Rayleigh-Love Wave Coupling in an Azimuthally Anisotropic Medium

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We present the generalized representation of the equations of motion for an elastic solid in a concise vector form with arbitrary orthogonal curvilinear coordinates and no assumption on symmetry of elastic moduli. For the particular case of the Cartesian coordinates, this representation leads to the generalized y-method for eigenvalues of surface wave dispersion in an anisotropic plane-stratified medium. The generalized y-method is the integration method to find eigenvalues of surface wave dispersion by iterating numerical depth-integration of first-order ordinary differential equations that are derived from the generalized representation of the equations of motion and Hookean law. Based on the generalized y-method, we present some numerical results for dispersion curves and azimuthal variations of surface wave velocities to fully display Rayleigh-Love wave coupling in Kawasaki's (1986) azimuthally anisotropic model for the upper mantle beneath the Pacific ocean.

When Rayleigh-Love wave coupling takes place in a particular period range between a pair of nearby modes, surface waves display the following distinct singularities: (1) a difference of phase velocities of the pair of the modes is smaller than about 0.5 km/s, (2) a pair of dispersion curves of group velocities cross each other, (3) polarizations of particle motion directions are twisted along the pair of dispersion curves from Rayleigh- to Love-types and from Love- to Rayleigh-types. These singularities are dependent on the relative depth- and azimuthal-distribution of the upper mantle anisotropy. When Rayleigh-Love wave coupling does not take place, the first-order perturbation theory works well.

1. Introduction

From the time that the \(Pn\) velocity anisotropy was first recognized in the central Pacific Ocean by Hess (1964), evidence has been accumulated for upper mantle anisotropy in both oceanic and continental environments.

In the 1980s, the upper mantle anisotropy has been one of the prevalent issues in seismology. Dziewonski and Anderson (1981) first attempted a transversely isotropic inversion of a vast amount of seismological data from eigenperiods of Earth's free...
oscillations, surface wave dispersion and travel times of body waves. They obtained the novel Preliminary Reference Earth Model (PREM), with a notably less distinct low velocity zone (LVZ). In their inversion, *transverse isotropy* with five independent elastic moduli was assumed as a reference postulate for modeling the seismic observations. The vertical axis of elastic symmetry was assumed for the other 220 km of the upper mantle.

Regan and Anderson (1984) applied the *transversely isotropic inversion* to regional dispersion data found in the Pacific Ocean by Mitchell and Yu (1980). Kawasaki (1986) attempted to apply a *quasi-azimuthally anisotropic inversion* to the same dispersion data. They both suggested a very thin oceanic lithosphere of about 45–50 km for an average Pacific Ocean of ages of 20 to 100 Ma, which was far thinner than previous isotropic inversions, but consistent with the non-seismic definition by seamount loading (Watts et al., 1980).

Tanimoto and Anderson (1985) mapped *azimuthal anisotropy* of mantle Rayleigh wave velocities at periods between 100–250 s and concluded that the retrieved anisotropy pattern could simulate the pattern of return convection flow in the mesosphere at depths of 200–300 km, which was derived from kinematic consideration by Hager and O’connell (1979). Nishimura and Forsyth (1988) and Suetsugu and Nakanishi (1987) suggested similar results for the lithosphere and asthenosphere under the Pacific Ocean with regional surface wave dispersion at periods shorter than 100 s. Now, it is widely recognized that upper mantle anisotropy observations could have an ability to map mantle convection flow at various depths.

The *assumption of transverse isotropy* is that azimuthal averages of surface wave velocities in an *azimuthally anisotropic* structure can well be approximated by those for corresponding *transversely isotropic* model. “Since our data represent an average over many azimuths, any residual anisotropy will be effectively averaged out” (Dziewonski and Anderson, 1981). This seems valid only if the anisotropy is within the scope of the first-order perturbation theory (Smith and Dahlen, 1973), in which Rayleigh-Love wave coupling is ignored. In the mapping of *azimuthal anisotropy* of Rayleigh wave phase velocity (e.g., Tanimoto and Anderson, 1985), the first-order perturbation theory was also postulated. However, we will show in the following section that Rayleigh-Love wave coupling may take place in some situations of upper mantle anisotropy and, as a result, the first-order perturbation theory and the *assumption of transverse isotropy* may break down.

To retrieve the upper mantle anisotropy, it must be fully understood beforehand how anisotropy affects surface wave propagation. For this purpose, we need a general theory of surface wave propagation in a generally anisotropic medium.

Theoretical investigations of first-order perturbations on seismic wave propagation in an anisotropic plane-stratified medium have been made by many authors: e.g., an azimuthal dependence of body wave velocities (Backus, 1965), surface wave velocities (Smith and Dahlen, 1973) and eigenfunctions (Montagner and Nataf, 1986). However, these first-order perturbation theories cannot include Rayleigh-Love wave coupling.

Crampin (1970) developed the extended propagator matrix method (or the extended Haskell matrix method) to include general anisotropy and subsequent Rayleigh-Love wave coupling. However, this method requires each layer to be homogeneous and cannot
be applied to models having a vertical gradient within respective layers. This method searches for roots of complex higher-order algebraic equations and often involves a severe numerical difficulty for convergence of eigenvalues when many layers exist. Crampin and his associates (e.g., Crampin, 1967, 1975, 1977; Kirkwood and Crampin, 1981) already showed Rayleigh-Love wave coupling of surface waves in azimuthally anisotropic media by the use of the extended propagator matrix method. The purpose of the present paper is to extend their study to understand a more detailed picture of this phenomenon with inclusion of more complex and realistic upper mantle structure.

Eigenvalues of surface wave dispersion that satisfy the boundary conditions at Earth's free surface and at infinite depth may also be found by iterating numerical depth-integration of first-order ordinary differential equations that are obtained from the equations of motion and Hookean law. This has sometimes been called the y-method after Alterman et al. (1959) first used the symbol “y” to represent eigenfunctions for the ordinary differential equations. The y-method is commonly used in seismology for its computational convenience in dealing with smooth models for the Earth.

In the first half of this paper, we will generalize the representation for the y-method of Takeuchi and Saito (1972) to allow general anisotropy in a concise vector forms as a possible alternative to the extended propagator matrix method. Although many new quantities must be defined to develop the new representation, notations of Takeuchi and Saito (1972) will mostly be kept. The y-method with this generalized representation is called the generalized y-method hereafter. Among the advantages of this approach are thus:

1. we can automatically involve Rayleigh-Love wave coupling,
2. we can calculate dispersion of the Rayleigh-Love modes for an Earth model with arbitrary vertical velocity gradient, while the extended-propagator matrix method is limited to a stack of homogeneous layers,
3. we can easily calculated partial derivatives.

In the second half, we will also present some numerical results to demonstrate Rayleigh-Love wave coupling of surface waves in a realistic model for the upper mantle beneath the Pacific Ocean, using the generalized y-method.

2. Generalized Equations of Motion

The equations of motion of an elastic medium in curvilinear coordinates (e.g., Fung, 1965) have the form

\[ \rho \frac{\partial^2 \vec{U}^j}{\partial t^2} = \vec{S}^{jk} |_k \]  \hspace{1cm} (1) \]

where \( t \) is time, \( \rho \) density, \( \vec{U}^j \) and \( \vec{S}^{jk} \) tensor components of elastic displacements and stresses. The tilde indicates a quantity in the time domain. Subscripts and superscripts denote covariant- and contravariant-quantities, respectively. All indices vary over 1–3 unless otherwise noted. The Einstein summation convention over 1, 2, and 3 for repeated indices is implied unless otherwise specified. A boldface character indicates a matrix or a vector.

The symbol \( |_k \) denotes the covariant differential defined as
\[ T_{m_1m_2...m_p}^{n_1n_2...n_q} = \frac{\partial T_{n_1n_2...n_q}}{\partial X^k} + \Gamma_{jk}^{m_1} T_{n_1n_2...n_q}^{m_2...m_p} 
+ \Gamma_{jk}^{m_1} T_{n_1n_2...n_q}^{m_2...m_p} - \Gamma_{n_1n_2...n_q}^{m_1} T_{-n_1n_2...n_q}^{m_2...m_p} \]  
\[ \Gamma_{kp}^q = \frac{1}{2} \gamma^{as} \left( \frac{\partial g_{sp}}{\partial X^k} + \frac{\partial g_{sk}}{\partial X^p} - \frac{\partial g_{kp}}{\partial X^s} \right) \]  
where \( g_{pq} \) and \( g^{pq} \) are covariant and contravariant metric matrices. Thereby applying Fourier-transformation with respect to time \( t \) to the equations of motion (1) results in

\[ \rho \omega^2 U^j = S^{jk} \]  
where \( \omega \) is angular frequency. Hereafter throughout this paper, all quantities related to elastic deformation without a tilde are always Fourier-transformed with respect to time.

Tensor components \( E_{pq} \) of elastic strains are

\[ E_{pq} = \frac{1}{2} (U_p U_q + U_q U_p) = \frac{1}{2} \left( \frac{\partial U_p}{\partial X^q} + \frac{\partial U_q}{\partial X^p} \right) - \frac{1}{2} (\Gamma^q_{pq} + \Gamma^q_{qp}) U_s. \]  

The generalized stress-strain relation (generalized Hookean law) in curvilinear coordinates is expressed with tensor components \( A^{kpq} \) of elastic moduli as

\[ S^{jk} = A^{kpq} E_{pq}. \]  

Now, we introduce displacement and stress vectors

\[ W_A = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}, \quad W^A = \begin{pmatrix} U^1 \\ U^2 \\ U^3 \end{pmatrix}, \quad W^{BJ} = \begin{pmatrix} S^{1j} \\ S^{2j} \\ S^{3j} \end{pmatrix}. \]  

With Eqs. (5) and (7), Hookean law (6) is rewritten in a vector form as

\[ W^{BJ} = L^{jk}_0 (G^k_q + D^k_q) W_A, \]  
where

\[ D^k_q = \partial / \partial X^k, \quad (L^{jk}_0)^{st} = A^{jsk}, \quad (G^k_q)_{st} = - \Gamma^k_{st}. \]  

\( (L^{jk}_0)^{st} \) and \( (G^k_q)^{st} \) define the \( st \)-components of matrices \( L^{jk}_0 \) and \( G^k_q \), respectively. Introduction of the matrices \( L^{jk}_0 \) is the key element of the generalized representation in such a simple vector form.

The equations of motion (4) can be rewritten in the same manner, using the
displacement and stress vectors in (7) as

\[ -\rho \omega^2 \mathbf{w}^A = (-\mathbf{C}^A_{\beta} + J^y_{\beta}) \mathbf{w}^B, \]

where \( J^y_{\beta} = \Gamma^y_{\beta} \). An overbar indicates the Hermitian conjugate of an algebraic quantity.

Now, we proceed to replace tensor components of displacements and stresses with corresponding physical components that have a uniform physical dimension. Note that the physical components do not obey the tensor transformation law and thereby are not components of a tensor (Fung, 1965).

When the curvilinear coordinates are orthogonal, such replacements become much simpler, using the following orthogonality of metric matrices

\[ g_{jk} = g_{kj} = 0, \quad (j \neq k), \]

\[ g_{jj} = 1, \quad \text{(not summed for } j). \]

Physical components of elastic displacements \( u_j \), strains \( e_{jk} \), stresses \( \sigma^{jk} \), and moduli \( \lambda^{pjkq} \) are related to the corresponding tensor components as

\[ u_j = U_j g_j, \quad u^l = U^j g_j, \]

\[ e_{jk} = E_{jk} (g_j g_k), \quad \sigma^{jk} = S^{jk} g_j g_k, \]

\[ \lambda^{pjkq} = \Lambda^{pjkq} g_p g_q g_k g_l, \]

(not summed for \( j, k, p, \) and \( q \)),

where

\[ g_1 = \sqrt{g_{11}}, \quad g_2 = \sqrt{g_{22}}, \quad g_3 = \sqrt{g_{33}}. \]

Displacement and stress vectors of physical components of \( \mathbf{w}_A \), \( \mathbf{w}^A \), and \( \mathbf{w}^{Bj} \) are defined as

\[ \mathbf{w}_A = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \mathbf{w}^A = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}, \quad \mathbf{w}^{Bj} = \begin{pmatrix} \sigma^{1j} \\ \sigma^{2j} \\ \sigma^{3j} \end{pmatrix}. \]

Introducing a matrix \( \mathbf{R}_z \)

\[ \mathbf{R}_z = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix}, \]

the following simple relationships are obtained under the orthogonality condition (10):

\[ \mathbf{W}_A = \mathbf{R}_z \mathbf{w}_A, \quad \mathbf{w}^A = \mathbf{R}_z^{-1} \mathbf{w}^A, \]

\[ \mathbf{W}^{Bj} = \frac{1}{g_j} \mathbf{R}_z^{-1} \mathbf{w}^{Bj}, \quad \mathbf{w}_A = \mathbf{w}^A. \]

\( \mathbf{w}_A = (u_1, u_2, u_3) \) in the Cartesian coordinates \((x_1, x_2, x_3)\) (see Fig. 1) and \( \mathbf{w}_* = (u_\theta, u_\phi, u_z) \) in the spherical polar coordinates \((\theta, \phi, z)\). \( \mathbf{w}^A = \mathbf{w}_A \) in the orthogonal
Tensor components $L^j_k$ of elastic moduli can be related to the corresponding physical components $L^{jk}$ as

$$L^j_k = \left( \frac{1}{g_k} \right) R_z^{-1} L^{jk} R_z^{-1},$$

and

$$(L^{jk})_{pq} = \lambda^{jkq},$$

(not summed for $j$ and $k$).

Substituting equations in (11) and (12) into (8) and (9) results in

$$w^{Rj} = \left( \frac{L^{jk}}{g_k} \right) R_z^{-1} (G^j_k + D^j_k) R_z w_A,$$

and

$$-\rho \omega^2 w^A = \frac{1}{g_j} R_z (-\bar{G}^j_j + J^j_j + D^j_j) R_z^{-1} w^{Rj}. $$

Defining $G_j, D_j,$ and $J_j$ as

$$G_j = (1/g_j) [R_z^{-1} G^j_k R_z + R_z^{-1} (D^j_k R_z)],$$

$$D_j = (1/g_j) D^j_k,$$

$$J_j = (1/g_j) J^j_j,$$

(not summed for $j$),

we produce the following equalities

$$(1/g_k) R_z^{-1} (G^j_k + D^j_k) R_z = (G_k + D_k),$$

$$(1/g_j) R_z (-\bar{G}^j_j + J^j_j + D^j_j) R_z^{-1} = (-\bar{G}_j + J_j + D_j),$$

(not summed for $j$ and $k$),

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where $D_jg_j=0$ (not summed for $j$) in orthogonal curvilinear coordinates is implicitly used. $D_j$ denotes partial derivatives with respect to corresponding physical components of the reference coordinates.

Substituting (15) into (13) and (14) yields

\[ w^{Bj} = (L^{jk})(G_k + D_k)w_A, \quad (16) \]
\[ -\rho \omega^2 w^A = (-\vec{G}_j + J_j + D_j)w^{Bj}. \quad (17) \]

Substituting (16) into (17) yields

\[ -\rho \omega^2 w^A = (-\vec{G}_j + J_j + D_j)(L^{jk})(G_k + D_k)w_A. \quad (18) \]

Equations (16)–(18), which are first derived in this paper, are the completely generalized representation of Hookean law and the equations of motion written in terms of physical components of elastic properties in arbitrary orthogonal curvilinear coordinates. They contain no assumptions concerning the symmetry of elastic moduli and the homogeneity of the medium. Only restriction is the orthogonality of the reference coordinates.

We define matrices $B^{jk}$ as

\[ B^{23} = (L^{33})^{-1}, \]
\[ B^{j3} = L^{j3}(L^{33})^{-1}, \quad B^{3k} = (L^{33})^{-1}L^{3k}, \]
\[ B^{jk} = L^{jk} - L^{j3}(L^{33})^{-1}L^{3k}, \quad (j = 1, 2, k = 1, 2), \quad (19) \]

where $E$ is a unit matrix. Isolating a term of $D_3w_A$ in (16), we have the first half of reference equations for the generalized $y$-method as

\[ D_3w_A = -G_3w_A - \sum_{k=1}^{2} (B^{3k})(G_k + D_k)w_A + (B^{23})w^B. \quad (20) \]

Hereafter, $w^{B3}$ is written as $w^B$ for simplicity.

Substituting (20) into (16) gives

\[ w^{Bj} = \sum_{k=1}^{2} (B^{jk})(G_k + D_k)w_A + (B^{j3})w^B, \quad (j = 1, 2). \quad (21) \]

Rearranging (16)–(18), (20), and (21), and isolating a term of $D_3w^B$ in (17), we obtain the other half of reference equations as

\[ D_3w^B = -\rho \omega^2 w^A - \sum_{j=1}^{2} \sum_{k=1}^{2} (-\vec{G}_j + J_j + D_j)(B^{jk})(G_k + D_k)w_A \]
\[ -\sum_{j=1}^{2} (-\vec{G}_j + J_j + D_j)(B^{3j})w^B - (-\vec{G}_3 + J_3)w^B. \quad (22) \]

If variables are separated, $D_3w_A$ and $D_3w^B$ on the left-hand side of (20) and (22) are replaced with corresponding first-order ordinary derivatives $dw_A/dz$ and $dw^B/dz$, respectively. $z$ is a coordinate vertical to Earth’s free surface.

It is straightforward to derive partial derivatives with these generalized
representation as shown in the Appendix, Partial Derivatives.

3. The y-Equations for the Generalized Surface Waves

In the particular case of the Cartesian coordinates, the representation for the generalized y-method in the previous section 2 becomes simpler. Here, following the notational convention for the Cartesian coordinate system, all indices indicating covariant- or contravariant-types are written as subscripts in the following, since there is no difference between them in terms of tensor transformation law.

In the Cartesian coordinate system,

\[ X^1 = x_1, \quad X^2 = x_2, \quad X^3 = z, \]

\[ D_1 = \frac{\partial}{\partial x_1}, \quad D_2 = \frac{\partial}{\partial x_2}, \quad D_3 = \frac{\partial}{\partial z}, \]

\[ g_1 = g_2 = g_3 = 1, \]

\[ J_1 = J_2 = J_3 = 0, \]

\[ \Gamma^k_k = 0, \]

\[ R_x = E, \]

\[ G_1 = G_2 = G_3 = 0. \]

Substituting (23) into (20) and (22), we have

\[ D_3 w_A = -(B_{31}D_1 + B_{32}D_2)w_A + B_{Z3}w^B, \] \hspace{1cm} (24)

\[ D_3 w^B = -\rho \omega^2 w_A - (D_1 B_{13} + D_2 B_{23})w^B \]

\[ - (D_1 B_{11}D_1 + D_2 B_{21}D_1 + D_1 B_{12}D_2 + D_2 B_{22}D_2)w_A. \] \hspace{1cm} (25)

Equations (24) and (25) are similar to the representation of Kennett (1986) for a laterally heterogeneous medium, which was derived in Cartesian coordinates. In the case of a laterally heterogeneous medium, a wavenumber vector k turns out to be a function of coordinates and variables cannot be separated.

On the other hand, in a laterally homogeneous medium, the concept of plane surface waves can be applied (Crampin, 1970), which allows the separation of variables and subsequently the replacement of \( D_3 \) with \( d/dz \) even in a generally anisotropic medium. Thereby, \( \mu_j \) and \( \sigma_{j3} \) can be written as

\[ u_j = Y_{m(j)}(z) \exp[-i(\omega t - k_1 x_1 - k_2 x_2)], \]

\[ \sigma_{j3} = Y_{m(j+3)}(z) \exp[-i(\omega t - k_1 x_1 - k_2 x_2)], \] \hspace{1cm} (26)

where \( m(1) = 3, \ m(2) = 5, \ m(3) = 1, \ m(4) = 4, \ m(5) = 6, \) and \( m(6) = 2. \) \( k_1 \) and \( k_2 \) are two horizontal components of the wavenumber vector k. New vectors \( Y_A \) and \( Y_B \) are here introduced as

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Substituting (26) and (27) into (24) and (25), we have

\[
\frac{dY_A}{dz} = -(ik_1B_{31} + ik_2B_{32})Y_A + B_{33}Y_B,
\]

\[
\frac{dY_B}{dz} = -\rho \omega^2 Y_A - (ik_1B_{13} + ik_2B_{23})Y_B
\]

\[+(k_1^2B_{11} + k_1k_2B_{12} + k_2^2B_{22})Y_A.\]

A factor of $\exp[-i(\omega - k_1x_1 - k_2x_2)]$ is hereafter omitted for simplicity. We have now obtained the reference equations needed for the generalized $y$-method. Note that the $y$-method has been usually defined as an integration method to find an eigenvalue, a phase velocity of Rayleigh or Love waves, that satisfies boundary conditions of free elastic stresses at Earth's surface and no displacements at infinite depth, by iterating numerical depth-integration of such first-order ordinary differential equations as (28) and (29).

Substituting (26)-(29) into (A.8), we have energy integrals in the Cartesian coordinates

\[
I_1 = \int_{-\infty}^{H} \rho \bar{Y}_A Y_A dz,
\]

\[
I_2 = \int_{-\infty}^{H} \left[ \bar{Y}_A(k_1^2B_{11} + k_1k_2B_{12} + k_2^2B_{22})Y_A + \bar{Y}_B B_{23} Y_B \right] dz,
\]

with an equality relation $\omega^2 I_1 = I_2$. $H$ is a $z$-coordinate of the Earth’s surface. They reduce to (169) and (174) of Takeuchi and Saito (1972) for a transversely isotropic medium. $\varepsilon$ defined as $\varepsilon = I_2/\omega^2 I_1 - 1$ is conventionally used to display the consistency of the numerical depth-integrations of the ordinary differential equations (28) and (29).

Perturbing (28) and (29) with respect to wavenumber components $k_1$ and $k_2$, density and elastic moduli, taking similar operations to those for (A.9)-(A.13) and integrating from $z = -\infty$ to the surface $z = H$, we have

\[
2\omega \delta \omega I_1 + \omega^2 \frac{\partial I_1}{\partial \rho} \delta \rho = \delta k_1 I_3 + \delta k_2 I_4 + \frac{\partial I_5}{\partial \lambda_{jkl}} \delta \lambda_{jkl},
\]

where

\[
I_3 = \int_{-\infty}^{H} [\bar{Y}_A(2k_1B_{11} + k_2B_{12})Y_A - 2i\bar{Y}_A B_{13} Y_B] dz,
\]

\[
I_4 = \int_{-\infty}^{H} [\bar{Y}_A(k_1B_{11} + 2k_2B_{22})Y_A - 2i\bar{Y}_A B_{23} Y_B] dz,
\]
Considering perturbations of only $k_1$ and $k_2$ in (30) with $\lambda_{jkpq}$ unperturbed, the following two equations can be derived:

$$2\omega \frac{\partial \omega}{\partial k_1} = I_3,$$

$$2\omega \frac{\partial \omega}{\partial k_2} = I_4.$$

A group velocity vector $U$ is thus obtained for the plane surface waves,

$$U = (U, U_t) = \left( \frac{\partial \omega}{\partial k_1}, \frac{\partial \omega}{\partial k_2} \right) = \left( \frac{I_3}{I_0}, \frac{I_4}{I_0} \right), \quad (31)$$

where $I_0 = 2\omega I_1$. $U$ and $U_t$ are radial and transverse components of group velocity. As seen in (31), a direction of the group velocity is in general not parallel to that of the corresponding slowness, a wavefront defined by an inverse of a phase velocity of plane surface waves.

Taking perturbations of elastic moduli $\lambda_{jkpq}$ with $k_1$ and $k_2$ unperturbed in (30), and dividing both sides by $2\omega^2 I_1 = 2I_2$, we have

$$\frac{\delta \omega}{\omega} = -\frac{1}{2I_1} \frac{\partial I_1}{\partial \rho} + \frac{1}{2I_2} \frac{\partial I_5}{\partial \lambda_{jkpq}} - \delta \lambda_{jkpq}, \quad \text{(not summed for } j, k, p, \text{ and } q). \quad (32)$$

In the following, the summation convention for repeated indices is not applied for elastic moduli $\lambda_{jkpq}$ unless otherwise noted.

With (32) and the relationships

$$\frac{\delta c}{c} = \frac{\partial \omega}{\omega},$$

$$c = \frac{\omega}{k_1},$$

the following equations are produced

$$\frac{\partial c}{\partial \rho} = \left( \frac{c^2}{2UI_1} \right) \nabla_A \nabla_A ,$$

$$\frac{\partial c}{\partial \lambda_{jkpq}} = \left( \frac{c^2}{2I_2} \right) \left[ \nabla_A \left( k_1^2 \frac{\partial B_{11}}{\partial \lambda_{jkpq}} + k_1 k_2 \frac{\partial B_{12}}{\partial \lambda_{jkpq}} + k_2^2 \frac{\partial B_{22}}{\partial \lambda_{jkpq}} \right) \right] \nabla_A$$

$$-2 \nabla_A \left( ik_1 \frac{\partial B_{13}}{\partial \lambda_{jkpq}} + ik_2 \frac{\partial B_{23}}{\partial \lambda_{jkpq}} \right) \nabla_B - \nabla_B \left( \frac{\partial B_{23}}{\partial \lambda_{jkpq}} \right) \nabla_B \left( \frac{\partial B_{23}}{\partial \lambda_{jkpq}} \right), \quad (33)$$

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or, its equivalent from (A.15)

\[
\frac{\partial c}{\partial \lambda_{jkpq}} = \left( \frac{c^2}{2U_2} \right) \left[ Y_A \left( k^2 \frac{\partial L_{11}}{\partial \lambda_{jkpq}} + k_1 k_2 \frac{\partial L_{12}}{\partial \lambda_{jkpq}} + k_2^2 \frac{\partial L_{22}}{\partial \lambda_{jkpq}} \right) Y_A 
- 2Y_A \left( ik_1 \frac{\partial L_{13}}{\partial \lambda_{jkpq}} + ik_2 \frac{\partial L_{23}}{\partial \lambda_{jkpq}} \right) \frac{dY_A}{dz} + \frac{dY_A}{dz} \frac{\partial L_{33}}{\partial \lambda_{jkpq}} \frac{dY_A}{dz} \right].
\] (34)

If the elastic moduli in (28) and (29) are perturbed with sine and cosine terms representing azimuthal anisotropy, it is straightforward to obtain such first-order perturbations of azimuthal dependences of phase velocities and eigenfunctions as Smith and Dahlen (1973) and Montagner and Nataf (1986).

For plane surface waves propagating in the \( x_1 \)-direction, (28) and (29) become simpler with \( k_1 = k \) and \( k_2 = 0 \) as

\[
\frac{dY_A}{dz} = -ikB_{31}Y_A + B_{z3}Y_B, \quad \text{(35)}
\]

\[
\frac{dY_B}{dz} = -\rho \omega^2 Y_A + k^2 B_{11}Y_A - ikB_{13}Y_B. \quad \text{(36)}
\]

For a transversely isotropic medium, (35) and (36) further reduce to the familiar equations (34) and (62) of Takeuchi and Saito (1972), as shown in the Appendix, An Azimuthal Variation of Equation of Motion.

Now, horizontal axes are rotated by \( \phi \) measured from an \( x_1 \)-axis and new \( x_1 \)- and \( x_2 \)-axes are redefined \( x \)- and \( y \)-axes, respectively. Seismic waves are assumed to propagate in a direction of a new \( x_1 \)-axis. For \( k, Y_A, Y_B \), and \( B_{jk} \) in the new coordinates \( (x, y, z) \), we have the same expression as (35) and (36). Hereafter, (35) and (36) are taken for those propagating in the direction of an \( x \)-axis in the new coordinate system.

In order to derive propagation velocities of the generalized surface waves for an azimuth of \( \phi \), it seems convenient to rotate horizontal axis by \( \phi \) and to calculate eigenvalues using (35) and (36) rather than to use (28) and (29) for a pair of components \( (k_1, k_2) \) of a wavenumber.

The correspondence between \( Y_j \), conventional notations \( y^R_j \) and \( y^I_j \) of Takeuchi and Saito (1972), and elastic displacements \( u_j \) and stresses \( \sigma_{zj} \) are

\[
\begin{pmatrix}
  y^R_1 \\
  y^R_2 \\
  y^R_3 \\
  y^R_4 \\
  y^R_5 \\
  y^R_6 \\
  y^I_1 \\
  y^I_2 \\
  y^I_3 \\
  y^I_4 \\
  y^I_5 \\
  y^I_6
\end{pmatrix}
= \begin{pmatrix}
  Y_1 \\
  Y_2 \\
  Y_3 \\
  Y_4 \\
  Y_5 \\
  Y_6
\end{pmatrix}
\begin{pmatrix}
  u_z \\
  \sigma_{zz} \\
  u_x \\
  \sigma_{xz} \\
  u_y \\
  \sigma_{yz}
\end{pmatrix} \quad \text{(37)}
\]

The integral identities for plane surface waves are now
Fig. 2. Solid lines show an azimuthally averaged structure of the azimuthally anisotropic KB-Z model (Kawasaki, 1986) for the average Pacific Ocean of ages of 20–100 Ma. Broken lines show B-model of Regan and Anderson (1984).

Table 1. Elastic constants of KB-Z and KB-T models.

<table>
<thead>
<tr>
<th>Model</th>
<th>Depth</th>
<th>$\rho$</th>
<th>$A$</th>
<th>$C$</th>
<th>$L$</th>
<th>$N$</th>
<th>$A-2N$</th>
<th>$F$</th>
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<tbody>
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<td>crust</td>
<td>0-4.85</td>
<td>1.02</td>
<td>2.3</td>
<td>2.3</td>
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<td>4.85-5.03</td>
<td>2.00</td>
<td>5.4</td>
<td>5.4</td>
<td>2.0</td>
<td>2.0</td>
<td>1.4</td>
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<tr>
<td></td>
<td>5.03-6.65</td>
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<td></td>
<td>6.63-11.52</td>
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<td>134.1</td>
<td>44.1</td>
<td>44.1</td>
<td>45.9</td>
<td>45.9</td>
</tr>
<tr>
<td>KB-T</td>
<td>11.82-51.3</td>
<td>3.31</td>
<td>224.1</td>
<td>205.0</td>
<td>72.0</td>
<td>75.4</td>
<td>73.4</td>
<td>70.0</td>
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<tr>
<td></td>
<td>51.3</td>
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<tr>
<td></td>
<td>220.0</td>
<td>3.31</td>
<td>207.9</td>
<td>189.3</td>
<td>60.0</td>
<td>62.9</td>
<td>82.0</td>
<td>77.3</td>
</tr>
</tbody>
</table>

Axi-symmetric with the horizontal $x_3$-axis

<table>
<thead>
<tr>
<th>Model</th>
<th>Depth</th>
<th>$\rho$</th>
<th>$C_{11}$</th>
<th>$C_{33}$</th>
<th>$C_{44}$</th>
<th>$C_{66}$</th>
<th>$C_{12}$</th>
<th>$C_{23}$</th>
<th>$P_Z$</th>
</tr>
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<tbody>
<tr>
<td>KB-Z</td>
<td>11.82-51.3</td>
<td>3.31</td>
<td>244.0</td>
<td>205.0</td>
<td>69.0</td>
<td>75.0</td>
<td>73.0</td>
<td>67.0</td>
<td>0.4</td>
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<tr>
<td></td>
<td>51.3</td>
<td>3.31</td>
<td>218.7</td>
<td>184.4</td>
<td>61.1</td>
<td>65.7</td>
<td>65.8</td>
<td>62.1</td>
<td>1.1</td>
</tr>
<tr>
<td></td>
<td>220.0</td>
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<td>228.3</td>
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<td>57.9</td>
<td>62.0</td>
<td>81.1</td>
<td>73.5</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Units are km for depth, g/cm$^3$ for $\rho$ and GPa for $C_{ij}$ and $P_Z$. $P_Z$ equals to $(C_{11}+C_{33}-2C_{13}-4C_{66})/8$, which accounts for $4\phi$ variation of azimuthal anisotropy. In KB-Z model of Kawasaki (1986) which is axi-symmetric with a horizontal $x_3$-axis, $C_{33}=C_{44}$, $C_{55}=C_{66}$, and $C_{13}=C_{12}$. KB-T is an azimuthal average of the KB-Z model. In the transversely isotropic model KB-T, $A=C_{11}=C_{22}$, $C=C_{33}$, $L=C_{44}=C_{55}$, $N=C_{66}$, $A-2N=C_{12}$, and $F=C_{13}=C_{23}$. Elastic moduli $A$, $C$, $F$, $L$, and $N$ were introduced by Love (1944). A structure below 220 km depth is the same as PREM (Dziewonski and Anderson, 1981).
and we have the partial derivative

$$I_2 = \int_{-\infty}^{\infty} \left[ k^2 \tilde{Y}_A B_{11} Y_A + \tilde{Y}_B B_{23} Y_B \right] dz,$$

$$I_3 = \int_{-\infty}^{\infty} \left[ 2k \tilde{Y}_A B_{11} Y_A - 2i \tilde{Y}_A B_{13} Y_B \right] dz,$$

$$I_4 = \int_{-\infty}^{\infty} \left[ k \tilde{Y}_A B_{12} Y_A - 2i \tilde{Y}_A B_{23} Y_B \right] dz,$$

$$I_5 = \int_{-\infty}^{\infty} \left[ k^2 \tilde{Y}_A B_{11} Y_A - 2ik \tilde{Y}_A B_{13} Y_B - \tilde{Y}_B B_{23} Y_B \right] dz,$$

and we have the partial derivative

$$\frac{\partial c}{\partial \lambda_{jkpq}} = \left( \frac{c^2}{2U_1} \right) \left[ k^2 \tilde{Y}_A \frac{\partial B_{11}}{\partial \lambda_{jkpq}} Y_A - 2ik \tilde{Y}_A \frac{\partial B_{13}}{\partial \lambda_{jkpq}} Y_B - \tilde{Y}_B \frac{\partial B_{23}}{\partial \lambda_{jkpq}} Y_B \right],$$

or its equivalent,

$$\frac{\partial c}{\partial \lambda_{jkpq}} = \left( \frac{c^2}{2U_1} \right) \left[ k^2 \tilde{Y}_A \frac{\partial L_{11}}{\partial \lambda_{jkpq}} Y_A - 2ik \tilde{Y}_A \frac{\partial L_{13}}{\partial \lambda_{jkpq}} \frac{dY_A}{dz} + \frac{d\tilde{Y}_A}{dz} \frac{dL_{33}}{dz} \frac{dY_A}{dz} \right].$$

These expressions for partial derivatives also reduce to (191)–(197) of Takeuchi and Saito (1972) for a transversely isotropic medium.

4. Azimuthally Anisotropic Upper Mantle Models

Regan and Anderson (1984) applied transversely isotropic inversion of Dziewonski and Anderson (1981) to regional dispersion data in the Pacific Ocean of Mitchell and Yu (1980). Regan and Anderson proposed a new picture of an oceanic upper mantle with a thin lithosphere of about 45 km (broken lines in Fig. 2), which was far thinner than previous isotropic inversions, but consistent with the non-seismic definition by seamount loading (Watts et al., 1980).

Ophiolite is a part of a segment that was once a paleo-oceanic plate and is hence a fossil of the paleo-oceanic upper mantle. Christensen and his associates (e.g., Christensen and Salisbury, 1979) extensively studied petrofabrics and elastic features of ultramafic rocks from the ophiolite complex, which are summarized in Christensen (1984). The basic elastic configurations observed are (1) the axi-symmetry of P wave velocity anisotropy of up to 10 percent with respect to the paleo-plate spreading direction, in which the a-axis of olivine and the c-axis of pyroxene of the ultramafic rocks are aligned, and (2) a less distinct anisotropy of S wave velocities of 0–3 percent if they did exist.

Kawasaki (1986) attempted a quasi-azimuthally anisotropic inversion of the dispersion data and obtained an azimuthally anisotropic model of an oceanic upper mantle with a thin lithosphere of about 45 km (solid lines in Fig. 2). Hereafter, B-model of Kawasaki (1986) is called KB-Z model, where a label Z indicates an azimuthally anisotropic structure. Its elastic moduli are listed in Table 1. One of the basic configurations of the KB-Z model is the axi-symmetry with the paleo-spreading direction.
Fig. 3. Dispersion curves of Rayleigh-type generalized surface waves of $o_R$, $1_R$, and $2_R$ for two azimuths of $0^\circ$ and $90^\circ$ and those of Love-type generalized surface waves of $o_L$, $1_L$, and $2_L$ for two azimuths of $0^\circ$ and $45^\circ$ in KB-Z model. Azimuths in brackets are from the paleo-plate spreading direction in KB-Z model. Labels $n_R$ and $n_L$ denote the $n$-th higher-mode Rayleigh-type and Love-type generalized surface waves, respectively.

Fig. 4. Dispersion curves of $o_R$, $o_L$, $1_R$, and $1_L$ for an azimuth of $45^\circ$. Twisting of polarizations takes place between $o_L$ and $1_R$ in a boxed region. This is more distinct for group velocities (right), where a pair of dispersion curves cross each other in the boxed region.

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Rayleigh-Love Wave Coupling

in both the LVZ and the lithospheric plate above the LVZ. This feature is consistent with above-mentioned measurement of ultramafic rocks (e.g., Christensen and Salisbury, 1979), and with petrological modeling of Estey and Douglas (1986).

The overall feature of upper mantle anisotropy of KB-Z model is also quite consistent with seismic anisotropy observations in the Pacific Ocean; Pn velocity anisotropy of up to 8 percent (e.g., Shimamura et al. 1983), a small azimuthal anisotropy of Sn velocity (e.g., Shearer and Orcutt, 1986), SH-SV polarization anisotropy (e.g., Mitchell and Yu, 1980), azimuthal anisotropy of up to 4 percent of mantle Rayleigh wave velocities (Nishimura and Forsyth, 1988) and a small azimuthal anisotropy of mantle Love wave velocities (Nishimura and Forsyth, 1985).

In the following, surface wave dispersion is calculated for plane-stratified structure, based on KB-Z model using the generalized y-method. We do not take sphericity of the Earth into account, but basic features would hold good for a spherical Earth, since magnitudes of sphericity corrections are less than 0.1 km/s (Biswas and Knopoff, 1970; North and Dziewonski, 1976).

5. Twisting of Polarizations

Crampin and his associates (e.g., Crampin, 1975) mentioned Rayleigh-Love wave coupling of surface waves in azimuthally anisotropic media by the use of the extended propagator matrix method. We would like to extend their studies to understand a more detailed picture of this phenomenon.

In an isotropic medium, Rayleigh waves have vertical and radial components of displacements, displaying elliptic particle motions in a vertical plane parallel to a wave propagation direction, and Love waves have only a transverse component within a horizontal plane. However, in an azimuthally anisotropic medium, surface waves no longer display such pure polarizations, and their particle motion directions are obliquely tilted and inclined. In this sense, Crampin (1977) called them "generalized surface waves" and proposed a new naming convention, 0G, 1G, 2G, etc., in order of slowness of phase velocities. For examples, the fundamental-mode generalized surface waves 0G generally has a polarization close to that of the fundamental-mode Rayleigh waves, and 1G to that of the fundamental-mode Love waves in an isotropic medium.

However, we use traditional terms, 'Rayleigh' and 'Love,' and their abbreviations "R and L to represent the n-th higher-mode Rayleigh- and Love-type generalized surface waves, respectively, for the purpose of clarifying the types of polarizations. Crampin's notation is supplemented in the text as 1R(2G).

Eigenfunctions of Y1 to Y6 are normalized by a surface value Y1(H) for the Rayleigh-type and by Y3(HS) for the Love-type generalized surface waves, respectively, following a notational convention (e.g., Takeuchi and Saito, 1972). HS is the z-coordinate at the top of a solid part of the plane-stratified structure.

Figure 3 shows dispersion curves of phase velocities (left) and a radial component of group velocities (right) of the Rayleigh-type generalized surface waves of 0R(0G), 1R(2G) and 2R(4G) for two azimuths of 0° and 90° (see Fig. 1 for the reference coordinate system) and those of the Love-type generalized surface waves of 0L(1G), 1L(3G), and 2L(5G) for two azimuths of 0° and 45° for KB-Z model. The bracketed numerals are
MODE-TWISTING BETWEEN $\delta L$ AND $\delta R$

**Phase Velocity**

![Phase Velocity Diagram](image)

**Group Velocity**

![Group Velocity Diagram](image)

Fig. 5. Dispersion curves of phase (left) and group (right) velocities for the azimuth of 45°, enlarged from Fig. 4 for periods of 18–36 s. Labels $\delta L$ and $\delta R$ indicate that particle motion polarizations are of $\delta L$ and $\delta R$ types around the label locations along the dispersion curves, respectively. Eigenfunctions at locations (1)–(8) are given in Fig. 6.

Fig. 6. Eigenfunctions at the locations (1)–(8) along the pair of dispersion curves in Fig. 5. $Y_1$, $Y_3$, and $Y_5$ are vertical, radial and transverse components of displacement eigenfunctions, respectively. Surface wave polarizations are twisting from $\delta R$ at (1) to $\delta L$ at (4) and from $\delta L$ at (5) to $\delta R$ at (8), respectively.
azimuths of propagation directions of plane surface waves measured from an \( x_1 \)-axis. Surface wave velocities are usually smallest or largest in these azimuths except at 45° for Love waves. The direction of largest velocity of the Love-type generalized surface waves may differ from 45°, but the deviation would be 5° at the largest. Therefore, Fig. 3 illustrates possible ranges of azimuthal variations of dispersion curves of respective modes in KB-Z model.

Figure 4 shows dispersion curves for an azimuth of 45°, focusing on twisting of polarizations from \( 0L \) to \( 1R \) and from \( 1R \) to \( 0L \) along a pair of dispersion curves of \( 1G \) and \( 2G \) in the boxed region. The pair of dispersion curves for a radial component of group velocities (right) for the azimuth cross each other near a period of 30 s, whereas those of corresponding phase velocities (left) only approach each other within about 0.05 km/s or about 1 percent. In general, dispersion curves of phase velocities for an azimuth do not cross each other in an anisotropic medium. Details of basic features of the twisting of polarizations are examined in Figs. 5 and 6.

Figure 5 shows the dispersion curves for an azimuth of 45°, enlarged from Fig. 4 for periods between 18 and 36 s. Figure 6 shows depth-profiles of the eigenfunctions of \( Y_1, Y_3, \) and \( Y_5 \) at the locations of (1)–(8) in Fig. 5. Note that \( Y_1, Y_3, \) and \( Y_5 \) correspond to vertical \( u_z \), radial \( u_r \), and transverse \( u_t \) displacement components. Those at (1) and (8) are close to typical polarizations for the first higher-mode isotropic Rayleigh waves and those at (4) and (5) to the fundamental-mode isotropic Love waves. At (2), (3), (6), and (7), they are intermediate between the two modes. Twisting of polarizations of eigenfunctions is thus evidently observed from \( 1R \) at location (1) to \( 0L \) at (4) and from \( 0L \) at (5) to \( 1R \) at (8) along the respective dispersion curves.

Also in other azimuths, the twisting of polarizations takes place at periods between 20 and 30 s in KB-Z model.

6. Azimuthal Variations of Phase and Group Velocities

In this section, the other distinction of anisotropy effects, azimuth-dependent variations of surface wave velocities (hereafter called azimuthal-branches), will be discussed. The KB-Z model displays orthorhombic symmetry with the \( x_1-x_2, x_2-x_3, \) and \( x_1-x_3 \) planes leading to symmetry of surface wave velocities in directions of \( x_1 \)- and \( x_2 \)-axes. Azimuthal variations are thus shown only for azimuths between 0° and 90° in the following figures.

Figures 7 and 8 show azimuthal-branches of phase and group velocities at three periods of 30, 50, and 70 s. Magnitudes of azimuthal variations vary with modes, reflecting relative depth-distribution of upper mantle anisotropy.

The azimuthal-branches of phase velocities (Fig. 7) of \( 0L(1G) \) and \( 1R(2G) \) at a period of 30 s approach each other within 0.05 km/s for azimuths of 70°–90° where group velocities (Fig. 8) are drastically distorted by Rayleigh-Love wave coupling near the region indicated by (X). In general, for models with stronger anisotropy, the crossing of a pair of azimuthal-branches of group velocities and twisting of polarizations along the pair of azimuthal-branches takes place due to Rayleigh-Love wave coupling, similarly to those along dispersion curves. Similar twisting of split S waves was recently theoretically suggested by Shearer and Chapman (1989).
AZIMUTHAL VARIATION OF PHASE VELOCITY FOR KB-Z

Fig. 7. Azimuthal variations of phase velocities in KB-Z model at three periods of 30, 50, and 70 s. Azimuthal-branches of $0_L$ and $1_R$ at a period of 30 s are approaching close to each other for azimuths of $70^\circ-90^\circ$.

AZIMUTHAL VARIATION OF GROUP VELOCITY FOR KB-Z

Fig. 8. Azimuthal variations of radial component of group velocities in KB-Z model at three periods of 30, 50, and 70 s. Azimuthal-branches of $0_L$ and $1_R$ at a period of 30 s approach to each other for azimuths of $70^\circ-90^\circ$.

In Fig. 9, phase and group velocities of $0_L(1G)$ and $1_R(2G)$ for a period of 30 s are enlarged along with those of a corresponding transversely isotropic model KB-T (broken lines), whose elastic moduli $C_{jk}$ (azimuthal averages of $C_{jk}$ of KB-Z model) in Table 1 are obtained through (A.19). The group velocities are seriously distorted, compared with those in the corresponding transversely isotropic medium as seen in the

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Fig. 9. Solid lines are azimuthal-branches of phase velocities and radial component of group velocities of $\phi R$, $\phi L$, and $\phi R$ in KB-Z model at a period of 30 s, enlarged from Fig. 8. Broken lines are those in the corresponding transversely isotropic model KB-T whose elastic moduli are azimuthal averages of those of KB-Z.

Fig. 10. Azimuthal variations of transverse component $U_t$ of group velocities and eigenfunctions of $Y_1(HS)$ and $Y_5(HS)$ of $\phi L$ and $\phi R$ at the top of a solid part of a plane-stratified structure. Pertinent parameters are the same as Fig. 9.
right figure. The assumption of transverse isotropy completely breaks down for the case of group velocities of $0_L$ and $1_R$.

As is well known for isotropic Earth models, partial derivatives have a narrow crest in the particular depth-range for Rayleigh waves and a broad side-lobe for Love waves (e.g., Aki and Richards, 1980). Once the azimuthal anisotropy is invoked in the lithosphere/asthenosphere, the relative depth-locations of crests and troughs of partial derivatives become azimuth-dependent as in (A.14). This results in such an intricate dispersion behavior of group velocities as shown in Figs. 8 and 9. Care should be taken in inversion particularly for multi-mode surface wave dispersion.

Phase velocities of $0_L(1G)$ and $1_R(2G)$ for azimuths around 45° at the period of 30 s in Fig. 9 are smaller by about 0.03 km/s on an average than those predicted by a corresponding transversely isotropic model. Although this difference seems small, this could have an effect to reduce a thickness of lithosphere by about 7 km in an anisotropic inversion for phase velocities of surface waves (Kawasaki, 1985).

Figure 10 shows azimuthal-branches of $U_n$, $Y_1(HS)$ and $Y_5(HS)$ at a period of 30 s. For azimuths of 60°–80°, $Y_5(HS)$ on the azimuthal-branch of $1_R(2G)$ and $Y_1(HS)$ on the azimuthal-branch of $0_L(1G)$ are near to 1 or -1, which implies extremely oblique polarizations due to Rayleigh-Love wave coupling. $U_n$ is up to 0.2 km/s in this figure and is up to 0.3 km/s in other period ranges, which implies that possible magnitudes of deviations of group velocity directions is up to 4° from propagation directions of corresponding plane surface waves.

Polarizations of the generalized surface waves in the time domain, which are obtained by integrating those at respective Fourier periods, should also be dependent on a source depth. The generalized surface waves in the time domain should display peculiar particle motions in the period ranges of Rayleigh-Love wave coupling.

7. Conclusions

The generalized representation of the equations of motion for a generally anisotropic medium has been presented in a compact vector form. The merit of the generalized representation is that it automatically incorporates Rayleigh-Love wave coupling and can easily be written in computer code for forward modeling of dispersion of the generalized surface waves. It is obvious that seismological information can be retrieved by inverse approaches only within the bounds of our ability of the forward modeling of seismic wave propagation. We believe that this new representation will play an important role in the future development of studies of the structure of Earth’s deep interior.

The basic features of Rayleigh-Love wave coupling are as follows when it does occur. Once Rayleigh-Love wave coupling takes place between a pair of nearby modes with a difference of phase velocities smaller than about 1 percent or 0.05 km/s, the first-order perturbation theories break down. Rayleigh-Love wave coupling sometimes drastically affects group velocities in particular at shorter periods, where the assumption of transverse isotropy is no longer valid. In other words, the assumption of transverse isotropy for surface wave inversions is valid only if Rayleigh-Love wave coupling does not take place. If Rayleigh-Love wave coupling does not take place, surface wave
velocities can be predicted well by the first-order perturbation theory.

Another expression of Rayleigh-Love wave coupling is twisting of polarizations along a pair of dispersion curves or azimuthal-branches, where particle motion polarizations are oblique between isotropic Rayleigh and Love waves. In an azimuthally anisotropic medium, a single dispersion branch does not uniquely signify a single polarization of Rayleigh- or Love-types as in the case of isotropic surface waves.

A transverse component $U_t$ of a group velocity is in general up to 0.3 km/s, implying that possible magnitudes of deviations of group velocity directions is up to $4^\circ$ from propagation directions of corresponding plane surface waves.

We would like to emphasize that the anisotropy effects are a 0-th order phenomenon if the effects due to lateral heterogeneity are the 1-st order phenomena (Mochizuki, 1986).

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REFERENCES


APPENDIX

1. Partial Derivatives

In this section, we derive partial derivatives in arbitrary orthogonal curvilinear coordinates. First, we obtain integral identities, following the context of Takeuchi and Saito (1972). Taking a complex conjugate and transpose of (20) yields

$$D_3 \bar{w}_A = -\bar{w}_A \bar{G}_3 + \bar{w}^B (B^{23}) - \sum_{k=1}^2 \left[ (\bar{w}_A G_k B^{k3}) + (D_k \bar{w}_A)(B^{k3}) \right]. \quad (A.1)$$

Using the symmetry relations, $\bar{G}_k = -G_k$, $\bar{B}^{kj} = B^{kj}$ and $\bar{B}^{23} = B^{23}$, (A.1) is rewritten as

$$D_3 \bar{w}_A = \bar{w}_A G_3 + \bar{w}^B (B^{23}) - \sum_{k=1}^2 \left[ -\bar{w}_A G_k B^{k3} + (D_k \bar{w}_A)(B^{k3}) \right]. \quad (A.2)$$

Multiplying (A.2) by $w^B$ from the right-hand side of the matrices gives

$$(D_3 \bar{w}_A) w^B = \bar{w}_A G_3 w^B + \bar{w}^B (B^{23}) w^B - \sum_{k=1}^2 \left[ -\bar{w}_A (G_k + D_k)(B^{k3}) w^B + (D_k \bar{w}_A)(B^{k3} w^B) \right]. \quad (A.3)$$

Multiplying (22) by $\bar{w}_A$ from the left-hand side of the matrices results in

$$\bar{w}_A (D_3 w^B) = -\rho \omega^2 \bar{w}_A w^A - \bar{w}_A (-\bar{G}_3 + J_3) w^B - \sum_{j=1}^2 \sum_{k=1}^2 \bar{w}_A (-G_j + J_j + D_j)(B^{kj})(G_k + D_k) w_A \quad (A.4)$$

$$- \sum_{j=1}^2 \bar{w}_A (-G_j + J_j + D_j)(B^{j3}) w^B.$$

From (A.3) and (A.4), we obtain

$$D_3 (\bar{w}_A w^B) = (D_3 \bar{w}_A) w^B + \bar{w}_A (D_3 w^B)$$

$$= -\rho \omega^2 \bar{w}_A w^A - \sum_{j=1}^2 \sum_{k=1}^2 \bar{w}_A (-G_j + J_j + D_j)(B^{kj})(G_k + D_k) w_A \quad (A.5)$$

$$- \sum_{k=1}^2 (D_k \bar{w}_A B^{k3} w^B) - J_i \bar{w}_A (B^{i3}) w^B + \bar{w}^B (B^{23}) w^B.$$

From the boundary conditions of free stresses ($w^B = 0$) at the surface and the non-singularity ($w_A = 0$) at the center of the Earth or infinity, the depth-integration of the left-hand side of (A.5) vanishes. We, then, have the following equality

$$\omega^2 I_1 = I_2,$$  \quad (A.6)

where
When the Earth is transversely isotropic, the variables can be separated. Omitting a common factor, exp\[-i(\sqrt{\theta}t-k_1x_1-k_2x_2)\], for surface waves in the Cartesian coordinates and common spherical harmonic functions, [\partial Y^m_l/\partial \theta, (1/\sin\theta)\partial Y^m_l/\partial \phi, Y^m_l] for Earth's free oscillations in the spherical polar coordinates, Eqs. (A.7) and (A.8) reduce to (169) for Love waves and (174) for Rayleigh waves, and to (178) for torsional oscillations of Takeuchi and Saito (1972). However, (A.8) does not reduce to (180) of Takeuchi and Saito (1972) for spheroidal oscillations. Since their equation (180) is non-symmetric with respect to the three components of displacements, we prefer our equation (A.8) for its symmetry.

We proceed to obtain partial derivatives. Perturbing (20) and (22), taking the complex conjugate of (20), and multiplying (20) by \(w^B\) from the right-hand side and (22) by \(\bar{w}_A\) from left-hand side of the matrices, we have

\[
(D_3 \delta \bar{w}_A)w^B = \delta \bar{w}_A \bar{G}_3 w^B + \delta \bar{w}^B B^{23} w^B + \bar{w}_A \delta B^{23} w^B
\]

\[
+ \sum_{k=1}^2 \left[ \delta \bar{w}_A (G_k + D_k)(B^{k3})w^B + \bar{w}_A (G_k + D_k)(\delta B^{k3})w^B \right] - D_k \delta \bar{w}_A (B^{k3})w^B - D_k \bar{w}_A (\delta B^{k3})w^B
\]

\[
\bar{w}_A (D_3 \delta w^B) = -\delta \rho \omega^2 \bar{w}_A w^A - 2\rho \omega \delta \omega \bar{w}_A w^A - \rho \omega^2 \bar{w}_A \delta w^A
\]

\[
- \sum_{j=1}^2 \sum_{k=1}^2 \bar{w}_A (-G_j + J_j + D_j)(\delta B^j)(G_k + D_k)w_A
\]

\[
- \sum_{j=1}^2 \sum_{k=1}^2 \bar{w}_A (-G_j + J_j + D_j)(B^{j3})(G_k + D_k)\delta w_A
\]

\[
- \bar{w}_A (-G_3 + J_3) \delta w^B - \sum_{j=1}^2 \bar{w}_A (-G_j + J_j + D_j)(\delta B^{3j})w^B
\]

\[
- \sum_{j=1}^2 \bar{w}_A (-G_j + J_j + D_j)(B^{3j})\delta w^B
\]

respectively. Taking a complex conjugate of (20) and multiplying by \(\delta w^B\) from the right-hand sides of the matrices results in

\[
(D_3 \bar{w}_A) \delta w^B = \bar{w}_A \bar{G}_3 \delta w^B + \bar{w}^B (B^{23}) \delta w^B
\]

\[
+ \sum_{k=1}^2 \left[ \bar{w}_A (G_k + D_k)(B^{k3})\delta w^B - (D_k \bar{w}_A B^{k3} \delta w^B) \right].
\]
Multiplying (22) by $\delta \tilde{w}_A$ from the left-hand side yields
\begin{equation}
\delta \tilde{w}_A(D_3 w^B) = -\rho \omega^2 \delta \tilde{w}_A w^A + \sum_{j=1}^{2} \sum_{k=1}^{2} \delta \tilde{w}_A(-\tilde{G}_j + J_j + D_j)(B^{jk})(G_k + D_k)w_A \tag{A.11}
\end{equation}

\begin{equation}
-\sum_{j=1}^{2} \delta \tilde{w}_A(-\tilde{G}_j + J_j + D_j)(B^{j3})w^B - (-\tilde{G}_3 + J_3)\delta \tilde{w}_A w^B.
\end{equation}

Adding (A.9) to (A.11), subtracting (A.6) and (A.10) and canceling out perturbations of eigenfunctions, the following equation is obtained:
\begin{equation}
D_3(\delta \tilde{w}_A w^B - \tilde{w}_A \delta w^B) = (D_3 \delta \tilde{w}_A)w^B + \delta \tilde{w}_A(D_3 w^B) - (D_3 \tilde{w}_A)\delta w^B - \tilde{w}_A(D_3 \delta w^B)
\end{equation}

\begin{equation}
= \delta \rho \omega^2 \tilde{w}_A w^A + 2\rho \omega \delta \omega w_A w^A + \sum_{k=1}^{2} (D_k \tilde{w}_A) \delta B^{k3}w^B \tag{A.12}
\end{equation}

\begin{equation}
-\tilde{w}_B(\delta B^{23})w^B + \sum_{j=1}^{2} \tilde{w}_A J_j(\delta B^{j3})w^B
\end{equation}

\begin{equation}
+ \sum_{j=1}^{2} \sum_{k=1}^{2} \tilde{w}_A(-\tilde{G}_j + J_j + D_j)(\delta B^{jk})(G_k + D_k)w_A.
\end{equation}

Integrating (A.12) with the same boundary condition that was used to derive (A.7) and (A.8), we have
\begin{equation}
-\omega^2 \int \tilde{w}_A \delta \rho w^A dV - 2\omega \delta \omega \int \tilde{w}_A \rho w^A dV = \int \left[ \sum_{k=1}^{2} (D_k \tilde{w}_A) \delta B^{k3}w^B - \tilde{w}_B(\delta B^{23})w^B \right]
\end{equation}

\begin{equation}
+ \sum_{j=1}^{2} \tilde{w}_A J_j(\delta B^{j3})w^B + \sum_{j=1}^{2} \sum_{k=1}^{2} \tilde{w}_A(-\tilde{G}_j + J_j + D_j)(\delta B^{jk})(G_k + D_k)w_A]dV. \tag{A.13}
\end{equation}

Partial derivatives are thus formed such as
\begin{equation}
I_o \frac{\partial \omega}{\partial \rho} = -\omega^2 \tilde{w}_A w^A,
\end{equation}

\begin{equation}
I_o \frac{\partial \omega}{\partial \alpha} = -\sum_{k=1}^{2} D_k \tilde{w}_A(\frac{\partial B^{k3}}{\partial \alpha})w^B + \tilde{w}_B(\frac{\partial B^{23}}{\partial \alpha})w^B - \sum_{j=1}^{2} \tilde{w}_A J_j(\frac{\partial B^{j3}}{\partial \alpha})w^B \tag{A.14}
\end{equation}

\begin{equation}
- \sum_{j=1}^{2} \sum_{k=1}^{2} \tilde{w}_A(-\tilde{G}_j + J_j + D_j)(\frac{\partial B^{jk}}{\partial \alpha})(G_k + D_k)w_A,
\end{equation}

where $I_o = 2\omega I_1$.

By operating a similar procedure directly to (18), an alternative expression to (A.13) can be obtained as
\begin{equation}
I_o \delta \omega = \int [\delta \tilde{w}_A \delta \rho w^A]dV.
\end{equation}

An alternate expression for (A.14) is as
This is quite similar to those for Earth's free oscillations. This simple expression for partial derivatives of surface wave dispersion could be more useful for inversions than (A.14) and conventional expressions of Takeuchi and Saito (1972), due to its linearity in the perturbation of elastic moduli. Expressions for partial derivatives in this section are valid even if variables are not separated as in the case of a laterally heterogeneous medium.

2. Rotation of the Coordinate Axes for Elastic Moduli

In this section, we rotate coordinate axes to derive azimuthal dependence of elastic moduli in Cartesian coordinates. Following the notational convention in Cartesian coordinates, all indices indicating covariant- or contravariant-types are written as subscripts.

Rotation of elastic tensors with respect to a vertical x₃-axis is expressed (e.g., Fung, 1965) as

\[
\lambda_{pqrs}(\phi) = \Theta_{pj} \Theta_{qk} \Theta_{rn} \Theta_{sn} \lambda_{jkmn},
\]

where

\[
\Theta = \begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

\(\phi\) is an azimuth measured from the x₁- to x₂-axes in the horizontal plane (Fig. 1). Brackets denote elastic quantities in the new coordinate system rotated.

Hereafter, the medium is assumed to be orthorhombic without a loss of generality with x₁-x₂, x₂-x₃, and x₁-x₃ planes of symmetry for simplicity. In this case, there are nine independent elastic moduli of \(\lambda_{1111}, \lambda_{2222}, \lambda_{3333}, \lambda_{1122}, \lambda_{1133}, \lambda_{2233}, \lambda_{1212}, \lambda_{1313},\) and \(\lambda_{2323},\) for which we use scalar expression \(C_{pq}\) of Love (1944): \(C_{11} = \lambda_{1111},\) \(C_{22} = \lambda_{2222}, C_{33} = \lambda_{3333}, C_{44} = \lambda_{2323}, C_{55} = \lambda_{1313}, C_{66} = \lambda_{1212}, C_{12} = \lambda_{1122}, C_{13} = \lambda_{1133},\) and \(C_{23} = \lambda_{2233}.\) Rewriting (A.16) in a scalar form, we have \([C_{jk}]\) in the new coordinate system as

\[
\begin{align*}
[C_{11}] &= A + (C_{11} - C_{22}) \cos(2\phi)/2 + P_x \cos(4\phi), \\
[C_{22}] &= A - (C_{11} - C_{22}) \cos(2\phi)/2 + P_x \cos(4\phi), \\
[C_{33}] &= C, \\
[C_{12}] &= A - 2N - P_x \cos(4\phi), \\
[C_{13}] &= F + (C_{13} - C_{23}) \cos(2\phi)/2, \\
[C_{23}] &= F - (C_{13} - C_{23}) \cos(2\phi)/2, \\
[C_{44}] &= L + (C_{44} - C_{55}) \cos(2\phi)/2,
\end{align*}
\]

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\[
\begin{align*}
[C_{55}] &= L - (C_{44} - C_{55}) \cos(2\phi) / 2, \\
[C_{66}] &= N - P_z \cos(4\phi), \\
[C_{16}] &= -(C_{11} - C_{22}) \sin(2\phi) / 4 - P_z \sin(4\phi), \\
[C_{26}] &= -(C_{11} - C_{22}) \sin(2\phi) / 4 + P_z \sin(4\phi), \\
[C_{36}] &= -(C_{13} - C_{23}) \sin(2\phi) / 2, \\
[C_{45}] &= (C_{44} - C_{55}) \sin(2\phi) / 2, \\
[C_{jk}] &= 0, \text{ otherwise,}
\end{align*}
\]

(A.18)

where

\[
P_z = \frac{(C_{11} + C_{22} - 2C_{12} - 4C_{66})}{8}, \quad A = \frac{(C_{11} + C_{22})}{2} - P_z, \quad C = C_{33},
\]

\[
F = \frac{(C_{13} + C_{23})}{2}, \quad L = \frac{(C_{44} + C_{55})}{2}, \quad N = C_{66} + P_z, \quad A - 2N = C_{12} + P_z.
\]

(A.19)

Azimuthally averaging \([C_{jk}]\) in (A.18) results in five independent elastic moduli \(A, C, F, N,\) and \(L\) in (A.19) for a transversely isotropic medium with a vertical \(x_3\)-axis of symmetry, which were first introduced by Love (1944). Similar expressions to (A.18) and (A.19) were previously presented by many authors (e.g., Kumazawa, 1964).

Smith and Dahlen (1973) showed that first-order perturbations of azimuthal variations of phase velocities of surface waves can well be approximated by only \(2\phi\) and \(4\phi\) terms. Substituting (A.18) into (28)-(34) and (A.15) lead straightforward to similar first-order perturbations of phase velocities to Smith and Dahlen (1973) and eigenfunctions to Montagner and Nataf (1986), respectively.

Backus (1965) also derived quite similar representations to \([C_{11}], [C_{55}],\) and \([C_{66}]\) in (A.18). These elastic moduli can be directly connected with three body waves of \(qP, qSV,\) and \(qSH\) in an anisotropic medium with mutually orthogonal planes of particle motion directions (Keith and Crampin, 1977). These three body waves correspond to \(P, SV,\) and \(SH\) waves, respectively, in an isotropic medium.

One important consequence from (A.18) and (A.19) is that all terms related to a \(2\phi\) variation are canceled out by azimuthal averaging. Seismological information associated with the \(2\phi\) variation cannot be retrieved by a transversely isotropic inversion. On the other hand, information of a \(4\phi\) variation of azimuthal anisotropy, represented by \(P_z,\) is partly preserved in the azimuthally averaged elastic moduli.

To a first-order perturbation, propagation velocities of Love-type generalized surface waves are principally controlled by \(C_{66}\). The small azimuthal anisotropy of Love waves (Nishimura and Forsyth, 1985) is thus equivalent to observation that the \(Pn\) velocity anisotropy obtained by Ocean Bottom Seismometer (OBS) experiments in the northwest Pacific Ocean was modeled principally by \(2\phi\) terms (Shimamura et al., 1983).

3. An Azimuthal Variation of Equations of Motion

To show an example of the effects of azimuthal anisotropy, we rewrite the equations (24) and (25) for an orthorhombic medium with \(x_1-x_2, x_2-x_3\) and \(x_1-x_3\) planes of symmetry as in the previous section. If more general anisotropy is introduced, the following algebraic expression (A.20) becomes very complicated.
In the case of the orthorhombic symmetry, substituting (A.18) into (12) to obtain matrices $L_{jk}$ and then into (19), we have

$$B_{23} = \frac{1}{C C_{44} C_{55}} \begin{pmatrix} C[C_{44}] & -C[C_{45}] & 0 \\ -C[C_{45}] & C[C_{55}] & 0 \\ 0 & 0 & C_{44} C_{55} \end{pmatrix}$$

$$B_{13} = \begin{pmatrix} 0 & 0 & [C_{13}] / C \\ 0 & 0 & [C_{36}] / C \\ 1 & 0 & 0 \end{pmatrix}$$

$$B_{23} = \begin{pmatrix} 0 & 0 & [C_{36}] / C \\ 0 & 0 & [C_{23}] / C \\ 0 & 1 & 0 \end{pmatrix}$$

(A.20)

$$B_{11} = \begin{pmatrix} [C_{11}] - [C_{13}]^2 / C & [C_{26}] - [C_{23}] [C_{36}] / C & 0 \\ [C_{16}] - [C_{13}] [C_{16}] / C & [C_{66}] - [C_{36}]^2 / C & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_{22} = \begin{pmatrix} [C_{66}] - [C_{36}]^2 / C & [C_{26}] - [C_{23}] [C_{36}] / C & 0 \\ [C_{26}] - [C_{23}] [C_{36}] / C & [C_{22}] - [C_{32}]^2 / C & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_{12} = \begin{pmatrix} [C_{16}] - [C_{13}] [C_{36}] / C & [C_{12}] - [C_{13}] [C_{32}] / C & 0 \\ [C_{66}] - [C_{36}]^2 / C & [C_{62}] - [C_{32}] [C_{36}] / C & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

A combined matrix expression of Eqs. (24) and (25) is thus

$$\frac{d}{dz} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} 0 & 1/C & kF/C & 0 & 0 & 0 \\ -\rho \omega^2 & 0 & 0 & -k & 0 & 0 \\ -k & 0 & 0 & \gamma_2 & 0 & 0 \\ 0 & -kF/C & -\rho \omega^2 + k^2 \gamma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_2 \\ 0 & 0 & 0 & 0 & -\rho \omega^2 + k^2 \gamma & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{pmatrix}$$
where

\[ \gamma_1 = A - F/C^2, \quad \gamma_2 = L/(C_{44}C_{55}), \]

\[ P_{M13} = -P_{M42} = k\Delta_2 \cos(2\phi), \quad P_{M15} = -P_{M62} = -k\Delta_2 \sin(2\phi), \]

\[ P_{M34} = -P_{M56} = \Delta_3 \cos(2\phi), \quad P_{M36} = P_{M54} = -\Delta_3 \sin(2\phi), \]

\[ P_{M43}/k^2 = -\Delta_4 + [\Delta_1 - 2F\Delta_2] \cos(2\phi) + (P_z - \Delta_4) \cos(4\phi), \]

\[ P_{M45}/k^2 = P_{M63}/k^2 = \Delta_5 \sin(2\phi) + \Delta_6 \sin(4\phi) + P_z \Delta_2 \sin(6\phi)/2, \]

\[ P_{M65}/k^2 = -\Delta_4 - (P_z - \Delta_4) \cos(4\phi) \]

and

\[ \Delta_1 = (C_{11} - C_{22})/2, \quad \Delta_2 = (C_{13} - C_{23})/(2C), \]

\[ \Delta_3 = (C_{44} - C_{55})/(2C_{44}C_{55}), \quad \Delta_4 = CD_2^2/2, \]

\[ \Delta_5 = (-\Delta_1 + F\Delta_1/C + P_z \Delta_2)/2, \quad \Delta_6 = (-P_z + FP_z/C + \Delta_1 \Delta_2/4). \]

The first term in the right-hand side of (A.21) is the same as a combined matrix representation of (34) for Rayleigh and (36) for Love waves of Takeuchi and Saito (1972) except \( \gamma_2 \). If a medium is transversely isotropic, \( A = C_{11} = C_{22}, F = C_{13} = C_{23} \) and \( L = C_{44} = C_{55} \). The second term vanishes and (A.21) thus decouples into (34) for Rayleigh and (36) for Love waves of Takeuchi and Saito (1972).

Notation List

The general stipulation for notations in this paper is as follows. A tilde indicates a quantity in the time domain. Quantities without the tilde are always Fourier-transformed throughout this paper. Vectors and matrices are written in bold face type. A superscript and a subscript of a tensor denote contravariant- and covariant-types, respectively. An overbar indicates a Hermitian conjugate. An asterisk * indicates a transpose.

Tensors that are bracketed are those rotated by an angle \( \phi \) with a vertical axis to the Earth's surface.

The explanation for each symbol is as follows:

- \( A, \lambda^{1111} \) for a transversely isotropic medium;
- \( B^a \), coefficient matrix in the equations of motion;
- \( C, \lambda^{3333} \) for a transversely isotropic medium;
- \( c \), phase velocity;
- \( D_0 = \partial / \partial X^a \), derivative by a tensor component of an \( X^a \) coordinate;
- \( D_0 = (1/\gamma_p) \partial / \partial X^a \), derivative by a physical component of an \( X^a \) coordinate;
- \( E_{pa} \), tensor component of elastic strain;
- \( e_{pa} \), physical component of elastic strain;

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$F, \lambda^{1233} = \lambda^{2333}$ for a transversely isotropic medium; $G_N$, $N$-th higher-mode generalized surface waves;

$G_p$, coefficient matrix of covariant differentials, an $st$-component of which is $-\Gamma_{st}$;

$G_p = -(1/g_p)(Rz^{-1}G_{pq}Rz + Rz^{-1}(D_p Q));$

$\gamma_{pq}$, covariant-type metric matrix;

$g_{pq}$, $p$-th diagonal component of the covariant metric matrix in the orthogonal curvilinear coordinate system;

$H$, $z$-coordinate of Earth's free surface; $H_s$, $z$-coordinate of the top of a solid part of the Earth;

$I_p$, energy integrals;

$L_0 = \lambda^{1233}$ for a transversely isotropic medium;

$L_p = \lambda^{1233}$ for a transversely isotropic medium;

$L_p^0$, coefficient matrix of tensor component of elastic moduli in a vector expression of the equations of motion, an $st$-th component of which is $A_{st}^p$;

$L_p^0$, coefficient matrix of physical component of elastic moduli in a vector expression of the equations of motion, an $st$-th component of which is $A_{st}^p$;

$L_{ps}$, matrix elements representing effects of the Rayleigh-Love wave coupling;

$L_{ps}$, $m$-th higher-mode Rayleigh-type generalized surface waves;

$L_{pq}$, coefficient matrix of tensor component of elastic moduli in a vector expression of the equations of motion, an $st$-th component of which is $A_{st}^p$;

$L_{pq}$, coefficient matrix of physical component of elastic moduli in a vector expression of the equations of motion, an $st$-th component of which is $A_{st}^p$;

$N$, $\lambda^{1233}$ for a transversely isotropic medium;

$P_{jk}$, tensor component of elastic stress; $t$, time;

$U$, group velocity vector;

$U_p$, tensor component of elastic displacements; $U_p$, physical component of elastic displacements;

$U_t$, radial component of group velocity; $U_t$, transverse component of group velocity;

$\Omega_{at}$, covariant-type displacement vector of tensor components;

$\Omega_{at}$, contravariant-type displacement vector of tensor components;

\( \gamma_{pq} \), tensor component of elastic stress;

$w_A$, covariant-type displacement vector of physical components;

$w_A$, contravariant-type displacement vector of physical components;

$w_B$, covariant-type stress vector of tensor components;

$w_B$, contravariant-type stress vector of physical components;

$Y_p$, eigenfunction for generalized surface waves in a plane-stratified medium;

$Y_p = [iY_3, iY_5, Y_1]^*$; $Y_p = [iY_3, iY_5, Y_1]^*$;

$\gamma_{pq}$, eigenfunction for isotropic Rayleigh waves; $\gamma_{pq}$, eigenfunction for isotropic Love waves;

$\Theta$, matrix of rotation of the coordinate axes;

$\Delta$, deviation of elastic moduli from those of transversely isotropic Earth model;

$\phi$, azimuth measured from the $x_1$-axis; $\Gamma_{st}$, Christoffel symbol;

$A_{st}^p$, tensor component of an elastic modulus; $A_{st}^p$, physical component of an elastic modulus;

$[\lambda_{ps}]$, elastic moduli in the Cartesian coordinates rotated by $\phi$ around a vertical axis;

$\rho$, density; $\omega$, an angular frequency; $\nu$, covariant differential.