Scattering of P, SV Waves by Random Distribution of Aligned Open Cracks

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In our previous paper (Kawahara and Yamashita, 1992; hereafter referred to as Paper I), we theoretically studied the scattering of P, SV, and SH waves by a zonal distribution of aligned cracks, which simulates a fault fracture zone. It was investigated how the geometrical properties of the crack distribution and the frictional characteristics of the crack surface are reflected in the attenuation and dispersion of the incident waves due to the scattering. The boundary condition on the crack surface was assumed as follows:

(A1) the shear stress vanishes on the surface, or the viscous friction of Newtonian type acts on it, and

(A2) the opening displacement of the crack is negligibly small. Under the high confining pressure in the earth's crust, the cracks will be closed and the opening displacement will be suppressed, though the fluid inside the crack could keep the surface from adhesion and make the shear displacement possible. This is why we chose this condition.

There will exist, however, open cracks—cracks with small but finite aspect ratios—under the environment with very low confining pressure. Such cracks may be of interest from the viewpoint of the mechanics of shallow earthquakes and material science. Some experimental works (Brace and Bombalakis, 1963; Cox and Scholz, 1988) indicate that a typical shear crack does not grow in its own plane but generates secondary tensile cracks around its edge; the generation of such tensile, or open, cracks is thought to occur in the first stage of the fault formation in the upper part of the earth's crust (Cox and Scholz, 1988). The macroscopic dilatancy preceding the catastrophic fracture of compressed rock is also interpreted as being due to the development of pervasive open microcracking within the rock (Brace *et al.*, 1966). In the field of engineering, it is a crucial problem to preclude the catastrophic fracture of structural materials which is controlled by the behavior of preexisting flaws—such as microcracks, open as well as shear, induced by metal fatigue. The nondestructive testing for detecting such flaws is highly important; one of the most effective ways is to make use of the scattered waves due to the flaws (Thompson, 1983). In this paper, we theoretically study the scattering of P, SV waves by random distribution of aligned open cracks. For this, we suppose that

(B) both of the shear and normal components of stress vanish on the surface, instead of Conditions (A).

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Fig. 1. Geometry of the random distribution of cracks. Symbol u^0 denotes the P (or SV) wave incident on the cracked region ($X_2 > 0$) at an angle ϕ (or θ).

We consider the two-dimensional problem (Fig. 1). We assume that aligned cracks with the same length, $2a$, are randomly distributed within the upper half plane ($X_2 > 0$). The crack orientation is assumed to coincide with the X_1 -axis. The crack distribution is assumed to be uniform and its number density, ν , is assumed to be sufficiently small. On the cracked region, harmonic plane elastic (P or SV) waves are assumed to be incident at arbitrary angles. No fracture is assumed to be excited by the incidence of elastic waves.

Now we summarize the theoretical treatment of the scattering problem, which is detailed in Paper I. We employ the idea of the mean wave formalism as we (in Paper I) and others (e.g., Hudson, 1980; Yamashita, 1990) did. We also assume Foldy's (1945) approximation: the wave effectively incident on a crack is assumed to equal the mean total wave itself. This is known to be valid when the number of the scatterers is infinitely large though their distribution is very sparse, and it will give the solution accurately to $O(\nu a^2)$ (Ishimaru, 1978; Keller, 1964). Then the basic equation which is satisfied by the mean displacement wave, $\langle u \rangle$, is given as follows (Eq. (6) in Paper I):

$$\langle u_A \rangle = u_A^0 + \nu \int_{X_2 > 0} S_A^i \langle u_i \rangle dr_i, \quad (1)$$

where u^0 is the incident wave, A and i denote the observation point and the location of a representative crack respectively, and $S_A^i \langle u_i \rangle$ is the scattered wave generated by the crack for the sake of the action of $\langle u_i \rangle$. According to the representation theorem, $S_A^i \langle u_i \rangle$ can be written as the convolution of the displacement discontinuity across the crack and the traction corresponding to the Green function. The displacement discontinuity is determined by the boundary condition and, as mentioned above, we assume here Condition (B), that is,

$$\tau_{12}^E + \tau_{12}^S = \tau_{22}^E + \tau_{22}^S = 0 \quad \text{on the crack surface}, \quad (2)$$

where τ_{jk}^E and τ_{jk}^S are the stress components caused by $\langle u_i \rangle$ and $S_A^i \langle u_i \rangle$ respectively. Equations (1) and (2) yield the solution $\langle u \rangle$ for the given incident wave.

We assume first the incident P wave in the form

$$u_A^0 = A_0 e^{ihX_1 \sin \phi + ihX_2 \cos \phi} (\sin \phi, \cos \phi). \quad (3)$$

Here the time factor $\exp(-i\omega t)$ is omitted for brevity, where $\omega = h\alpha$, h is the wavenumber and α is the P-wave velocity of the matrix. We also assume the mean wave within the cracked region as an attenuating P wave:

$$\langle u_A \rangle = A e^{i h X_1 \sin \phi + i(h \cos \phi + \kappa_P) X_2} (\sin \phi, \cos \phi + \kappa_P/h), \quad (4)$$

where $\text{Im } \kappa_P \geq 0$ is required and the symbol Im stands for the imaginary part. Although there will exist the attenuating SV wave due to the P-S conversion, it will be negligible for large X_2 . Then the unknown constants A and κ_P , the latter of which is of interest now, are determined by rewriting explicitly the right-hand side of Eq. (1) through the representation theorem and the condition (2) and comparing it with the left-hand side. Here we only show the result; we get

$$\kappa_P \simeq v \Phi_1 f h \sin 2\phi \sin \phi + v \Phi_2 \frac{h}{2f \cos \phi} (1 - 2f \sin^2 \phi)^2, \quad (5)$$

where $f = \beta^2/\alpha^2$, β is the S-wave velocity of the matrix, and

$$\Phi_j(h, \phi) = \int_{-a}^a D_j(\xi) e^{-i h \xi \sin \phi} d\xi, \quad j=1, 2. \quad (6)$$

Note that the constant f is related to Poisson's ratio s as $2f = (1-2s)/(1-s)$, and the physical requirement $0 < s < 1/2$ constrains f as $0 < f < 1/2$. The function $D_j(\xi)$, which represents the j -th component of the normalized displacement discontinuity across the crack, is determined by Condition (B); explicitly, it is

$$\int_{-a}^a D_j(\xi) T_{j2j}(x, 0 | \xi, 0) d\xi - e^{i h x \sin \phi} = 0, \quad |x| < a, \quad j=1, 2, \quad (7)$$

where the functional form of $T_{jkl}(\cdot)$ is given in Appendix, and the summation convection is *not* applied to the subscript j . This result means that the scattering effect consists of two components—the ones derived from the shear and opening displacement of the crack respectively; if the latter is neglected ($D_2 = \Phi_2 = 0$), the result coincides with that of Paper I in case of no friction. Equation (7) can be solved numerically, and the accuracy of our method to solve it has been checked in Paper I in comparison with Mal's (1970) result in case of normal ($\phi = 0^\circ$) P-wave incidence.

The incident and mean waves are assumed as

$$u_A^0 = B_0 e^{i k X_1 \sin \theta + i k X_2 \cos \theta} (\cos \theta, -\sin \theta), \quad (8)$$

$$\langle u_A \rangle = B e^{i k X_1 \sin \theta + i(k \cos \theta + \kappa_{SV}) X_2} (\cos \theta + \kappa_{SV}/k, -\sin \theta), \quad (9)$$

for the SV-wave incidence, where $k = \omega/\beta = h/\sqrt{f}$, and B and κ_{SV} are unknown constants. In this case, we obtain

$$\kappa_{SV} \simeq v \Phi_1 k \frac{\cos^2 2\theta}{2 \cos \theta} + v \Phi_2 k \sin 2\theta \sin \theta, \quad (10)$$

where Φ_j is given by Eq. (6) together with Snell's law $h \sin \phi = k \sin \theta$. This also agrees with the result of Paper I again when the crack opening is neglected.

Equations (5) and (10) give the attenuation coefficients Q^{-1} and the phase velocities v of $\langle u \rangle$ for the P- and SV-wave incidence (see Appendix D in Paper I). The values of Q^{-1} are

$$Q_{\{SV\}}^{-1} = Q_{\{SV\}}^{(1)-1} + Q_{\{SV\}}^{(2)-1}, \quad (11.1)$$

$$Q_P^{(1)-1} = v \operatorname{Im} \Phi_1 f \sin^2 2\phi, \quad (11.2)$$

$$Q_P^{(2)-1} = v \operatorname{Im} \Phi_2 f^{-1} (1 - 2f \sin^2 \phi)^2, \quad (11.3)$$

$$Q_{SV}^{(1)-1} = v \operatorname{Im} \Phi_1 \cos^2 2\theta, \quad (11.4)$$

$$Q_{SV}^{(2)-1} = v \operatorname{Im} \Phi_2 \sin^2 2\theta. \quad (11.5)$$

These equations indicate the following properties of Q^{-1} . Each attenuation coefficient consists of two components— $Q^{(1)-1}$ and $Q^{(2)-1}$, which are derived from the shear and opening displacement of the crack respectively. Both of $Q_P^{(1)-1}$ and $Q_{SV}^{(2)-1}$ vanish for the normal incidence ($\phi, \theta = 0^\circ$) and in the limit of grazing incidence ($\phi, \theta \rightarrow 90^\circ$) while $Q_{SV}^{(1)-1} = 0$ for $\theta = 45^\circ$; this is easily understandable from the characteristics of P- and SV-wave motions. Concerning $Q_P^{(2)-1}$, it decreases monotonically with ϕ and it vanishes only for $f \rightarrow 1/2$, $\phi \rightarrow 90^\circ$. Although it appears singular at $f \rightarrow 0$ at first sight, it is not because of $T_{222}(\cdot) \propto f^{-1}$ and hence $\Phi_2 \propto f$. The phase velocities are obtained by Eq. (11) if we define Δv as

$$\Delta v_{\{SV\}} = \left\{ \begin{array}{c} \alpha \\ \beta \end{array} \right\} - v_{\{SV\}} = \Delta v_{\{SV\}}^{(1)} + \Delta v_{\{SV\}}^{(2)}, \quad (12)$$

and replace " $Q_P^{(j)-1}$," " $Q_{SV}^{(j)-1}$ " and " $\operatorname{Im} \Phi_j$ " by " $\Delta v_P^{(j)}/\alpha$," " $\Delta v_{SV}^{(j)}/\beta$ " and " $\operatorname{Re} \Phi_j/2$ "

Fig. 2. (a) Dependence of Q_P^{-1} on ϕ . Curve parameters are the values of ϕ ; among them, "90°" especially means the limit $\phi \rightarrow 90^\circ$. (b) Each component of Q_P^{-1} for $\phi = 45^\circ$. Symbols "(1)," "(2)," and "(1)+(2)" denote $Q_P^{(1)-1}$, $Q_P^{(2)-1}$, and $Q_P^{-1} = Q_P^{(1)-1} + Q_P^{(2)-1}$, respectively. Note that the abscissa indicates ka , not ha , in each figure.

respectively, where the symbol Re means the real part.

Now we assume the Poisson solid ($s=1/4$, $f=1/3$) hereafter. The dependence of Q_p^{-1} on ϕ is shown in Fig. 2(a). The figure indicates that the peak wavenumber, at which Q_p^{-1} takes the peak value, is insensitive to ϕ ; it satisfies $ka \approx 1.1$. The magnitude decreases with ϕ though it does not vanish for $\phi \rightarrow 90^\circ$ because of the contribution of $Q_p^{(2)-1}$. The magnitude of $Q_p^{(2)-1}$ is always larger than that of $Q_p^{(1)-1}$ for any incidence; an example is shown in Fig. 2(b) for $\phi=45^\circ$, at which the magnitude of $Q_p^{(1)-1}$ is close to the maximum. This figure also denotes that the asymptotic relation $Q^{-1} \propto k^{-1}$ in the high wavenumber range, observed in the results of Yamashita (1990) and Paper I, is satisfied for each component. The dependence of Q_{sv}^{-1} is shown in Fig. 3(a). Its peak wavenumber as well as its peak magnitude is denoted not to be very sensitive to θ , though the peak magnitude of each component, $Q_{sv}^{(j)-1}$, strongly depends on θ , as indicated in Eq. (11). The values of the peak wavenumbers are in the range $1.1 < ka < 1.8$. The high dependence on θ is, however, observed in the high wavenumber range ($ka > 3$); as θ increases, the deviation of the slope from the asymptotic curve $Q_{sv}^{-1} \propto k^{-1}$ develops.

Fig. 3. Dependence of θ , of (a) Q_{sv}^{-1} , (b) $Q_{sv}^{(1)-1}$, and (c) $Q_{sv}^{(2)-1}$. Curve parameters are the values of θ ; among them, "90°" especially means the limit $\theta \rightarrow 90^\circ$.

Fig. 4

Fig. 5

Fig. 4. Dependence of Δv_p on ϕ . Curve parameters are the values of ϕ .

Fig. 5. Dependence of Δv_{sv} on θ . Curve parameters are the values of θ .

This deviation is derived from the both components, as shown in Fig. 3(b) and (c). Figure 3(b) indicates that, when θ is sufficiently small, the high wavenumber slope of $Q_{sv}^{(1)-1}$ satisfies the relation $Q_{sv}^{(1)-1} \propto k^{-1}$ well; it becomes less steep as θ approaches $\theta_c = \sin^{-1} \sqrt{f} = 35.26^\circ$ and, beyond θ_c , there appears irregular winding on the slope. The similar irregularity is also observed in case of $Q_{sv}^{(2)-1}$ for $\theta > 45^\circ$ (Fig. 3(c)). Concerning the phase velocities, the results for Δv_p and Δv_{sv} are shown in Figs. 4 and 5. The general tendency that Δv is insensitive to k for $ka < 1$ and decreases with k for $ka > 1$, as seen in Paper I, is again common in these cases, and the dependence on the incident angles is analogous to that of Q^{-1} . Although Δv appears to have sharp peaks in the range $ka > 1$ for some incident angles, they are just ripples exaggerated in log-scale plotting.

The characteristic ripples observed on the curves of Q^{-1} and Δv in the high wavenumber range directly reflect the fluctuation of Φ_j ; this is interpreted as the variation of the interference pattern between the diffracted fields originated from both of the crack tips, as mentioned in Paper I. The wavelengths of the ripples are regulated only by the crack length because we assume here Foldy's (1945) approximation, under which the interaction between the mean wave and an isolated representative crack is dealt with. The reason why the ripple patterns are highly complicated for some incidence will be that the interaction between the crack tips occurs through Rayleigh and body waves (Keogh, 1985); the ripples which appear in the result of Paper I for SH mode are nearly sinusoidal where the crack tips interact with each other only through SH waves. The high wavenumber irregularity observed in Fig. 3 will also reflect the multiple interaction between the crack tips.

In conclusion, the attenuation and dispersion of P, SV waves due to the scattering by open cracks are highly distinct from those by cracks which cannot open; the contribution of $Q^{(2)-1}$ and $\Delta v^{(2)}$ to Q^{-1} and Δv is significant in the case of open cracks. Hence it is necessary to know the detailed information about the boundary conditions

on the crack surface for the sake of the correct interpretation of the observed scattering phenomena. It is also shown that there appears the characteristic irregularity in the high wavenumber slopes of Q^{-1} and Δv for SV waves incident at large angles. Although the cracks with the free surface are assumed in this paper, our methodology can be straightforwardly applied to other types of cracks. For the purpose, Eq. (7) should be modified in accordance with the boundary condition though Eqs. (6), (11), and (12) remain valid throughout.

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APPENDIX. EXPRESSION FOR T_{jk}

As given in Appendix B of Paper I, it is

$$\begin{aligned}
T_{jkl}(x_1, x_2 | \xi_1, \xi_2) &= T_{jkl}^*(x_1, x_2 | \xi_1, \xi_2) \\
&+ \frac{i}{4} \left[\left(\delta_{jl} \frac{\partial^2}{\partial x_k \partial x_2} + \delta_{k2} \frac{\partial^2}{\partial x_j \partial x_1} + \delta_{kl} \frac{\partial^2}{\partial x_j \partial x_2} + \delta_{j2} \frac{\partial^2}{\partial x_k \partial x_1} \right) H_0^{(1)}(kR) \right. \\
&\left. - \frac{4}{k^2} \frac{\partial^4}{\partial x_j \partial x_k \partial x_l \partial x_2} (H_0^{(1)}(hR) - H_0^{(1)}(kR)) \right], \quad (A.1)
\end{aligned}$$

where $R^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2$, $H_0^{(1)}(\cdot)$ is the Hankel function, and

$$\begin{aligned}
T_{111}^*(x_1, x_2 | \xi_1, \xi_2) &= \frac{i}{2} (1 - 2f) \frac{\partial^2}{\partial x_1 \partial x_2} H_0^{(1)}(hR) \\
&= T_{122}^*(x_1, x_2 | \xi_1, \xi_2) = T_{212}^*(x_1, x_2 | \xi_1, \xi_2) = T_{221}^*(x_1, x_2 | \xi_1, \xi_2), \quad (A.2)
\end{aligned}$$

$$T_{112}^*(x_1, x_2 | \xi_1, \xi_2) = \frac{i}{4} \frac{1 - 2f}{f} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) H_0^{(1)}(hR), \quad (A.3)$$

$$T_{121}^*(x_1, x_2 | \xi_1, \xi_2) = T_{211}^*(x_1, x_2 | \xi_1, \xi_2) = 0, \quad (A.4)$$

$$T_{222}^*(x_1, x_2 | \xi_1, \xi_2) = \frac{i}{4} f^{-1} \left[(1 - 2f)^2 \frac{\partial^2}{\partial x_1^2} + (1 - 4f^2) \frac{\partial^2}{\partial x_2^2} \right] H_0^{(1)}(hR). \quad (A.5)$$