	Does not exist	Applicable	Not applicable

Cauchy equation (1) and the boundary condition Eq. (2) listed below. The equation of motion called in this comment the (Newton-Euler-)Cauchy equation is

$$\rho \ddot{u}_i = (C_{ijkl} u_{k,l})_{,j} + f_i, \quad (1)$$

or in the frequency domain,

$$\omega^2 \rho u_i + (C_{ijkl} u_{k,l})_{,j} = -f_i. \quad (G35)$$

The boundary condition for the surface traction T_i is expressed as:

$$(C_{ijkl} u_{k,l}) n_j = T_i. \quad (2)$$

Equation set No.2 means the combination of Eqs. (3) and (4) below:

$$\rho \ddot{u}_i = \left[\frac{1}{2} (C_{ijkl} + C_{klij}) u_{k,l} \right]_{,j} + f_i, \quad (3)$$

or in the frequency domain,

$$\omega^2 \rho u_i + \left[\frac{1}{2} (C_{ijkl} + C_{klij}) u_{k,l} \right]_{,j} = -f_i; \quad (G32)$$

and

$$\frac{1}{2}(C_{ijkl} + C_{klij})u_{k,l}n_j = T_i. \quad (4)$$

These Eqs. (3), (G32), and (4) are derived from the variational principle for the conservative system, the potential energy of which is assumed to be given by Eq. (G9), i.e., $\frac{1}{2} \int dV u_{i,j} C_{ijkl} u_{k,l}$. Of course the equation set No. 2 coincides with No. 1 for the $C_{klij} = C_{ijkl}$ material.

It is evident that no difference exists between Tables 1 and 2 for the $C_{klij} = C_{ijkl}$ material. But contradictory conclusions are derived for the $C_{klij} \neq C_{ijkl}$ material. Therefore attention is focused on the latter case.

The definitions of the potential energy of the elastic body used by Geller (1988) are discussed in some detail as follows.

His original definition of the potential energy is Eq. (G8):

$$PE = -\frac{1}{2} \int dV u_i (C_{ijkl} u_{k,l})_{,j}. \quad (G8)$$

Using Eqs. (G5) and (G6), this can be written as

$$PE = -\frac{1}{2} \int dV u_i \sigma_{ij,j} \quad (5)$$

$$= -\frac{1}{2} \int dV u_i F_i. \quad (6)$$

Note that $F_i = \sigma_{ij,j}$ (G6) is the elastic force per unit volume and is definitely different from the external body force per unit volume f_i . I believe that the body force f_i is assumed to be absent in the discussion by Geller (1988) until he introduces f_i in (G32). We see that these expressions (5) and (6) cannot be true by considering a simple static situation where the elastic body is deformed by some surface tractions. Since the body force is absent, the equation of equilibrium becomes

$$\sigma_{ij,j} = 0 \quad (7)$$

or

$$F_i = 0. \quad (8)$$

The expressions (5) and (6) give always the value 0 in spite of the fact that the elastic body is strained. Therefore the equation (G8) cannot be accepted as the proper definition of the potential energy of the elastic body.

Next, we consider his second expression of the potential energy, i.e., Eq. (G9):

$$PE = \frac{1}{2} \int dV u_{i,j} C_{ijkl} u_{k,l}. \quad (G9)$$

This expression is, of course, correct for the elastic body with 21 independent elastic

Fig. 1. Three different paths from the same initial state (0) to the same final state (3).

constants C_{ijkl} , i.e., $C_{klij} = C_{ijkl}$. Geller's (1988) assertion that this expression gives the potential energy for the elastic body with 36 independent elastic constants C_{ijkl} , i.e., $C_{klij} \neq C_{ijkl}$, is wrong. Two simple examples below show that the mechanical work done by the surface tractions (external forces) to unit volume of the elastic body, which has the same final strain values, depends on the path whereby the final strain is reached, unless $C_{klij} = C_{ijkl}$ condition holds. Therefore, for the $C_{klij} \neq C_{ijkl}$ material, we cannot define the strain energy function (potential energy per unit volume), which is expressed by the final strain and elastic constants, independently of the strain path.

In the examples, the initial equilibrium state with no strain or stress is denoted by the state (0), from which displacement is measured. The two intermediate states and the final state are called state (1), (2), and (3), respectively. Three paths from state (0) to state (3) are considered, as shown schematically in Fig. 1:

Path I: state (0) → state (1) → state (3)

II: state (0) → state (2) → state (3)

III: state (0) → state (3).

The mechanical works done by the surface tractions along the three paths I, II, and III are denoted by A_I , A_{II} , and A_{III} , respectively.

Example 1.

state (1): $u_1 = \varepsilon_1 x_1$, $u_2 \equiv u_3 \equiv 0$

state (2): $u_2 = \varepsilon_2 x_2$, $u_1 \equiv u_3 \equiv 0$

state (3): $u_1 = \varepsilon_1 x_1$, $u_2 = \varepsilon_2 x_2$, $u_3 \equiv 0$,

where ε_1 and ε_2 are some constants.

The detail of the calculation of mechanical works along three paths is explained in the Appendix. The results are:

$$A_I = \frac{1}{2} (C_{1111} \varepsilon_1^2 + C_{2222} \varepsilon_2^2) + C_{2211} \varepsilon_1 \varepsilon_2$$

$$A_{II} = \frac{1}{2}(C_{1111}\varepsilon_1^2 + C_{2222}\varepsilon_2^2) + C_{1122}\varepsilon_1\varepsilon_2$$

$$A_{III} = \frac{1}{2}(C_{1111}\varepsilon_1^2 + C_{2222}\varepsilon_2^2) + \frac{1}{2}(C_{1122} + C_{2211})\varepsilon_1\varepsilon_2.$$

The right-hand side of Eq. (G9) gives the value A_{III} for the unit volume in the above example.

Example 2.

$$\text{state (1): } u_1 = \varepsilon_1 x_1, \quad u_2 \equiv u_3 \equiv 0$$

$$\text{state (2): } u_1 = 0, \quad u_2 = \frac{1}{2}\varepsilon_2 x_3, \quad u_3 = \frac{1}{2}\varepsilon_2 x_2$$

$$\text{state (3): } u_1 = \varepsilon_1 x_1, \quad u_2 = \frac{1}{2}\varepsilon_2 x_3, \quad u_3 = \frac{1}{2}\varepsilon_2 x_2.$$

The calculation of mechanical works is similar to the former example.

$$A_I = \frac{1}{2}(C_{1111}\varepsilon_1^2 + C_{2323}\varepsilon_2^2) + C_{2311}\varepsilon_1\varepsilon_2$$

$$A_{II} = \frac{1}{2}(C_{1111}\varepsilon_1^2 + C_{2323}\varepsilon_2^2) + C_{1123}\varepsilon_1\varepsilon_2$$

$$A_{III} = \frac{1}{2}(C_{1111}\varepsilon_1^2 + C_{2323}\varepsilon_2^2) + \frac{1}{2}(C_{1123} + C_{2311})\varepsilon_1\varepsilon_2.$$

The expression for A_{III} coincides with the right-hand side of Eq. (G9) for the unit volume.

The above two examples show that the mechanical work done by the surface tractions depends on the path, unless $C_{klij} = C_{ijkl}$ condition holds. I do not think more examples are necessary to show that the potential energy does not exist for the $C_{klij} \neq C_{ijkl}$ material. We cannot apply the variational principle for the conservative system to such a material.

I conclude that Geller's (1988) discussion for the $C_{klij} \neq C_{ijkl}$ material contains an error about the existence of the potential energy. The traditional conclusions summarized in Table 2 remain true for the Hookean elastic body.

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APPENDIX

We consider a unit volume of the elastic body with six surfaces: $x_1 = \pm \frac{1}{2}$, $x_2 = \pm \frac{1}{2}$, and $x_3 = \pm \frac{1}{2}$. Here the mechanical works done by the surface tractions in Example 1 along the three paths are calculated.

Displacement components are expressed as:

$$u_1 = f(t)\varepsilon_1 x_1, \quad u_2 = g(t)\varepsilon_2 x_2, \quad u_3 \equiv 0,$$

where $f(t)$ and $g(t)$ are arbitrary continuous, differentiable functions of time parameter t , which satisfy

$$f(0) = g(0) \quad \text{and} \quad f(1) = g(1) = 1.$$

Then $e_{11} = f(t)\varepsilon_1$, $e_{22} = g(t)\varepsilon_2$, and other strain components are all zero. Stress component is given by:

$$\sigma_{ij} = C_{ij11}f(t)\varepsilon_1 + C_{ij22}g(t)\varepsilon_2.$$

Mechanical works done by surface tractions through six surfaces are calculated. Surface traction is given by the following formula by Cauchy.

$$T_i = \sigma_{ij}n_j,$$

where n_j is the outward unit normal vector of the surface.

(i) Across the surfaces $x_1 = \pm \frac{1}{2}$.

$$T_i = \pm \sigma_{i1} = \pm (C_{i111}f(t)\varepsilon_1 + C_{i122}g(t)\varepsilon_2)$$

$$du_1 = \pm \frac{1}{2} \varepsilon_1 df(t), \quad du_2 = \varepsilon_2 x_2 dg(t), \quad du_3 \equiv 0.$$

Incremental work done by surface traction through a small area $dx_2 dx_3$ during the time parameter interval dt is given by

$$T_i du_i = (C_{1111}f(t)\varepsilon_1 + C_{1122}g(t)\varepsilon_2) \frac{1}{2} \varepsilon_1 df(t) dx_2 dx_3 \\
+ (C_{2111}f(t)\varepsilon_1 + C_{2122}g(t)\varepsilon_2) \varepsilon_2 x_2 dg(t) dx_2 dx_3$$

for both surfaces. Since the second term in the right-hand side of the above equation

is an odd function of x_2 , this term contributes nothing in the integral with respect to x_2 from $-\frac{1}{2}$ to $\frac{1}{2}$.

The mechanical work through the surfaces $x_1 = \pm \frac{1}{2}$ is given by

$$\begin{aligned} A\left(x_1 = \pm \frac{1}{2}\right) &= 2 \int (C_{1111}f(t)\varepsilon_1 + C_{1122}g(t)\varepsilon_2) \frac{1}{2} \varepsilon_1 df(t) \iint dx_2 dx_3 \\ &= \frac{1}{2} C_{1111} \varepsilon_1^2 + C_{1122} \varepsilon_1 \varepsilon_2 \int g(t) df(t). \end{aligned}$$

(ii) Across the surfaces $x_2 = \pm \frac{1}{2}$.

$$T_i = \pm \sigma_{i2} = \pm (C_{i211}f(t)\varepsilon_1 + C_{i222}g(t)\varepsilon_2)$$

$$du_1 = \varepsilon_1 x_1 df(t), \quad du_2 = \pm \frac{1}{2} \varepsilon_2 dg(t), \quad du_3 \equiv 0.$$

The mechanical work through the surfaces $x_2 = \pm \frac{1}{2}$ is given by

$$\begin{aligned} A\left(x_2 = \pm \frac{1}{2}\right) &= 2 \int (C_{2211}f(t)\varepsilon_1 + C_{2222}g(t)\varepsilon_2) \frac{1}{2} \varepsilon_2 dg(t) \\ &= \frac{1}{2} C_{2222} \varepsilon_2^2 + C_{2211} \varepsilon_1 \varepsilon_2 \int f(t) dg(t). \end{aligned}$$

(iii) Across the surfaces $x_3 = \pm \frac{1}{2}$.

$$du_1 = \varepsilon_1 x_1 df(t), \quad du_2 = \varepsilon_2 x_2 dg(t), \quad du_3 \equiv 0$$

The mechanical work through the surfaces $x_3 = \pm \frac{1}{2}$ is given by

$$A\left(x_3 = \pm \frac{1}{2}\right) = 0.$$

Adding the three values above, we get the total mechanical work done by the surface tractions to the unit volume of the body.

$$\begin{aligned} A &= \frac{1}{2} (C_{1111} \varepsilon_1^2 + C_{2222} \varepsilon_2^2) \\ &\quad + \varepsilon_1 \varepsilon_2 \left(C_{1122} \int g(t) df(t) + C_{2211} \int f(t) dg(t) \right). \end{aligned} \quad (A1)$$

Now selecting adequate forms for the two functions $f(t)$ and $g(t)$, we can get the mechanical work along the three paths I, II, and III.

Path I: state (0) $\rightarrow$ state (1) $\rightarrow$ state (3)

$$f(t)=1 \quad \text{for } \frac{1}{2} \leq t \leq 1, \quad g(t)=0 \quad \text{for } 0 \leq t \leq \frac{1}{2}.$$

Then,

$$\int g(t)df(t)=0 \quad \text{and} \quad \int f(t)dg(t)=1.$$

Finally we get

$$A_I = \frac{1}{2} (C_{1111}\varepsilon_1^2 + C_{2222}\varepsilon_2^2) + C_{2211}\varepsilon_1\varepsilon_2.$$

Path II: state (0) $\rightarrow$ state (2) $\rightarrow$ state (3)

Interchanging the suffixes 1 and 2 in the expression for A_I , we get

$$A_{II} = \frac{1}{2} (C_{1111}\varepsilon_1^2 + C_{2222}\varepsilon_2^2) + C_{1122}\varepsilon_1\varepsilon_2.$$

Path III: state (0) $\rightarrow$ state (3)

$$f(t)=g(t) \quad \text{for } 0 \leq t \leq 1.$$

Then,

$$\int g(t)df(t) = \int f(t)dg(t) = \frac{1}{2}.$$

Finally we get

$$A_{III} = \frac{1}{2} (C_{1111}\varepsilon_1^2 + C_{2222}\varepsilon_2^2) + \frac{1}{2} (C_{1122} + C_{2211})\varepsilon_1\varepsilon_2.$$