Seismic Wavefields in Multi-Layered Media
Calculated by Hybrid Combination of
Boundary Element Method and Thin-Layer
Finite Element Method
——The Case of Two-Dimensional P-SV-Wavefields——

Hiroyuki Fujiwara
National Research Institute for Earth Science and Disaster Prevention,
Tsukuba 305, Japan

Two-dimensional P-SV-wavefields in laterally heterogeneous multi-layered media are calculated by using the hybrid combination of the boundary element method (BEM) and the thin-layer finite element method (TLFEM). The BEM can be combined with the TLFEM by matching the boundary conditions on vertical boundaries. Expansion of wavefield into summation of normal modes is used to match the boundary conditions. This method is suitable to calculate propagation of surface waves in laterally heterogeneous multi-layered media. It is easy to treat incident surface waves from distant sources, which is difficult to calculate by other boundary methods. Since this method does not require the assumption of periodic structure, it can be applicable to non-periodical problems. Wavefields in a multi-layered slope structure are investigated. When a point source is in a thicker layer side, the amplitude of scattered waves is large and backscattered Rayleigh waves exist. When a point source is in a thinner layer side, Rayleigh waves whose dispersion corresponds to that for an averaged structure of both sides mainly propagate in the forward direction.

1. Introduction

Many methods have been studied to calculate elastic wavefields in laterally heterogeneous multi-layered media. These methods are divided into two categories. One is the domain method and the other is the boundary method. The finite element method (FEM) and the finite difference method (FDM) belong to the domain method. The domain method can be applied to almost all problems in calculation of seismic wavefields. One of the major problems in the calculation of wavefields by the domain method is that an artificial boundary must be introduced. Absorbing boundary conditions have been studied to eliminate the non-physical reflections caused by the artificial boundary (e.g., Clayton and Engquist, 1977; Smith, 1974). The energy transmitting boundary (Lysmer and Drake, 1973) is one of the powerful absorbing boundaries to calculate the
wavefields in multi-layered media by using FEM. The energy transmitting boundary is constructed by using the thin-layer finite element method (TLFEM) (Lysmer and Waas, 1972; Tajimi and Shimomura, 1973). The TLFEM is a kind of FEM which is efficient to calculate the wavefields in horizontally layered media.

On the other hand, the Aki-Larner method (Aki and Larner, 1970), the boundary integral-discrete wavenumber method (Bouchon et al., 1989), the indirect boundary element method (IBEM) (Sánchez-Sesma et al., 1993), and the direct boundary element method (BEM) belong to the boundary method. The boundary method has been applied to many engineering seismology problems as the seismic response of a sedimentary basin and the effect of topography on the amplification of the seismic waves because of advantages over the domain method such as a reduction of the dimension of the discretized domain and applicability to infinite domains. The boundary method, however, has a limitation of applicability. The Aki-Larner method and the boundary integral-discrete wavenumber method require the assumption of periodic configuration of a medium. It is difficult to calculate the wavefield in a non-periodical multi-layered medium by these methods. By using BEM and IBEM, it is enough to discretize a finite domain for the calculation of a wavefield in an infinite domain if a suitable Green's function can be adopted. In general, however, it is difficult to adopt a Green's function that satisfies all boundary conditions because the calculation of such a Green's function requires much CPU time. The Green's function for full-space has a simple analytical form and can be calculated in a short CPU time. If we use the full-space Green's function to calculate a wavefield in a multi-layered medium, however, it is necessary to discretize infinite domains. Therefore, a problem arises by the truncation of elements. Fujiwara and Takenaka (1993) proposed the reference solution approach for BEM to reduce the truncation error. Yokoi and Takenaka (1995) applied this method to IBEM. These methods, however, cannot be applied to the multi-layer problem in which the layered structure of the right side of the medium differs from that of the left side.

Fujiwara (1996) proposed an alternative method of calculating the two-dimensional SH-wavefields in multi-layered structure. In the method, the energy transmitting boundary which is constructed by using the TLFEM is combined with BEM. Expansion of wavefield into summation of normal modes is used for the combination. Since this method does not require the assumption of periodic structure, it can be applicable to non-periodical problems and is suitable to calculate surface wave propagation in laterally heterogeneous multi-layered media. In this article, a method is developed to calculate the two-dimensional P-SV-wavefields in multi-layered media.

2. Expansion of Wavefield into Summation of Normal Modes

We consider a wavefield in a horizontally multi-layered half-space. Assuming an absorbing boundary condition (Engquist and Majda, 1977; Maeda, 1992) at depth $H$ which is deep enough to approximate the half-space, we can regard the problem as wave propagation in a layered medium whose thickness is finite. We can express the wavefield for such a problem as an infinite sum of normal modes. The wavefield consists of waves propagating towards the positive direction of the $x$-axis and waves propagating
towards the negative direction of the x-axis.

\begin{equation}
  u_x(x, z) = u^-_x(x, z) + u^+_x(x, z),
  \tag{1}
\end{equation}

\begin{equation}
  u_z(x, z) = u^-_z(x, z) + u^+_z(x, z),
  \tag{2}
\end{equation}

Each term in Eqs. (1) and (2) can be expressed as an infinite sum of normal modes.

\begin{equation}
  u^+_x(x, z) = \sum_{n=1}^{\infty} a_n^+ r_n^{(1)}(z) e^{+ik_n x},
  \tag{3}
\end{equation}

\begin{equation}
  u^-_z(x, z) = \sum_{n=1}^{\infty} i a_n^- r_n^{(2)}(z) e^{-ik_n x},
  \tag{4}
\end{equation}

where \( k_n \) is an eigenvalue for horizontal wavenumber and \( r_n^{(1)} \) and \( r_n^{(2)} \) are eigenvectors corresponding to \( k_n \).

Based on the above expression, we can derive a relation between the displacement and the traction across a vertical boundary in multi-layered medium. The traction across the vertical boundary is given by

\begin{equation}
  t^+_x(x, z) = \sum_{n=1}^{\infty} i a_n^+ \left\{ \lambda(z) \frac{dr_n^{(2)}(z)}{dz} + (\lambda(z) + 2\mu(z))k_n r_n^{(1)}(z) \right\} e^{+ik_n x},
  \tag{5}
\end{equation}

\begin{equation}
  t^-_z(x, z) = \sum_{n=1}^{\infty} a_n^- \mu(z) \left\{ \frac{dr_n^{(1)}(z)}{dz} - k_n r_n^{(2)}(z) \right\} e^{-ik_n x}.
  \tag{6}
\end{equation}

The inner product of the normal modes is defined by

\begin{equation}
  \langle r_m, r_n \rangle = \frac{1}{2} \int_0^H \left[ (k_m + k_n) \left\{ \lambda(z) + 2\mu(z) \right\} r_n^{(1)}(z) r_m^{(1)}(z) + \mu(z) r_n^{(2)}(z) r_m^{(2)}(z) \right]
  \nonumber
  + \lambda(z) \left( r_n^{(1)}(z) \frac{dr_m^{(2)}(z)}{dz} + r_n^{(1)}(z) \frac{dr_m^{(1)}(z)}{dz} \right)
  \nonumber
  - \mu(z) \left( r_m^{(2)}(z) \frac{dr_n^{(1)}(z)}{dz} + r_m^{(2)}(z) \frac{dr_n^{(2)}(z)}{dz} \right) \, dz,
  \tag{7}
\end{equation}

where we put \( r_n = (r_n^{(1)}, i r_n^{(2)}) \). The following orthogonality holds.

\begin{equation}
  \langle r_m, r_n \rangle = \delta_{mn}.
  \tag{8}
\end{equation}

By calculating the inner product, we have

\begin{equation}
  \langle u^\pm, r_n \rangle = a_n^\pm e^{\mp ik_n x}.
  \tag{9}
\end{equation}

Then displacement and traction can be written as

\begin{equation}
  u^\pm_x(x, z) = \sum_{n=1}^{\infty} \langle u^\pm, r_n \rangle r_n^{(1)}(z),
  \tag{10}
\end{equation}

\begin{equation}
  u^\pm_z(x, z) = \mp \sum_{n=1}^{\infty} i \langle u^\pm, r_n \rangle r_n^{(2)}(z),
  \tag{11}
\end{equation}

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The infinite sums in the above equations can be approximated by finite sums. Then we have

\begin{align}
  u^\pm_x(x, z) &= \sum_{n=1}^{M} \langle u_\pm, r_n \rangle r_n^{(1)}(z), \\
  u^\pm_z(x, z) &= \sum_{n=1}^{M} i\langle u_\pm, r_n \rangle r_n^{(2)}(z), \\
  t^\pm_x(x, z) &= \sum_{n=1}^{M} i\langle u_\pm, r_n \rangle r_n^{(1)}(z), \\
  t^\pm_z(x, z) &= \sum_{n=1}^{M} \langle u_\pm, r_n \rangle r_n^{(2)}(z).
\end{align}

These equations are expressed by the following matrix forms:

\begin{align}
  u_\pm &= U_\pm e_\pm, \\
  t_\pm &= T_\pm e_\pm,
\end{align}

where

\begin{align}
  U_\pm &= \begin{bmatrix}
    U_x^{(1)}(z_1) & \cdots & U_x^{(M)}(z_1) \\
    \vdots & \ddots & \vdots \\
    U_x^{(1)}(z_N) & \cdots & U_x^{(M)}(z_N) \\
    U_z^{(1)}(z_1) & \cdots & U_z^{(M)}(z_1) \\
    \vdots & \ddots & \vdots \\
    U_z^{(1)}(z_N) & \cdots & U_z^{(M)}(z_N)
  \end{bmatrix}, \\
  T_\pm &= \begin{bmatrix}
    T_x^{(1)}(z_1) & \cdots & T_x^{(M)}(z_1) \\
    \vdots & \ddots & \vdots \\
    T_x^{(1)}(z_N) & \cdots & T_x^{(M)}(z_N) \\
    T_z^{(1)}(z_1) & \cdots & T_z^{(M)}(z_1) \\
    \vdots & \ddots & \vdots \\
    T_z^{(1)}(z_N) & \cdots & T_z^{(M)}(z_N)
  \end{bmatrix},
\end{align}

\begin{align}
  u_\pm &= [u_x^\pm(x, z_1), \cdots, u_x^\pm(x, z_N), u_z^\pm(x, z_1), \cdots, u_z^\pm(x, z_N)]^T, \\
  t_\pm &= [t_x^\pm(x, z_1), \cdots, t_x^\pm(x, z_N), t_z^\pm(x, z_1), \cdots, t_z^\pm(x, z_N)]^T, \\
  c_\pm &= [\langle u_\pm, r_1 \rangle, \langle u_\pm, r_2 \rangle, \cdots, \langle u_\pm, r_M \rangle]^T.
\end{align}
In Eqs. (20) and (21), we put
\begin{align}
U^{(n)}_\pm(z) &= r^{(1)}_n(z), \\
U^{(n)}_x(z) &= \mp r^{(2)}_n(z), \\
T^{(n)}_x(\pm)(z) &= \mp i \left\{ (\lambda(z) + 2\mu(z))k_n r^{(1)}_n(z) + \lambda(z) \frac{dr^{(2)}_n(z)}{dz} \right\}, \\
T^{(n)}_z(\pm)(z) &= \mu(z) \left( \frac{dr^{(1)}_n(z)}{dz} - k_n r^{(2)}_n(z) \right).
\end{align}

The inverse operator of $U_\pm$ can be derived as follows. By calculating the inner product $\langle u_\pm, r_n \rangle$, we have
\begin{equation}
\langle u_\pm, r_n \rangle = \frac{i}{2} \int_0^R \left[ T^{(n)}_x(\pm)(z) u^\pm_x(x, z) + T^{(n)}_z(\pm)(z) u^\pm_z(x, z) \right] dz + \sum_{m=1}^\infty \langle u_\pm, r_m \rangle I_{nm},
\end{equation}
where
\begin{equation}
I_{nm} = \frac{1}{2} \int_0^R \left[ k_m \{(\lambda(z) + 2\mu(z))r^{(1)}_n(z)r^{(1)}_m(z) + \mu(z)r^{(2)}_n(z)r^{(2)}_m(z)\} \\
+ \lambda(z)r^{(1)}_n(z) \frac{dr^{(2)}_m(z)}{dz} - \mu(z)r^{(2)}_n(z) \frac{dr^{(1)}_m(z)}{dz} \right] dz,
\end{equation}
\begin{equation}
\overline{T^{(n)}_x(\pm)(z)} = \pm i \left\{ (\lambda(z) + 2\mu(z))k_n r^{(1)}_n(z) + \lambda(z) \frac{dr^{(2)}_n(z)}{dz} \right\},
\end{equation}
\begin{equation}
\overline{T^{(n)}_z(\pm)(z)} = \mu(z) \left( \frac{dr^{(1)}_n(z)}{dz} - k_n r^{(2)}_n(z) \right).
\end{equation}

Discretizing Eq. (29), we have the following matrix form:
\begin{equation}
c_\pm = F T_\pm^D u_\pm,
\end{equation}
where
\begin{equation}
F = \frac{i}{2} \begin{bmatrix}
-\frac{1}{2} & I_{12} & \cdots & I_{1M} \\
I_{21} & -\frac{1}{2} & \cdots & I_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
I_{M1} & I_{M2} & \cdots & -\frac{1}{2}
\end{bmatrix}^{-1},
\end{equation}
We define operators $E_\pm$ and $N_\pm$ as

$$T^*_\pm = \begin{bmatrix} T^{(1)}_{x\pm}(z_1) & \cdots & T^{(M)}_{x\pm}(z_1) \\ \vdots & \ddots & \vdots \\ T^{(1)}_{x\pm}(z_N) & \cdots & T^{(M)}_{x\pm}(z_N) \\ T^{(1)}_{\pm}(z_1) & \cdots & T^{(M)}_{\pm}(z_1) \\ \vdots & \ddots & \vdots \\ T^{(1)}_{\pm}(z_N) & \cdots & T^{(M)}_{\pm}(z_N) \end{bmatrix}^T,$$

(35)

$$\text{diag } \mathbf{D} = [d_1 \ d_2 \ \cdots \ d_N \ d_1 \ d_2 \ \cdots \ d_N].$$

(36)

We define operators $E_\pm$ and $N_\pm$ as

$$E_\pm \equiv FT^*_\pm D,$$

(37)

$$N_\pm \equiv T^*_\pm E_\pm.$$  

(38)

Operator $E_\pm$ is the inverse operator of $U_\pm$. Operator $N_\pm$ is called traction operator. Then we have

$$E_\pm U_\pm = I,$$

(39)

$$N_\pm U_\pm = T_\pm.$$  

(40)

We calculate the eigenvalues and the eigenvectors by using the TLFEM to construct the operators $E_\pm$, $U_\pm$, and $T_\pm$. The TLFEM is a kind of FEM which is efficient to calculate wavefields in horizontally layered media. The total domain is divided into thin layer elements for the discretization. The analytic function which satisfies an elastic wave equation is adopted as the interpolation function for the horizontal direction. The linear spline function is used for the vertical direction. After the discretization, we can obtain a generalized eigenvalue problem with respect to the horizontal wavenumber. Although we have to solve a generalized complex eigenvalue problem when the absorbing boundary condition is applied, we can now use an efficient subroutine for such a problem in many computers.

3. Hybrid Combination of BEM and TLFEM

We consider a two-dimensional P-SV-wavefield in a layered medium as shown in Fig. 1. Domain $V (= \bigcup_{i=1}^{N} V^{(i)})$ is laterally heterogeneous. Domains $V^{(A)}$ and $V^{(B)}$ are horizontally multi-layered media. The part of the boundary of $V^{(i)}$ that contacts $V^{(i)}$ is denoted by $S^{(i)}_{j}$. The BEM in which the full-space Green's functions are adopted is used for discretization of $V$ and the TLFEM is used for $V^{(A)}$ and $V^{(B)}$. The BEM can be combined with the TLFEM by matching the boundary conditions on the vertical boundaries $S^{(A)} (= \bigcup_{i=1}^{N} S^{(A)}_{i})$ and $S^{(B)} (= \bigcup_{i=1}^{N} S^{(B)}_{i})$.

We consider the wavefield in the frequency domain by assuming the time dependence $e^{i\omega t}$. The boundary integral equations for domains $V^{(i)} (i=1, \cdots, N-1)$ are given by

$$c^{(j)}_{jk}(x, z)u^{(j)}_{k}(x, z) = PV \int_{S^{(i)}} [G^{(j)}_{jk}(x', z'; x, z)J^{(j)}_{k}(x', z')] dS,$$

(41)

$$-H^{(j)}_{rj}(x', z'; x, z)u^{(j)}_{k}(x', z')] dS,$$

(41)

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Fig. 1. Configuration of the two-dimensional multi-layered medium used in this study. Domain $V = \bigcup_{i=1}^{N} V(i)$ is laterally heterogeneous multi-layered medium. Domains $V^{(A)}$ and $V^{(B)}$ are horizontally homogeneous layered media.

for $V^{(N)}$,

$$c_{jk}^{(N)}(x, z)u_{j}^{(N)}(x, z) = PV \int_{S^{(N)}} \left[ G_{jk}^{(N)}(x', z'; x, z) t_{k}^{(N)}(x', z') - H_{jk}^{(N)}(x', z'; x, z) u_{j}^{(N)}(x', z') \right] dS + u_{j}^{(N)}_{N,k}(x, z),$$

where $u_{k}^{(i)}$ and $t_{k}^{(i)}$ are the $k$-components of displacement and traction on the boundary $S^{(i)}$, respectively (e.g., Kobayashi, 1987). The symbol $PV$ denotes the Cauchy principal value. $c_{jk}^{(i)}$ is a function determined by the shape around $(x, z)$ and is equal to $1/2 \delta_{jk}$ in the case of a smooth boundary. The incident wave is denoted by $u_{k}^{(N)}$. We assume that a source is in $V^{(N)}$. The boundaries $S^{(i)} (i=1, \cdots, N-1)$ are defined by

$$S^{(i)} = S_{i-1}^{(i)} \cup S_{i+1}^{(i)} \cup S_{A}^{(i)} \cup S_{B}^{(i)};$$

and $S^{(N)}$ is given by

$$S^{(N)} = S_{N-1}^{(N)} \cup S_{A}^{(N)} \cup S_{B}^{(N)}.\$$

The integral kernel $G_{jk}^{(i)}$ and $H_{jk}^{(i)}$ are given by

$$G_{x}^{(i)}(x', z'; x, z) = -\frac{i}{4\mu^{(i)}} \left\{ H_{1}^{(2)}(k_{\beta^{(i)}}r) - \frac{k_{\alpha^{(i)}}}{k_{\beta^{(i)}}} H_{1}^{(1)}(k_{\alpha^{(i)}}r) \right\},$$

$$G_{zz}^{(i)}(x', z'; x, z) = -\frac{i}{4\mu^{(i)}} \left\{ \frac{(x'-x)(z'-z)}{r^2} \left\{ H_{2}^{(2)}(k_{\beta^{(i)}}r) - \frac{k_{\alpha^{(i)}}}{k_{\beta^{(i)}}} H_{2}^{(1)}(k_{\alpha^{(i)}}r) \right\} \right\},$$

$$G_{x}^{(i)}(x', z'; x, z) = -\frac{i}{4\mu^{(i)}} \left\{ \frac{(x'-x)(z'-z)}{r^2} \left\{ H_{2}^{(2)}(k_{\beta^{(i)}}r) - \frac{k_{\alpha^{(i)}}}{k_{\beta^{(i)}}} H_{2}^{(1)}(k_{\alpha^{(i)}}r) \right\} \right\},$$

$$G_{zz}^{(i)}(x', z'; x, z) = -\frac{i}{4\mu^{(i)}} \left\{ \frac{(x'-x)(z'-z)}{r^2} \left\{ H_{2}^{(2)}(k_{\beta^{(i)}}r) - \frac{k_{\alpha^{(i)}}}{k_{\beta^{(i)}}} H_{2}^{(1)}(k_{\alpha^{(i)}}r) \right\} \right\},$$

$$G_{x}^{(i)}(x', z'; x, z) = -\frac{i}{4\mu^{(i)}} \left\{ \frac{(x'-x)(z'-z)}{r^2} \left\{ H_{2}^{(2)}(k_{\beta^{(i)}}r) - \frac{k_{\alpha^{(i)}}}{k_{\beta^{(i)}}} H_{2}^{(1)}(k_{\alpha^{(i)}}r) \right\} \right\},$$

$$G_{zz}^{(i)}(x', z'; x, z) = -\frac{i}{4\mu^{(i)}} \left\{ \frac{(x'-x)(z'-z)}{r^2} \left\{ H_{2}^{(2)}(k_{\beta^{(i)}}r) - \frac{k_{\alpha^{(i)}}}{k_{\beta^{(i)}}} H_{2}^{(1)}(k_{\alpha^{(i)}}r) \right\} \right\},$$

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where $G_{jk}^{(i)}$ is the first derivative of $G_{jk}^{(i)}$ with respect to the $l$-component, $\mu^{(i)}$ and $\lambda^{(i)}$ are the Lamé constants of $V^{(i)}$, $\alpha^{(i)}$ and $\beta^{(i)}$ are P- and S-wave velocities of $V^{(i)}$, respectively. $H_{n}^{(2)}$ is the Hankel function of the second kind and order $n$, and $n_{x}^{(i)}$ and $n_{z}^{(i)}$ are the components of the normal vector $n^{(i)}(x', z')$, and $k_{x0}$, $k_{\beta 0}$, and $r$ are given by

\begin{align}
k_{x0} &= \frac{\omega}{\alpha^{(i)}}, & k_{\beta 0} &= \frac{\omega}{\beta^{(i)}}, \\
r &= \sqrt{(x'-x)^2 + (z'-z)^2}.
\end{align}

We can rewrite the above equations in the following operator equations. For domains $V^{(i)} (i=1, \cdots, N-1)$,

\begin{align}C_{i-1}^{(i)}u_{i-1}^{(i)} &= G_{i-1}^{(i)}t_{i-1}^{(i)} + G_{i-1}^{(i)}t_{i+1}^{(i)} + G_{i-1}^{(i)}t_{i}^{(i)} + G_{i-1}^{(i)}u_{B}^{(i)} \\
&- H_{i-1}^{(i)}u_{i+1}^{(i)} - H_{i-1}^{(i)}u_{i}^{(i)} - H_{i-1}^{(i)}u_{A}^{(i)} - H_{i-1}^{(i)}u_{B}^{(i)},
\end{align}

\begin{align}C_{i+1}^{(i)}u_{i+1}^{(i)} &= G_{i+1}^{(i)}t_{i+1}^{(i)} + G_{i+1}^{(i)}t_{i}^{(i)} + G_{i+1}^{(i)}t_{i-1}^{(i)} + G_{i+1}^{(i)}t_{B}^{(i)} \\
&- H_{i+1}^{(i)}u_{i-1}^{(i)} - H_{i+1}^{(i)}u_{i}^{(i)} - H_{i+1}^{(i)}u_{A}^{(i)} - H_{i+1}^{(i)}u_{B}^{(i)},
\end{align}

\begin{align}C_{A}^{(i)}u_{A}^{(i)} &= G_{A}^{(i)}u_{A}^{(i)} + G_{A}^{(i)}u_{A}^{(i)} + G_{A}^{(i)}u_{A}^{(i)} + G_{A}^{(i)}u_{B}^{(i)} \\
&- H_{A}^{(i)}u_{A}^{(i)} - H_{A}^{(i)}u_{A}^{(i)} - H_{A}^{(i)}u_{A}^{(i)} - H_{A}^{(i)}u_{B}^{(i)}.
\end{align}

\begin{align}C_{B}^{(i)}u_{B}^{(i)} &= G_{B}^{(i)}u_{B}^{(i)} + G_{B}^{(i)}u_{B}^{(i)} + G_{B}^{(i)}u_{B}^{(i)} + G_{B}^{(i)}u_{B}^{(i)} \\
&- H_{B}^{(i)}u_{B}^{(i)} - H_{B}^{(i)}u_{B}^{(i)} - H_{B}^{(i)}u_{B}^{(i)} - H_{B}^{(i)}u_{B}^{(i)}.
\end{align}

and for $V^{(N)}$,

\begin{align}C_{N-1}^{(N)}u_{N-1}^{(N)} &= G_{N-1}^{(N)}u_{N-1}^{(N)} + G_{N-1}^{(N)}u_{N-1}^{(N)} + G_{N-1}^{(N)}u_{N-1}^{(N)} + u_{N-1}^{(N)} \\
&- H_{N-1}^{(N)}u_{N-1}^{(N)} - H_{N-1}^{(N)}u_{N-1}^{(N)} - H_{N-1}^{(N)}u_{N-1}^{(N)} + u_{N-1}^{(N)},
\end{align}

\begin{align}C_{A}^{(N)}u_{A}^{(N)} &= G_{A}^{(N)}u_{A}^{(N)} + G_{A}^{(N)}u_{A}^{(N)} + G_{A}^{(N)}u_{A}^{(N)} + u_{A}^{(N)} \\
&- H_{A}^{(N)}u_{A}^{(N)} - H_{A}^{(N)}u_{A}^{(N)} - H_{A}^{(N)}u_{A}^{(N)} + u_{A}^{(N)},
\end{align}

\begin{align}C_{B}^{(N)}u_{B}^{(N)} &= G_{B}^{(N)}u_{B}^{(N)} + G_{B}^{(N)}u_{B}^{(N)} + G_{B}^{(N)}u_{B}^{(N)} + u_{B}^{(N)} \\
&- H_{B}^{(N)}u_{B}^{(N)} - H_{B}^{(N)}u_{B}^{(N)} - H_{B}^{(N)}u_{B}^{(N)} + u_{B}^{(N)}.
\end{align}

The definitions of the operators are described in Appendix 1. It is assumed that a source exists in $V^{(N)}$.

From the boundary conditions, we have

\begin{align}u_{i}^{(j)} &= u_{i}^{(j)}, \\
t_{i}^{(j)} &= 0, \\
t_{i}^{(j)} &= -t_{i}^{(j)}.
\end{align}
By using the operators defined in the previous section, we have

\[ u_A^{(i)} = U_{A_{i}}^{(i-1)}c_{A_{i}}^{(i-1)}, \]  
\[ t_A^{(i)} = T_{A_{i}}^{(i-1)}c_{A_{i}}^{(i-1)}, \]  
\[ u_B^{(i)} = U_{B_{i}}^{(i-1)}c_{B_{i}}^{(i-1)} , \]  
\[ t_B^{(i)} = T_{B_{i}}^{(i-1)}c_{B_{i}}^{(i-1)}. \]

After operating \( E_A^{(i)} \) to Eqs. (54) and (57) and operating \( E_B^{(i)} \) to Eqs. (55) and (58), we substitute Eqs. (59)–(65) into Eqs. (52)–(58), then we have

\[ -G_{00}^{(i)}t_1^{(i)} + (H_{00}^{(i)} + C_{00}^{(i)})u_0^{(i)} + H_{02}^{(i)}u_2^{(i)} + (G_{04}^{(i)}T_{A_{i}}^{(i)} + H_{05}^{(i)}U_{A_{i}}^{(i)})c_{A_{i}}^{(i)} + (G_{06}^{(i)}T_{B_{i}}^{(i)} + H_{07}^{(i)}U_{B_{i}}^{(i)})c_{B_{i}}^{(i)} = 0, \]  
\[ -G_{20}^{(i)}t_2^{(i)} + H_{22}^{(i)}u_0^{(i)} + (H_{22}^{(i)} + C_{22}^{(i)})u_2^{(i)} + (G_{24}^{(i)}T_{A_{i}}^{(i)} + H_{25}^{(i)}U_{A_{i}}^{(i)})c_{A_{i}}^{(i)} + (G_{26}^{(i)}T_{B_{i}}^{(i)} + H_{27}^{(i)}U_{B_{i}}^{(i)})c_{B_{i}}^{(i)} = 0, \]  
\[ -G_{i-1}^{(i)}t_{i-1}^{(i-1)} - G_{i+1}^{(i)}t_{i+1}^{(i-1)} + (H_{i-1}^{(i)} + C_{i-1}^{(i)})u_{i-1}^{(i)} + H_{i+1}^{(i)}u_{i+1}^{(i)} + (G_{i-1}^{(i)}T_{A_{i}}^{(i)} + H_{i-1}^{(i)}U_{A_{i}}^{(i)})c_{A_{i}}^{(i)} + (G_{i+1}^{(i)}T_{B_{i}}^{(i)} + H_{i+1}^{(i)}U_{B_{i}}^{(i)})c_{B_{i}}^{(i)} \]  
\[ = 0 \quad (i = 1, \cdots, N-1), \]  
\[ -G_{i+1}^{(i)}t_{i+1}^{(i-1)} - G_{i-1}^{(i)}t_{i-1}^{(i-1)} + H_{i+1}^{(i)}u_{i+1}^{(i-1)} + (H_{i-1}^{(i)} + C_{i-1}^{(i)})u_{i-1}^{(i)} + (G_{i+1}^{(i)}T_{A_{i}}^{(i)} + H_{i+1}^{(i)}U_{A_{i}}^{(i)})c_{A_{i}}^{(i)} + (G_{i-1}^{(i)}T_{B_{i}}^{(i)} + H_{i-1}^{(i)}U_{B_{i}}^{(i)})c_{B_{i}}^{(i)} \]  
\[ = 0 \quad (i = 1, \cdots, N-1), \]  
\[ -G_{N-1}^{(N)}t_{N-1}^{(N)} + (H_{N-1}^{(N)} + C_{N-1}^{(N)})u_{N-1}^{(N)} + (G_{N-1}^{(N)}T_{A_{i}}^{(N)} + H_{N-1}^{(N)}U_{A_{i}}^{(N)})c_{A_{i}}^{(N)} + (G_{N-1}^{(N)}T_{B_{i}}^{(N)} + H_{N-1}^{(N)}U_{B_{i}}^{(N)})c_{B_{i}}^{(N)} \]  
\[ = u_{N-1}^{(N)}, \]  
\[ E_A^{(i)}H_A^{(i)}u_0^{(i)} + \sum_{i=2}^{N} \left[ E_A^{(i-1)}H_A^{(i-1)} + E_A^{(i)}H_A^{(i)} \right]u^{(i)} \]  
\[ + \sum_{i=2}^{N} \left[ -E_A^{(i-1)}g_A^{(i-1)} + E_A^{(i)}g_A^{(i)} \right]t^{(i-1)} \]  
\[ + \sum_{i=1}^{N} E_A^{(i)}[G_{A_A}^{(i)}T_{A_{i}}^{(i)} + (H_{A_A}^{(i)} + C_{A_A}^{(i)})U_{A_{i}}^{(i)}]c_{A_{i}}^{(i)} \]  
\[ + \sum_{i=1}^{N} E_A^{(i)}[G_{A_B}^{(i)}T_{B_{i}}^{(i)} + H_{A_B}^{(i)}U_{B_{i}}^{(i)}]c_{B_{i}}^{(i)} = E_A^{(N)}u_{N}^{(N)}, \]  
\[ E_B^{(i)}H_B^{(i)}u_0^{(i)} + \sum_{i=2}^{N} \left[ E_B^{(i-1)}H_B^{(i-1)} + E_B^{(i)}H_B^{(i)} \right]u^{(i)} \]  
\[ + \sum_{i=2}^{N} \left[ -E_B^{(i-1)}g_B^{(i-1)} + E_B^{(i)}g_B^{(i)} \right]t^{(i-1)} \]  
\[ + \sum_{i=1}^{N} E_B^{(i)}[G_{B_A}^{(i)}T_{B_{i}}^{(i)} + H_{B_A}^{(i)}U_{B_{i}}^{(i)}]c_{B_{i}}^{(i)} = E_B^{(N)}u_{N}^{(N)}. \]
Solving the above system of Eqs. (66)-(72), we can obtain $u^{(i)}$ and $t^{(i)}$ on the boundaries. The systems of equations for the cases where a source is in $V^{(A)}$ or $V^{(B)}$ are described in Appendix 2.

4. Calculation of Wavefields in Layered Media

At first a wavefield in a horizontally layered medium is calculated to investigate the accuracy of the hybrid combination of BEM and TLFEM (TLFE-BEM). The model used for calculation consists of horizontally stratified three layers over a half space as shown in Fig. 2. The domain $V (-12.5 \leq x \leq 12.5 \text{ (km)})$ is discretized by using the BEM and the TLFEM is applied to the domains $V^{(A)} (x \leq -12.5 \text{ (km)})$ and $V^{(B)} (x \geq 12.5 \text{ (km)})$. The boundary conditions are matched on the boundaries $S^{(A)}$ and $S^{(B)} (0 \leq z \leq 32.5 \text{ (km)})$. The location of a point source is $(0, 3) \text{ km}$. The source is a point force applied in the $x$-direction with a time history of a Ricker wavelet with the characteristic period of 6 s. In Fig. 3, the results of calculation by using the following three methods are compared; the discrete wavenumber method (solid line), the hybrid combination of BEM and TLFEM (broken line), and the ordinary BEM (dotted line). We can consider that the results of the discrete wavenumber method are the exact solutions. The waveforms of the ordinary BEM include non-physical waves caused by the truncation of elements. The amplitude of the waveform of the ordinary BEM at 12.0 km is smaller than that of the other methods.
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Fig. 3. Comparison of the results of the hybrid combination of BEM and TLFEM (broken line), the ordinary BEM (dotted line), and the discrete wavenumber method (solid line). The source is a point force applied in the x-direction at the center of the model with a depth of 3 km. The source function is a Ricker wavelet with the characteristic period of 6 s.

of the other methods because the boundary conditions are not satisfied at the truncation point. On the other hand, the waveforms of TLFE-BEM are in good agreement with those of the discrete wavenumber method.

We consider a multi-layered slope structure model as shown in Fig. 4. The waveforms calculated by using the TLFE-BEM are compared with waveforms by using the ordinary BEM in Fig. 5. The source is a point force applied in the x-direction at (0, 5) km. The source function is a Ricker wavelet with the characteristic period of 6 s. The z-component waveforms of the ordinary BEM are shown in Fig. 5(a) and those of the TLFE-BEM are shown in Fig. 5(b). The non-physical waves excited by the truncation of elements exist in Fig. 5(a). The waveforms calculated by using the TLFE-BEM shown in Fig. 5(b) are greatly improved.

Wavefields in the model of Fig. 4 are calculated to investigate characteristics of Rayleigh waves in a multi-layered slope structure for the following three source locations: (a) (−30, 5) km, (b) (0, 5) km, and (c) (30, 5) km. The source is a point force applied in the x-direction with a time history of a Ricker wavelet with the characteristic period of 6 s. The waveforms of the x-component and the z-component are shown in Fig. 6 and Fig. 7, respectively. In case (a) where the source is in a thicker layer side, the amplitude of scattered waves is large. There is a wave group whose amplitude becomes small as the wave group propagates in Fig. 7(a). When the source is just under the slope structure, Rayleigh waves propagate in both directions. The duration time is shortest near the right edge of the slope structure. In case (c) where the source is in a
Fig. 4. Multi-layered slope structure model used for the calculations.

<table>
<thead>
<tr>
<th>$H$ (km)</th>
<th>$\rho$ (g/cm$^3$)</th>
<th>$\alpha$ (km/s)</th>
<th>$\beta$ (km/s)</th>
</tr>
</thead>
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<td>1.3</td>
<td>0.5</td>
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<td>1.8</td>
<td>0.8</td>
</tr>
<tr>
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<td>2.2</td>
<td>2.5</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>4.5</td>
<td>2.6</td>
</tr>
<tr>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 5. The $z$-components of synthetic waveforms calculated by using (a) the ordinary BEM and (b) the TLFE-BEM for the model of Fig. 4. The waveforms of (b) are greatly improved.

thinner layer side, dispersed Rayleigh waves propagate forward and no back-scattered wave is found.

The amplitude of the $z$-component in the frequency-wavenumber domain is shown in Fig. 8. The solid lines are the dispersion curves for the structure of $V^{(A)}$ and the dash lines are those of $V^{(B)}$. The diameter of the dots is proportional to the logarithm of the amplitude. The location of the locally maximum amplitude near the fundamental mode.
Fig. 6. The $x$-components of synthetic waveforms at the ground surface. The location of source is (a) $(-30, 5)$ km, (b) $(0, 5)$ km, and (c) $(30, 5)$ km. The source is a point force applied in the $x$-direction with a time history of a Ricker wavelet with the characteristic period of 6 s.

Fig. 7. The $z$-components of synthetic waveforms at the ground surface.
Fig. 8. The frequency-wavenumber spectra of the z-components. The location of source is (a) \((-30, 5)\) km, (b) \((0, 5)\) km, and (c) \((30, 5)\) km. The diameter of the dot is proportional to the logarithm of the amplitude. The solid curves are dispersion curves for the layered structure of the thicker layer side \(V(A)\) and the dash curves are those of the thinner layer side \(V(B)\). The location of the locally maximum amplitude near the fundamental mode of Rayleigh wave is denoted by the symbol +. The disturbance of the location of the symbol + in (b) is due to numerical error.

of a Rayleigh wave is denoted by the symbol +. We can consider the location of the maximum amplitude as the fundamental mode of Rayleigh wave in the laterally heterogeneous multi-layered medium. In Fig. 8(a), there are waves that have negative \(k_x\). The dispersion of the waves of negative \(k_x\) corresponds to that for a structure between \(V(A)\) and \(V(B)\) but slightly closer to \(V(A)\). These waves are interpreted as the back-scattered Rayleigh waves due to the slope structure. In Fig. 8(b), the amplitude of body waves is large. Rayleigh waves of both positive \(k_x\) and negative \(k_x\) exist. The amplitude of Rayleigh waves of negative \(k_x\) is larger than that of positive \(k_x\). The dispersion of the Rayleigh waves of negative \(k_x\) corresponds to that for a structure between \(V(A)\) and \(V(B)\) but slightly closer to \(V(A)\), and that of positive \(k_x\) corresponds to that for a structure close to \(V(B)\). In Fig. 8(c), there is no wave of positive \(k_x\). The dispersion of the Rayleigh waves of negative \(k_x\) corresponds to that for an averaged structure between \(V(A)\) and \(V(B)\) but slightly closer to \(V(A)\). These characteristics of propagation of Rayleigh waves are similar to those of Love waves (Fujiwara, 1996).

5. Conclusion

Two-dimensional P-SV-wavefields in laterally heterogeneous multi-layered media were calculated by using the hybrid combination of BEM and TLFEM. The laterally heterogeneous part is discretized by the BEM in which the full-space Green's function

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is adopted as the integral kernel and the horizontally layered parts are discretized by the TLFEM. The BEM can be combined with the TLFEM by matching the boundary conditions on vertical boundaries. Normal mode expansion of the wavefield is used to match the boundary conditions. Using this method, we can solve the problem of an incident surface wave from a distant point source, which is difficult for other methods to calculate. Since this method does not require the assumption of periodic structure, it can be applicable to non-periodic problems.

Wavefields in multi-layered slope structure are calculated for different source locations. In the case where the source is in a thicker layer side, the amplitude of scattered waves is large. There are back-scattered Rayleigh waves. The dispersion of the back scattered Rayleigh waves corresponds to that for a structure between the thicker layer part and the thinner layer part but slightly closer to the thicker layer part. In the case that the source is just under the slope structure, Rayleigh waves propagate towards the positive and negative direction of the x-axis. The amplitude of Rayleigh waves propagating in the negative direction, which is the thicker layer side, is larger than that in the positive direction. The dispersion of the Rayleigh waves propagating in the negative direction corresponds to that for a structure between two sides but slightly closer to the thicker layer side and that in the positive direction corresponds to that for a structure close to the thinner layer side. When the source is in the thicker layer side, dispersed Rayleigh waves propagate forward and no back-scattered wave is found. The dispersion corresponds to that for an averaged structure of the two sides but slightly closer to the thicker layer side. These characteristics of propagation of Rayleigh waves are similar to those of Love waves.

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REFERENCES


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APPENDIX 1

The Eq. (41) can be rewritten to the operator Eqs. (52) to (55) in the following manner. From Eq. (41) we have

\[
c^{(l)}_{jk}(x, z) u^{(l)}(x, z) = PV \int_{S^{(l-1)}} \left[ G^{(l)}_{jk}(x', z'; x, z) t^{(l)}(x', z') - H^{(l)}_{jk}(x', z'; x, z) u^{(l)}(x', z') \right] dS \\
+ \int_{S^{(l+1)}} \left[ G^{(l)}_{jk}(x', z'; x, z) t^{(l)}(x', z') - H^{(l)}_{jk}(x', z'; x, z) u^{(l)}(x', z') \right] dS \\
+ \int_{S^{(l)}_A} \left[ G^{(l)}_{jk}(x', z'; x, z) t^{(l)}(x', z') - H^{(l)}_{jk}(x', z'; x, z) u^{(l)}(x', z') \right] dS \\
+ \int_{S^{(l)}_B} \left[ G^{(l)}_{jk}(x', z'; x, z) t^{(l)}(x', z') - H^{(l)}_{jk}(x', z'; x, z) u^{(l)}(x', z') \right] dS,
\]

for \((x, z) \in S^{(l)}_{t+1} \); where \(S^{(l)}_{t+1} \) is the set of points on the free surface.

\[
c^{(l)}_{jk}(x, z) u^{(l)}(x, z) = \int_{S^{(l+1)}} \left[ G^{(l)}_{jk}(x', z'; x, z) t^{(l)}(x', z') - H^{(l)}_{jk}(x', z'; x, z) u^{(l)}(x', z') \right] dS \\
+ PV \int_{S^{(l+1)}} \left[ G^{(l)}_{jk}(x', z'; x, z) t^{(l)}(x', z') - H^{(l)}_{jk}(x', z'; x, z) u^{(l)}(x', z') \right] dS \\
+ \int_{S^{(l)}_A} \left[ G^{(l)}_{jk}(x', z'; x, z) t^{(l)}(x', z') - H^{(l)}_{jk}(x', z'; x, z) u^{(l)}(x', z') \right] dS \\
+ \int_{S^{(l)}_B} \left[ G^{(l)}_{jk}(x', z'; x, z) t^{(l)}(x', z') - H^{(l)}_{jk}(x', z'; x, z) u^{(l)}(x', z') \right] dS,
\]

for \((x, z) \in S^{(l)}_{t+1} \);

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\[ c_{jk}^{(0)}(x, z) u_j^{(0)}(x, z) = \int_{S_{l-1}^0} \left[ G_{jk}^{(0)}(x', z'; x, z) t_j^{(0)}(x', z') - H_{jk}^{(0)}(x', z'; x, z) u_j^{(0)}(x', z') \right] dS \]

\[ + \int_{S_{l+1}^0} \left[ G_{jk}^{(0)}(x', z'; x, z) t_j^{(0)}(x', z') - H_{jk}^{(0)}(x', z'; x, z) u_j^{(0)}(x', z') \right] dS \]

\[ + PV \int_{S_A^1} \left[ G_{jk}^{(0)}(x', z'; x, z) t_j^{(0)}(x', z') - H_{jk}^{(0)}(x', z'; x, z) u_j^{(0)}(x', z') \right] dS \]

\[ + \int_{S_B^1} \left[ G_{jk}^{(0)}(x', z'; x, z) t_j^{(0)}(x', z') - H_{jk}^{(0)}(x', z'; x, z) u_j^{(0)}(x', z') \right] dS, \]

for \((x, z) \in S_A^0;\)

\[ c_{jk}^{(0)}(x, z) u_j^{(0)}(x, z) = \int_{S_{l-1}^0} \left[ G_{jk}^{(0)}(x', z'; x, z) t_j^{(0)}(x', z') - H_{jk}^{(0)}(x', z'; x, z) u_j^{(0)}(x', z') \right] dS \]

\[ + \int_{S_{l+1}^0} \left[ G_{jk}^{(0)}(x', z'; x, z) t_j^{(0)}(x', z') - H_{jk}^{(0)}(x', z'; x, z) u_j^{(0)}(x', z') \right] dS \]

\[ + \int_{S_A^1} \left[ G_{jk}^{(0)}(x', z'; x, z) t_j^{(0)}(x', z') - H_{jk}^{(0)}(x', z'; x, z) u_j^{(0)}(x', z') \right] dS \]

\[ + PV \int_{S_B^1} \left[ G_{jk}^{(0)}(x', z'; x, z) t_j^{(0)}(x', z') - H_{jk}^{(0)}(x', z'; x, z) u_j^{(0)}(x', z') \right] dS, \]

for \((x, z) \in S_B^0.\)

Using an operator form,

\[ \int_{S_{l-1}^0} \left[ G_{jk}^{(0)}(x', z'; x, z) t_j^{(0)}(x', z') - H_{jk}^{(0)}(x', z'; x, z) u_j^{(0)}(x', z') \right] dS, \]

for \((x, z) \in S_l^1;\)

can be rewritten as

\[ G_{i+1,l-1}^{(0)} t_l^{(0)} - H_{i+1,l-1}^{(0)} u_l^{(0)} \]

Then Eq. (41) can be rewritten to the operator Eqs. (52) to (55):

\[ C_{l-1}^{(0)} u_{l-1}^{(0)} = G_{l-1,l-1}^{(0)} t_{l-1}^{(0)} + G_{l-1,l+1}^{(0)} t_{l+1}^{(0)} + G_{l-1, A}^{(0)} A_{l-1}^{(0)} + G_{l-1, B}^{(0)} B_{l-1}^{(0)} \]

\[ - H_{l-1,l-1}^{(0)} u_{l-1}^{(0)} - H_{l-1,l+1}^{(0)} u_{l+1}^{(0)} - H_{l-1, A}^{(0)} u_{l-1}^{(0)} - H_{l-1, B}^{(0)} u_{l+1}^{(0)} \]

\[ C_{l+1}^{(0)} u_{l+1}^{(0)} = G_{l+1,l-1}^{(0)} t_{l-1}^{(0)} + G_{l+1,l+1}^{(0)} t_{l+1}^{(0)} + G_{l+1, A}^{(0)} A_{l+1}^{(0)} + G_{l+1, B}^{(0)} B_{l+1}^{(0)} \]

\[ - H_{l+1,l-1}^{(0)} u_{l-1}^{(0)} - H_{l+1,l+1}^{(0)} u_{l+1}^{(0)} - H_{l+1, A}^{(0)} u_{l-1}^{(0)} - H_{l+1, B}^{(0)} u_{l+1}^{(0)} \]

\[ C_{A}^{(0)} A_{l}^{(0)} = G_{A,l-1}^{(0)} t_{l-1}^{(0)} + G_{A,l+1}^{(0)} t_{l+1}^{(0)} + G_{A, A}^{(0)} A_{l}^{(0)} + G_{A, B}^{(0)} B_{l}^{(0)} \]

\[ - H_{A,l-1}^{(0)} u_{l-1}^{(0)} - H_{A,l+1}^{(0)} u_{l+1}^{(0)} - H_{A, A}^{(0)} u_{l-1}^{(0)} - H_{A, B}^{(0)} u_{l+1}^{(0)} \]

\[ C_{B}^{(0)} A_{l}^{(0)} = G_{B,l-1}^{(0)} t_{l-1}^{(0)} + G_{B,l+1}^{(0)} t_{l+1}^{(0)} + G_{B, A}^{(0)} A_{l}^{(0)} + G_{B, B}^{(0)} B_{l}^{(0)} \]

\[ - H_{B,l-1}^{(0)} u_{l-1}^{(0)} + H_{B,l+1}^{(0)} u_{l+1}^{(0)} - H_{B, A}^{(0)} u_{l-1}^{(0)} - H_{B, B}^{(0)} u_{l+1}^{(0)}. \]
When a source exists in $V^{(4)}$, the system of the equations is given by

$$
-\mathbf{G}^{(1)}_{002} + (\mathbf{H}^{(1)}_{01} + \mathbf{C}^{(1)}_0)\mathbf{u}^{(1)}_0 + \mathbf{H}^{(1)}_{01}\mathbf{u}^{(1)}_2 \\
+ (\mathbf{G}^{(1)}_{011}\mathbf{T}^{(1)}_A + \mathbf{H}^{(1)}_{011}\mathbf{U}^{(1)}_A)\mathbf{c}_A - (\mathbf{G}^{(1)}_{011}\mathbf{T}^{(1)}_B + \mathbf{H}^{(1)}_{011}\mathbf{U}^{(1)}_B)\mathbf{c}_B \\
= -[\mathbf{G}^{(1)}_{011}\mathbf{T}^{(1)}_A + \mathbf{H}^{(1)}_{011}\mathbf{U}^{(1)}_A]a^{IN+},
$$

$$
-\mathbf{G}^{(1)}_{0111} - (\mathbf{H}^{(1)}_{0111} + \mathbf{C}^{(1)}_{01})\mathbf{u}^{(1)}_1 + \mathbf{H}^{(1)}_{0111}\mathbf{u}^{(1)}_2 \\
+ (\mathbf{G}^{(1)}_{0111}\mathbf{T}^{(1)}_A + (\mathbf{H}^{(1)}_{0111} + \mathbf{C}^{(1)}_{01})\mathbf{u}^{(1)}_A)\mathbf{c}_A - (\mathbf{G}^{(1)}_{0111}\mathbf{T}^{(1)}_B + \mathbf{H}^{(1)}_{0111}\mathbf{U}^{(1)}_B)\mathbf{c}_B \\
= -[\mathbf{G}^{(1)}_{0111}\mathbf{T}^{(1)}_A + \mathbf{H}^{(1)}_{0111}\mathbf{U}^{(1)}_A]a^{IN+},
$$

$$
-\mathbf{G}^{(1)}_{011111} - (\mathbf{H}^{(1)}_{011111} + \mathbf{C}^{(1)}_{011})\mathbf{u}^{(1)}_1 \\
+ (\mathbf{G}^{(1)}_{011111}\mathbf{T}^{(1)}_A + (\mathbf{H}^{(1)}_{011111} + \mathbf{C}^{(1)}_{011})\mathbf{u}^{(1)}_A)\mathbf{c}_A - (\mathbf{G}^{(1)}_{011111}\mathbf{T}^{(1)}_B + \mathbf{H}^{(1)}_{011111}\mathbf{U}^{(1)}_B)\mathbf{c}_B \\
= -[\mathbf{G}^{(1)}_{011111}\mathbf{T}^{(1)}_A + \mathbf{H}^{(1)}_{011111}\mathbf{U}^{(1)}_A]a^{IN+},
$$

$$
-\mathbf{G}^{(N)}_{N-1N-1} - (\mathbf{H}^{(N)}_{N-1N-1} + \mathbf{C}^{(N)}_{N-1})\mathbf{u}^{(N)}_N \\
+ (\mathbf{G}^{(N)}_{N-1N-1}\mathbf{T}^{(N)}_A + (\mathbf{H}^{(N)}_{N-1N-1} + \mathbf{C}^{(N)}_{N-1})\mathbf{u}^{(N)}_A)\mathbf{c}_A - (\mathbf{G}^{(N)}_{N-1N-1}\mathbf{T}^{(N)}_B + \mathbf{H}^{(N)}_{N-1N-1}\mathbf{U}^{(N)}_B)\mathbf{c}_B \\
= -[\mathbf{G}^{(N)}_{N-1N-1}\mathbf{T}^{(N)}_A + \mathbf{H}^{(N)}_{N-1N-1}\mathbf{U}^{(N)}_A]a^{IN+},
$$

\[\mathbf{E}^{(1)}_A\mathbf{H}^{(1)}_{A0}\mathbf{u}^{(1)}_0 + \sum_{i=2}^{N}[\mathbf{E}^{(i-1)}_A\mathbf{H}^{(i-1)}_{A0}\mathbf{u}^{(i-1)} - \mathbf{E}^{(i)}_A\mathbf{H}^{(i)}_{A,i-1}\mathbf{u}^{(i-1)}]u^{(i-1)}_0 \]

\[+ \sum_{i=1}^{N}[\mathbf{E}^{(i-1)}_A\mathbf{G}^{(i-1)}_{A0}\mathbf{u}^{(i-1)} - \mathbf{E}^{(i)}_A\mathbf{G}^{(i)}_{A,i-1}\mathbf{u}^{(i-1)}]t^{(i-1)}_0 \]

\[+ \sum_{i=1}^{N}\mathbf{E}^{(i)}_A - \mathbf{G}^{(i)}_{A0}\mathbf{u}^{(i)}_A - (\mathbf{H}^{(i)}_{A0} + \mathbf{C}^{(i)}_{A0})\mathbf{u}^{(i)}_A \]

\[+ \sum_{i=1}^{N}\mathbf{E}^{(i)}_A - \mathbf{G}^{(i)}_{A0}\mathbf{u}^{(i)}_A - (\mathbf{H}^{(i)}_{A0} + \mathbf{C}^{(i)}_{A0})\mathbf{u}^{(i)}_A \]

\[= \sum_{i=1}^{N}\mathbf{E}^{(i)}_A - \mathbf{G}^{(i)}_{A0}\mathbf{u}^{(i)}_A - (\mathbf{H}^{(i)}_{A0} + \mathbf{C}^{(i)}_{A0})\mathbf{u}^{(i)}_A \]

\[\mathbf{E}^{(1)}_B\mathbf{H}^{(1)}_{B0}\mathbf{u}^{(1)}_0 + \sum_{i=2}^{N}[\mathbf{E}^{(i-1)}_B\mathbf{H}^{(i-1)}_{B0}\mathbf{u}^{(i-1)} - \mathbf{E}^{(i)}_B\mathbf{H}^{(i)}_{B,i-1}\mathbf{u}^{(i-1)}]u^{(i-1)}_0 \]

\[+ \sum_{i=1}^{N}[\mathbf{E}^{(i-1)}_B\mathbf{G}^{(i-1)}_{B0}\mathbf{u}^{(i-1)} - \mathbf{E}^{(i)}_B\mathbf{G}^{(i)}_{B,i-1}\mathbf{u}^{(i-1)}]t^{(i-1)}_0 \]

\[+ \sum_{i=1}^{N}\mathbf{E}^{(i)}_B - \mathbf{G}^{(i)}_{B0}\mathbf{u}^{(i)}_B - (\mathbf{H}^{(i)}_{B0} + \mathbf{C}^{(i)}_{B0})\mathbf{u}^{(i)}_B \]

\[= \sum_{i=1}^{N}\mathbf{E}^{(i)}_B - \mathbf{G}^{(i)}_{B0}\mathbf{u}^{(i)}_B - (\mathbf{H}^{(i)}_{B0} + \mathbf{C}^{(i)}_{B0})\mathbf{u}^{(i)}_B \]

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When a source exists in $V^{(b)}$, the system of the equations is given by

\[-G^{(1)}_{00}u_0^{(1)} + (H^{(1)}_{00} + C_0^{(1)})u_0^{(1)} + H^{(1)}_{02}u_2^{(1)} + (G^{(1)}_{01}A^{(1)} + H^{(1)}_{01}U^{(1)}_{A^{(1)}})c_{A^{(1)}} - (G^{(1)}_{01}B^{(1)} + H^{(1)}_{01}U^{(1)}_{B^{(1)}})c_{B^{(1)}} = 0,\]

\[-G^{(1)}_{20}u_0^{(1)} + (H^{(1)}_{20} + C_2^{(1)})u_2^{(1)} + (G^{(1)}_{21}A^{(1)} + H^{(1)}_{21}U^{(1)}_{A^{(1)}})c_{A^{(1)}} - (G^{(1)}_{21}B^{(1)} + H^{(1)}_{21}U^{(1)}_{B^{(1)}})c_{B^{(1)}} = 0,\]

\[-G^{(1)}_{i-1}u_{i-1}^{(1)} + (H^{(1)}_{i-1} + C_{i-1})u_{i-1}^{(1)} + (H^{(1)}_{i-1}U^{(1)}_{A^{(1)}})c_{A^{(1)}} - (G^{(1)}_{i-1}B^{(1)} + H^{(1)}_{i-1}U^{(1)}_{B^{(1)}})c_{B^{(1)}} = 0,\]

\[-G^{(1)}_{N-1}u_{N-1}^{(1)} + (H^{(1)}_{N-1} + C_{N-1})u_{N-1}^{(1)} + (G^{(1)}_{N-1}B^{(1)} + H^{(1)}_{N-1}U^{(1)}_{B^{(1)}})c_{B^{(1)}} = 0,\]

\[\sum_{i=1}^{N} E^{(1)}_{B^{(1)}}[G^{(1)}_{B^{(1)}B^{(1)}}U^{(1)}_{B^{(1)}} + (H^{(1)}_{B^{(1)}B^{(1)}} + C^{(1)}_{B^{(1)}})c^{(1)}_{B^{(1)}} = 0.\]
\[ + \sum_{i=1}^{N} \mathbf{F}^{(i)}_{B+} \left[ \mathbf{G}^{(i)}_{B, A} \mathbf{T}^{(i)}_{A-} + \mathbf{H}^{(i)}_{B, A} \mathbf{U}^{(i)}_{A-} \right] \mathbf{c}_{A-} \]

\[ + \sum_{i=1}^{N} \mathbf{F}^{(i)}_{B+} \left[ \mathbf{G}^{(i)}_{B, B} \mathbf{T}^{(i)}_{B+} + \left( \mathbf{H}^{(i)}_{B, B} + \mathbf{C}^{(i)}_{B} \right) \mathbf{U}^{(i)}_{B+} \right] \mathbf{c}_{B+} \]

\[ = \sum_{i=1}^{N} \mathbf{F}^{(i)}_{B-} \left[ \mathbf{G}^{(i)}_{B, B} \mathbf{T}^{(i)}_{B-} + \left( \mathbf{H}^{(i)}_{B, B} + \mathbf{C}^{(i)}_{B} \right) \mathbf{U}^{(i)}_{B-} \right] \mathbf{b}_{IN-} \]