Tidal Deformation of the Moon

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Tides on the surface of the Moon are investigated. The tidal potential, gravity and displacements on the surface of the Moon are expanded in harmonics depending on general forms of the lunar motion, which would accept the orbital evolution. These are expressed in numerical formulas more exactly for uses in correction for precise determination of the lunar motions and for possible observations of the tides of the Moon. Love numbers are computed for some expected models of the lunar interior. Possibilities are shown to discriminate the core effect in the tidal observations.

1. Introduction

Observations with lunar laser ranging have brought novel information on the lunar motion. The free librations of the Moon were found to consist of three different periods which depend on the lunar figure (see Mulholland, 1980). Recent studies suggest the detection of effects of the lunar solid and/or core-mantle friction in the 18.6 year precession of the lunar figure (Yoder, 1990), which would discriminate the dynamical effects of the core.

In these analyses, corrections for the tides of the Moon are considerably important as pointed out by Han (1963) since the precessional motions overlap with the lunar tidal deformation in nearly monthly periods. The tidal dissipation in the Moon is also one of the keys to estimate the orbital evolution, especially on the variation of the eccentricity (Groves, 1960; Goldreich, 1963; Goldreich and Soter, 1966). If we can measure more precisely the tides of the Moon through new techniques, such as a differential VLBI or gravimetry on the Moon, the tidal dissipation as well as the core-mantle dynamics can be directly observed. In these studies, we need a precise model of the tides on the Moon.

The Moon keeps a synchronous rotation on the spin and the revolution. The tides are mainly induced by the eccentricity and the inclination of the lunar orbit and the perturbation due to the Sun. The tidal force of the Sun itself is one order smaller than that of the Earth. Up to the present many works were achieved on development of the tidal theory of the Moon. Harrison (1963) presented the theoretical expansion of the gravity tides. Cheng and Toksoz (1978) presented the variation of the tidal stresses

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estimating Love numbers depending on lunar seismic observations. Lambeck (1988) presented expressions of the radial displacements. These have well served for exploration and researches in the seismic activity in the Moon, but seem not accurate enough for studies of the tides, and for correction for a more precise observation of the libration and the deformation of the Moon. Hanada et al. (1990, 1991) showed that observations with the differential VLBI will achieve a precision of a few centimeters in determination of the lunar motion by deploying artificial radio sources on the Moon. We develop a more exact expression on the tidal displacements and the gravity variation for these observations taking a general form as far as possible to permit to follow the orbital evolution.

2. General Formulas of Tides

General expressions on the tides of the Earth are given by several authors (Bartels, 1957; Harrison, 1971; etc.). We develop tides of the second order harmonics on the Moon. Let $R$ be the distance between the gravitational center of the Moon and the Earth, and $K$ and $M$ be the gravitational constant and the mass of the Earth, respectively. We express the coordinates of a point $P$ on the surface of the Moon by $r$ (radius), $\theta$ (colatitude), and $\lambda$ (east longitude). The tidal potential of the second order harmonics on a rigid moon induced by the Earth is given by $V = KMr^2/R^3(3/2 \cos^2 z - 1/2)$, where $z$ is the zenith distance of the Earth at the point $P$. Let $H$ be the hour angle of the Earth at the point $P$. Let $\delta$ be the declination of the Earth with respect to the equator of the Moon, $g$ the surface gravity of the Moon, and $k_2, l_2,$ and $h_2$ the Love numbers of the Moon. The tides of the second order harmonics are separated into the zonal, the tesseral and the sectorial components. The radial displacement is given by $V$ multiplied by $h_2/g$. The gravity variation and the tangential displacements are given by taking derivatives of $V$ along the radius, the meridian and the colatitude-line after multiplying respective factors.

Then the zonal components of the tidal potential, displacements, and gravity variation are given as follows:

1. Potential

$$V = k_2 r^2 (3 \cos^2 \theta - 1) F_{20} ,$$

2. Gravity variation

$$\Delta g = -2G_2 r (3 \cos^2 \theta - 1) F_{20} ,$$

3. Displacements

$$u_r = h_2 (r^2/g)(3 \cos^2 \theta - 1) F_{20} ,$$
$$u_\theta = -l_2 (r^2/g) 3 \sin 2\theta F_{20} ,$$
$$u_\lambda = 0 ,$$

where $F_{20}$ and $G_2$ are given by

$$F_{20} = (KM/4R^3)(3 \sin^2 \delta - 1)$$

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The tesseral components are given by

\[ V = k_2(1/2)r^2 \sin 2\theta F_{21} \cos H, \]
\[ \Delta g = -G_2 r \sin 2\theta F_{21} \cos H, \]
\[ u_r = h_2(r^2/2g) \sin 2\theta F_{21} \cos H, \]
\[ u_\theta = l_2(r^2/g) \cos 2\theta F_{21} \cos H \]

and

\[ u_z = -l_2(r^2/g) \cos \theta F_{21} \sin H, \]

where

\[ F_{21} = (3/2)(KM/R^3) \sin 2\delta. \]

The sectorial components are

\[ V = k_2 r^2 \sin^2 \theta F_{22} \cos 2H, \]
\[ \Delta g = -2G_2 r \sin^2 \theta F_{22} \cos 2H, \]
\[ u_r = h_2(r^2/g) \sin^2 \theta F_{22} \cos 2H, \]
\[ u_\theta = l_2(r^2/g) \sin 2\theta F_{22} \cos 2H \]

and

\[ u_z = -2l_2(r^2/g) \sin \theta F_{22} \sin 2H, \]

where

\[ F_{22} = (3/4)(KM/R^3) \cos^2 \delta. \]

3. Lunar Motion and Selenographic Coordinates of the Earth

The poles of the ecliptic, the lunar orbit and the lunar equator are in the same plane if the physical libration is ignored (which is less than about 130°). The Moon has attained to the second status of Cassini's law passing through the first and the fourth status maybe at the early stage of the evolution (Ward, 1975). Thus at present, the lunar equator intersects the lunar orbit at the ascending node. We define

- \( \Omega \): mean longitude of the ascending node counted from the mean vernal equinox of date,
- \( M' \): mean anomaly,
- \( \Gamma \): mean longitude of the perigee,
- \( l \): mean longitude of the Moon; \( e \): eccentricity,
- \( \beta \): ecliptic latitude of the Moon
- \( l' \): true longitude of the Moon
Fig. 1. Geocentric coordinate of the lunar orbit and equator.

\( a \): semi-major axis of the lunar orbit around the gravitational center of the Earth and the Moon,
\( i \): inclination of the lunar orbit with respect to the ecliptic,
\( \varepsilon \): inclination of the lunar equator with respect to the ecliptic,
\( L \): mean longitude of the Sun,
\( m \): ratio of the mean motions of the Sun and the Moon.

The true longitude of the Moon can be expressed in a general form as

\[
I' = I + 2e \sin M' + (5/4)e^2 \sin 2M' + (15/4)me \sin(2D - M') + (11/8)m^2 \sin 2D,
\]
adding the perturbation terms due to the Sun where \( D \) is the difference of \( I \) and the mean longitude of the Sun. The latitude of the Moon is given by

\[
\tan \beta = s \sin(I - \Omega) - se \sin(2I - \Omega) - se \sin(I' - \Omega) + (3/8)sm \sin\{2(l - L) - (l - \Omega)\},
\]
with

\[
s = \tan i.
\]

The distance \( R \) is given in general by

\[
a/R = \cos \beta \{(1 - (1/2)m^2) - e^2 - (3/4)e^2\} + e \cos M' + e^2 \cos 2M' - (1/4)s^2 \cos(2I - 2\Omega) + (15/8)me \cos(l - 2L + \Gamma) + m^2 \cos(2l - 2L)\].
\]

We turn the coordinate system to a lunar-central. This transformation needs to add 180° to the mean longitude of the ascending and the argument of the perigee. Thus we easily know that "a mean longitude" of the Earth with respect to the lunar-central coordinates is still equal to \( I \). In section 2, we defined the selenographic declination (latitude) of the Earth. Next we have to define the selenographic right-ascension (longitude) of the Earth, letting it be counted along the lunar equator from the intersection of the lunar equator and the ecliptic. Let \( \alpha \) be thus defined right-ascension of the Earth. These are expressed in terms of the inclinations, the mean longitude of the Moon etc. after omitting smaller terms as follows:

\[
\alpha = I' - \Omega - (1/4)e(\varepsilon + 2I) \sin(2I' - 2\Omega) + O[\{s(\varepsilon + 2\Omega)\}^3],
\]
The hour angle of the Earth at the point P of the Moon is

\[ H = (\sigma T + \text{const.}) - \Omega + \lambda - \alpha, \]

where \( \sigma \) is the spin velocity of the Moon, \( \lambda \) the east longitude of the point P as defined previously. Here, \((\sigma T + \text{const.})\) is nearly equal to \( l \) because of the synchronous rotation of the Moon. In this case \( H \) is expressed approximately by

\[ H = l + \lambda - l' + (1/4)\varepsilon (\varepsilon + 2l) \sin(2l' - 2\Omega). \]

The above formulas can follow the change of the lunar orbit brought by the evolution, as far as the Moon stays at the second status of the Cassini’s law. Hence we can compute the tides of the Moon inserting Eqs. (22), (24), and (26) into Eqs. (1) to (19). These formulas are enough to estimate the tidal effects in the evolution of the orbit.

These might, however, be not accurate enough for observations of the Moon because of simplification in the perturbation terms due to the Sun. Hence, we have to adopt numerical coefficients presented by Brown (1910) for the expression of the present tides of the Moon. Those are

\[ l' = l + 6^\circ.289 \sin M' + 0^\circ.214 \sin 2M' + 1^\circ.274 \sin (2D - M') + 0^\circ.658 \sin 2D + \cdots, \]

\[ \beta = 5^\circ.13 \sin(l - \Omega) + 0^\circ.28 \sin(2l - \Gamma - \Omega) - 0^\circ.28 \sin(\Gamma - \Omega) \]

\[ + 0.17 \sin\{2(l - L) - l + \Omega\} + \cdots \]

and

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Differences appear in the terms which include $D$ or $L$ in the arguments.

4. Numerical Results

The orbital elements of the present day are given as

$$e = 0.0549, \quad \varepsilon = 1^\circ 32'33'', \quad i = 5^\circ 7'47''.41, \quad a = 3.844 \times 10^5 \text{ km}.$$ 

These lead us to

$$H = \lambda - 6^\circ.289 \sin M' - 1^\circ.274 \sin(2D - M') - 0^\circ.658 \sin 2D - 0^\circ.214 \sin 2M' + 0^\circ.080 \sin(2I - 2\Omega) + \cdots$$

and

$$\sin \delta = -0.1164 \sin(l - \Omega) - 0.0063 \sin(2I - \Gamma - \Omega) + 0.0063 \sin(\Gamma - \Omega) + \cdots.$$ 

Inserting (29), (30), and (31) into (6), (13), and (19), we have

$$F_{20} = 1.754 \times 10^{-12}\{ -0.9845 - 0.1602 \cos M' - 0.01314 \cos 2M' - 0.0295 \cos(M' - 2L + 2\Gamma) - 0.0243 \cos(2I - 2L) - 0.0200 \cos(2I - 2\Omega) - 0.0022 \cos(3M' - 2\Gamma + 2\Omega) + 0.0022 \cos(M' + 2\Gamma - 2\Omega) \},$$

and

$$F_{21} = -2.105 \times 10^{-11} \{ 0.1156 \sin(l - \Omega) + 0.0157 \sin(2I - \Gamma - \Omega) + 0.0031 \sin(l - M' - \Omega) + 0.0063 \sin(M' - l + \Omega) + 0.0094 \sin(l - \Omega - M') \}.$$ 

and

$$F_{22} = 5.263 \times 10^{-12}\{ 0.9978 + 0.1624 \cos M' + 0.0133 \cos 2M' + 0.0299 \cos(M' - 2L + 2\Gamma) + 0.0246 \cos(2I - 2L) + 0.0067 \cos(2I - 2\Omega) + 0.0007 \cos(3I - \Gamma - 2\Omega) - 0.0007 \cos(l + \Gamma - 2\Omega) \}.$$ 

Thus the zonal component of the gravity variation at the point P is, in the unit of $\mu$gal,

$$\Delta g = G_2(3/2) \cos^2 \theta - (1/2) \{ 1200.7 + 195.4 \cos M' + 16.0 \cos 2M' + 36.0 \cos(M' - 2L + 2\Gamma) + 29.6 \cos(2I - 2L) + 24.4 \cos(2I - 2\Omega) \}.$$ 

The displacements are in the unit of cm as follows:
The tesseral components in the same units used for the zonal component are as follows:

\[ \Delta g = G_2 \sin 2\theta \left[ \cos \lambda \left( 422.8 \sin (l - \Omega) + 56.2 \sin (2l - \Gamma - \Omega) + 11.3 \sin (\Gamma - \Omega) \right) \right. \]
\[ + \sin \lambda \left( -23.2 \cos (2l - \Gamma - \Omega) + 23.2 \cos (\Gamma - \Omega) - 3.1 \cos (2l + M' - \Gamma - \Omega) + 3.1 \cos (l - 2\Gamma - \Omega) \right) \right] , \tag{38} \]

\[ u_a = h_2 \sin 2\theta \left[ - \cos \lambda \left( 226.4 \sin (l - \Omega) + 30.8 \sin (2l - \Gamma - \Omega) + 6.1 \sin (\Gamma - \Omega) \right) \right. \]
\[ + \sin \lambda \left( 12.4 \cos (2l - \Gamma - \Omega) - 12.4 \cos (\Gamma - \Omega) \right) \right] , \tag{39} \]

\[ u_\theta = l_2 \cos 2\theta \left[ - \cos \lambda \left( 452.9 \sin (l - \Omega) + 61.5 \sin (2l - \Gamma - \Omega) + 12.1 \sin (l - M' - \Omega) \right) \right. \]
\[ + \sin \lambda \left( 24.7 \cos (2l - \Gamma - \Omega) - 24.7 \cos (\Gamma - \Omega) \right) \right] \tag{40} \]

and

\[ u_\lambda = l_2 \cos \theta \left[ \sin \lambda \left( 452.9 \sin (l - \Omega) + 61.5 \sin (2l - \Gamma - \Omega) + 12.1 \sin (l - M' - \Omega) \right) \right. \]
\[ + \cos \lambda \left( 24.7 \cos (2l - \Gamma - \Omega) - 24.7 \cos (\Gamma - \Omega) \right) \right] . \tag{41} \]

The sectorial components are as follows:

\[ \Delta g = -G_2 \sin^2 \theta \left[ \cos 2\lambda \left( 1825.0 + 297.0 \cos M' + 24.3 \cos 2M' \right) \right. \]
\[ + 54.7 \cos (M' - 2L + 2\Gamma) + 45.0 \cos (2l - 2L) \right] \]
\[ + \sin 2\lambda \left( 400.7 \sin M' - 81.0 \sin (M' - 2L + 2\Gamma) + 41.9 \sin (2l - 2L) \right) \right] , \tag{42} \]

\[ u_a = h_2 \sin^2 \theta \left[ \cos 2\lambda \left( 977.2 + 159.1 \cos M' + 13.0 \cos 2M' \right) \right. \]
\[ + 29.3 \cos (M' - 2L + 2\Gamma) + 24.1 \cos (2l - 2L) \right] \]
\[ + \sin 2\lambda \left( 214.5 \sin M' - 43.4 \sin (M' - 2L + 2\Gamma) + 22.4 \sin (2l - 2L) \right) \right] , \tag{43} \]

\[ u_\theta = l_2 \sin 2\theta \left[ \cos 2\lambda \left( 977.2 + 159.1 \cos M' + 13.0 \cos 2M' \right) \right. \]
\[ + 29.3 \cos (M' - 2L + 2\Gamma) + 24.1 \cos (2l - 2L) \right] \]
\[ + \sin 2\lambda \left( 214.5 \sin M' + 43.4 \sin (M' - 2L + 2\Gamma) + 22.4 \sin (2l - 2L) \right) \right] \tag{44} \]

and
\[ u_{2} = -2l_{2} \sin \theta \left[ \sin 2\lambda \{ 977.2 + 159.1 \cos M' + 13.0 \cos 2M' \right. \\
+ 29.3 \cos(M' - 2L + 2\lambda) + 24.1 \cos(2L - 2\lambda) \left. \\
- \cos 2\lambda \{ 141.5 \sin M' + 43.4 \sin(M' - 2L + 2\lambda) + 22.4 \sin(2L - 2\lambda) \} \right]. \] \quad (45)

The Love numbers \( l_{2} \) and \( h_{2} \) and also the factor \( G_{2} \) vary with lunar models. The Love numbers reflect the density, bulk modulus and rigidity of the lunar interiors. We calculated the Love numbers for four different simple lunar models with several layers by using the program based on the formula derived by Okubo and Saito (1983). Two extreme models may set bounds to the real structure. Model 1: uniform density 3.34 g/cm\(^3\) and the rigidity \( 7.38 \times 10^{11} \) dyne/cm\(^2\) and no core; Model 2: mantle density 3.0 g/cm\(^3\) and rigidity \( 4.0 \times 10^{11} \) dyne/cm\(^2\) with a fluid core of the density 5.72 g/cm\(^3\) filling up to the half of the lunar radius. These models presented the estimations: Model 1, \( h_{2} = 0.0331, G_{2} = 1.0033 \); Model 2, \( h_{2} = 0.1006, G_{2} = 1.0143 \) (Harrison, 1963). Then we computed the Love numbers with two more realistic models. Model 3: 5-layered model with the core radius of 450 km given by Nakamura (1983). Model 4: 6-layered model with the core radius of 340 km given by Bill and Ferrari (1977). These models lead us to estimations: Model 3, \( h_{2} = 0.038, G_{2} = 1.0020 \); Model 4, \( h_{2} = 0.068, G_{2} = 1.0095 \) (Hanada et al., 1991).

5. Discussion and Summary

We developed general formulas of tides on the Moon and selenographic coordinates of the Earth, which allow us to estimate ancient tides (potential, gravity variation, and displacements) on the Moon as far as the Moon stayed at the second status of Cassini.

### Table 1. Major terms of the radial displacement and the gravity variation on the Moon.

<table>
<thead>
<tr>
<th>Colatitude</th>
<th>Zonal</th>
<th>Tesseral</th>
<th>Sectorial</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>((3/2) \cos^{2} \theta - (1/2) \cos X)</td>
<td>(\sin 2\theta \cos \lambda \sin X)</td>
<td>(\sin^{2} \theta \cos 2\lambda \cos X, \sin^{2} \theta \sin 2\lambda \sin X)</td>
</tr>
<tr>
<td>(0^\circ)</td>
<td>(u_{r} = -4.0 \text{ cm}) (\Phi_{2} = 7.2 \text{ cm})</td>
<td>(\Delta g = 196 \mu \text{ gal}) (</td>
<td>197</td>
</tr>
<tr>
<td>(45^\circ)</td>
<td>(u_{r} = -8.6 \text{ cm}) (\Phi_{2} = 15.4 \text{ cm})</td>
<td>(\Delta g = 423 \mu \text{ gal}) (</td>
<td>427</td>
</tr>
<tr>
<td>(90^\circ)</td>
<td>(u_{r} = 6.1 \text{ cm}) (\Phi_{2} = 10.9 \text{ cm})</td>
<td>(\Delta g = -298 \mu \text{ gal}) (</td>
<td>-300</td>
</tr>
</tbody>
</table>

Astron. argument \( X \)

| \(M'\) | \(l - \Omega\) | \(M'\) |

Upper figures of numerals are for Model 3. Lower figures (in parentheses) for Model 4.

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We also developed precise formulas of tides at present day for study of the lunar interior and for correction for the observations of the libration and deformation of the Moon.

The major components of tides are given in Table 1 as an example. In the table, we listed only the radial displacement and the gravity variation in order to show how they are affected by the Love numbers. As the Love number $l_2$ is expected to be close to 0.05, the magnitude of the tangential displacement is the same order with the radial one. We can estimate the tidal displacements with better than millimeter accuracy by using the formulas described in this paper. These formulas will be useful for interpretation of observations of the lunar deformation with advanced space technology such as VLBI, since differential VLBI technique can measure the variation in the distance between two artificial radio transmitters on the Moon with an accuracy of a few centimeters (Hanada et al., 1991). The results of Table 1 suggest to us possibilities to set a constraint on the model of the lunar internal structure.

Some problem in data analyses would be that two major periods of tides are close to each other (27.55 and 27.21 days) and then the rather long period of observations shall be needed to separate tidal constituents.

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