The Relation between the FOURIER Series Method and the \( \frac{\sin x}{x} \) Method for Gravity Interpretations.

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Abstract
The relation between the FOURIER series method and the \( \frac{\sin x}{x} \) method for gravity interpretations has been investigated and it is shown that the two methods are representing the same thing in the limiting case.

\( \text{§} \, 1. \) In 1937, the senior author (Tsuboi: 1937) introduced a direct and objective method for deducing the underground mass distribution \( M(x, y) \) that will produce a gravity anomaly distribution \( \mathcal{G}g(x, y) \) given along the earth's surface. Of course, there can be no unique solution for the problem. In order that a definite answer may be obtained, a simplifying and yet plausible assumption had to be introduced in that method regarding the mass distribution. The assumption made was that the mass \( M(x, y) \) is distributed on a single horizontal plane situated at a depth \( d \) below the earth's surface. This is a justifiable assumption, because in many actual cases, the underground mass anomaly is caused by an undulation of the interface between the lighter and the underlying denser materials and from the gravimetrical point of view, this arrangement can be replaced with a good approximation by a plane mass which is situated at the average depth of the interface and which is equal to the density difference of the two materials multiplied by the amplitude of undulation, so far as the undulation of interface is not very large compared with its average depth. With such an assumption as stated above, it was shown that if \( \mathcal{G}g(x, y) \) is expressed by a double FOURIER series such as

\[
\mathcal{G}g(x, y) = \frac{1}{2\pi k^2} \times \sum_{m,n} B_{mn} \cos \frac{mx}{\sin m \sin n} \exp \sqrt{m^2 + n^2} d, \tag{1}
\]

where \( k^2 \) is the NEWTONIAN constant of gravitation and \( d \) is the depth of the mass plane. In two dimensional case, (1) and (2) reduce themselves respectively into

\[
\mathcal{G}g(x) = \sum_{m} B_{m} \cos \frac{m x}{\sin m}, \tag{2}
\]

\[
M(x) = \frac{1}{2\pi k^2} \sum_{m} B_{m} \cos \frac{m x}{\sin m} \exp md. \tag{3}
\]

In order to deduce \( M(x, y) \) from a given \( \mathcal{G}g(x, y) \) according to this method, one proceeds as follows:
1) \( \mathcal{G}g(x, y) \) is analysed into a double [single] FOURIER series,
2) Each of the FOURIER coefficients in this series is multiplied by appropriate \( \exp \sqrt{m^2 + n^2} d \) [exp \( md \)],
3) The FOURIER series having these new coefficients is synthesized,
4) The value of the series at \( x=x, \, y=y \), \( [x=x] \) is divided by \( 2\pi k^2 \),
5) The final result is the required \( M(x, y) \) \( [M(x)] \).

Although this FOURIER series method for gravity interpretation has proved itself to be very useful and competent in applications (Tsuboi, 1938, 1939, 1940, 1941, 1942, 1948, 1950), admittedly it has a kind of awkwardness.
coming from the very nature of the Fourier series. As in many other instances in which it is used, the Fourier series expression of a quantity implies periodic repetition of the same pattern of distribution of that quantity even outside the domain in which we are interested. This circumstance has caused a kind of hindrance for unconditional usage of the method. In order to get over this difficulty, T. Rikitake (1952) suggested the usage of Hermite function while Y. Sato (1954) proposed a modification of the Fourier series method for this purpose.

§ 2. In 1955, the junior author (Tomoda) and and K. Aki (Tomoda and Aki, 1955) introduced the \( \sin x/x \) method for gravity interpretations with the hope of getting rid of the said awkwardness of the Fourier series method. This function \( \sin x/x \) is characteristic in that it takes a unit value at \( x=0 \) and vanishes at all other grid points which are \( \pi \) apart. It is therefore fitted for representing the distribution of such a quantity as vanishes at all equally spaced grid points except at one of them where it takes a finite value. If applied to gravity problems,

\[
\Delta g(x) = b \frac{\sin x}{x} \quad (5)
\]

represents such a distribution of \( \Delta g(x) \) as it is \( b \) at \( x=0 \) and is 0 at \( x=n\pi \), where \( n \) can be any integer other than zero. Since

\[
\sin x/x = \int_0^1 \cos mx \exp dm \quad (6)
\]

it is readily seen that no harmonic components having higher frequencies than 1 are contained in this function. Making use of the formula (3) and (4), the mass distribution \( M(x) \) that will give rise to the gravity anomaly distribution:

\[
\Delta g(x) = b \frac{\sin x}{x}
\]

is given by

\[
M(x) = \frac{b}{2\pi k^2} \int_0^1 \cos mx \exp mdm. \quad (7)
\]

This can be integrated into

\[
M(x) = \frac{b}{2\pi k^2} \frac{1}{x^2 + d^2} \times \{(d \cos x + x \sin x) \exp d - d\}. \quad (8)
\]

Since \( \sin x \) is zero and \( \cos x \) is either +1 or -1 at \( x=n\pi \) according as \( n \) is even or odd, \( M(x) \) at these points becomes simply

\[
M(n\pi) = \frac{b}{2\pi k^2} \frac{d}{(n\pi)^2 + d^2} (\pm \exp d - 1). \quad (9)
\]

In this formula, \( d \) is expressed in radian in case when the distance between two consecutive grid points is taken as \( \pi \). If this spacing is \( a \) and the depth is \( \delta \) in the actual scale, we may write

\[
d = \delta \pi. \quad (10)
\]

Inserting this in (9), we get

\[
M(na) = \frac{b}{2\pi k^2} \frac{1}{a} \frac{\delta/a}{\pi n^2 + (\delta/a)^2} (\pm \exp \delta \pi/a - 1). \quad (11)
\]

The numerical values of

\[
\phi(n, \delta/a) = \frac{1}{\pi} \frac{\delta/a}{n^2 + (\delta/a)^2} (\pm \exp \delta \pi/a - 1) \quad (12)
\]

for \( \delta/a = 0.5 \) and 1 are given in Table I.

Table I.

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<tr>
<th>( \pm n )</th>
<th>( \delta/a = 0.5 )</th>
<th>( \delta/a = 1.0 )</th>
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Any gravity anomaly distribution can be regarded as the linear superposition of
\[ b_i \sin \frac{x}{x_i}, \]
with the origin of \( x \) shifted appropriately for each \( l \). In a single mathematical expression, the whole distribution can be written as
\[ A_n(x) = \sum_i b_i \sin \left( \frac{x - x_i}{x_i} \right), \quad (13) \]

where \( b_i \) is the anomaly at \( x = x_i \). Consequently the underground mass which is needed for producing this anomaly distribution will take the value
\[ M(x_i) = \frac{1}{2\pi k^2} \sum_{m=1}^{2p-1} \frac{b_i}{(l-m)^2 + \delta^2} \left( \pm \exp \frac{\delta}{a} - 1 \right) \]
\[ = \frac{1}{2\pi k^2} \sum_{m=1}^{2p-1} b_i \Phi(l-m, \delta), \quad (14) \]
beneath \( x = x_i \).

\section{3.}
Thus there have been introduced two methods for the same purpose. The question naturally arises what the relationship between the two methods, the Fourier series method and the \( \sin x/x \) method, is. In what follows, it will be shown that the two methods are all representing the same thing in the limiting case.

The finite Fourier series expression of a function which takes a unit value at \( x=0 \) and which vanishes at all the \((2p-1)\) equally spaced intervening points lying between \( x=0 \) and \( x=2\pi \) is as follows:
\[ f(x) = \frac{1}{2} \left\{ \frac{1}{2} + \cos x + \cos 2x + \cdots \right\} \]
\[ \cdots + \cos (p-1)x + \frac{1}{2} \cos px \}. \quad (15) \]

If this function is regarded to be representing a gravity anomaly distribution, the mass distribution which will give rise to it is given by
\[ M(x) = \frac{1}{2\pi k^2} \frac{1}{\theta} \left\{ \frac{1}{2} \cos x \exp d \right. \]
\[ + \cos 2x \exp 2d + \cdots \]
\[ \cdots + \cos (p-1)x \exp (p-1)d \]
\[ \left. + \frac{1}{2} \cos px \exp pd \right\}, \quad (16) \]

the mass being assumed to be condensed on a single horizontal plane at a depth \( d \) as before. The series (16) can be written as
\[ M(x) = \frac{1}{2\pi k^2} \frac{1}{\theta} \left\{ 1 + \exp (d+ix) \right. \]
\[ + \exp 2(d+ix) + \cdots \]
\[ \cdots + \exp [(p-1)(d+ix)] \]
\[ \left. - \frac{1}{2} + \frac{1}{2} \exp (d+ix) \right\}, \quad (17) \]

where \( R \) means the real part of the expression within the brackets which follow it. Since the terms in the brackets but the last two form a geometrical series with a common ratio \( \exp (d+ix) \), (17) can be written as
\[ M(x) = \frac{1}{2\pi k^2} \frac{1}{\theta} \left\{ 1 - \exp (d+ix) \right. \]
\[ - \frac{1}{2} + \frac{1}{2} \exp (d+ix) \right\}. \quad (18) \]

If \( d \) and \( x \) are both small as compared with 1, this expression is transformed as follows:
\[ M(x) = \frac{1}{2\pi k^2} \frac{1}{\theta} \left\{ 1 - (\cos px + i \sin px) \exp pd \right. \]
\[ - \frac{1}{2} + \frac{1}{2} (\cos px + i \sin px) \exp pd \right\} \]
\[ = \frac{1}{2\pi k^2} \frac{1}{\theta} \left\{ 1 - d + (d \cos px + x \sin px) \exp pd \right. \]
\[ \left. - \frac{1}{2} + \frac{1}{2} \cos px \exp pd \right\}. \quad (19) \]

At the grid points where \( x \) is integral multiple of \( 2\pi/2\theta \), say
\[ x = n \frac{2\pi}{2\theta} \]
\[ = n \frac{2\pi}{2\theta}, \quad (20) \]
\sin \( px \) is zero and \cos \( px \) is either 1 or \(-1\) according as \( n \) is even or odd. Consequently, we get
\[ M\left( \frac{n\pi}{p} \right) = \frac{1}{2\pi k^2} \frac{1}{\theta} \left\{ \frac{d}{d^2 + \left( \frac{n\pi}{p} \right)^2} \right. \]
\[ \left. - \frac{1}{2} + \frac{1}{2} \exp pd \right\} \quad (21) \]

In this expression, \( d \) is expressed in radian. If \( a \) is the actual distance between two consecutive grid points, \( 2\pi \) in radian corresponds to \( 2a \) in the actual scale. So, if \( \delta \) is the actual depth of the mass plane, there is a relation
\[ d = \pi \delta \frac{p}{2a}. \quad (22) \]
Putting (22) into (21) and making \( \rho \) very large, we get finally
\[
M \left( \frac{\pi \rho}{\rho} \right) = \frac{1}{2\pi k^2} \phi(n, \sigma/a)
\]
where
\[
\phi(n, \sigma/a) = \frac{1}{\sigma^a} \frac{\partial}{\partial a} \left( \pm \exp \frac{\sigma}{a} - 1 \right).
\] (23)

This is exactly nothing but the formula (11), which was derived already. Thus it has been proved that the \( \sin \pi x/a \) method may be regarded as the limiting case of the Fourier series method in case the harmonic component in the series is taken up to a very high one.

\section{4.} This conclusion is useful if we wish to extend the \( \sin x/a \) method so as to be applied to three dimensional gravity interpretations. In three dimensional cases, we have to evaluate the integral
\[
\phi = \left[ \int_0^1 \int_0^1 \cos m x \cos n y \exp \sqrt{m^2 + n^2} d \right] d m \, d n,
\]
which corresponds to (7) for two dimensional cases. This expression does not appear to be integrable analytically. From the conclusion obtained in the above, this integral can safely be replaced by a double Fourier series such as
\[
\frac{1}{\rho^2} \left[ \frac{1}{2} + \cos \frac{\pi \rho}{\rho} \exp \frac{\pi}{\rho} \sqrt{\rho^2 + 1^2} d + \cos \frac{2\pi}{\rho} \exp \frac{\pi}{\rho} \sqrt{\rho^2 + 2^2} d + \cdots \right.
\]
\[
\left. + \cos \frac{\pi}{\rho} \exp \frac{\pi}{\rho} \sqrt{\rho^2 + 0^2} d + \cos \frac{\pi}{\rho} \exp \frac{\pi}{\rho} \sqrt{1^2 + 1^2} d + \cdots \right]
\]
\[
+ \cos (\rho - 1) \frac{\pi}{\rho} \exp \frac{\pi}{\rho} \sqrt{(\rho - 1)^2 + 0^2} d + \cos \frac{\pi}{\rho} \exp \frac{\pi}{\rho} \sqrt{(\rho - 1)^2 + 1^2} d + \cdots \right]
\]
\[
+ \cos \frac{\rho - 1}{\rho} \frac{\pi}{\rho} \exp \frac{\pi}{\rho} \sqrt{0^2 + 0^2} d + \cos \frac{\pi}{\rho} \exp \frac{\pi}{\rho} \sqrt{1^2 + 1^2} d + \cdots \right]
\]
\[
+ \cos \frac{1}{\rho} \frac{\pi}{\rho} \exp \frac{\pi}{\rho} \sqrt{\rho^2 + 0^2} d + \cos \frac{\pi}{\rho} \exp \frac{\pi}{\rho} \sqrt{\rho^2 + 1^2} d + \cdots \right]
\]
\[
+ \frac{1}{2} \cos \rho \exp \frac{\pi}{\rho} \sqrt{\rho^2 + 0^2} d + \cos \frac{\pi}{\rho} \exp \frac{\pi}{\rho} \sqrt{\rho^2 + 1^2} d + \cdots \right]
\]
\[
+ \frac{1}{2} \cos \rho \exp \frac{\pi}{\rho} \sqrt{\rho^2 + 1^2} d + \cdots \right)
\] (25)

This function, if divided by \( 2\pi k^2 \), represents a mass distribution at a depth \( d \) which will produce a unit gravity at \( x=0, y=0 \) and zero gravity at all other two dimensional grid points. This double Fourier series can be evaluated numerically (Tsuboi, Oldham, Waitman: 1958).

\section*{References}


The Relation between the FOURIER Series Method and the sin x/x Method.


 Dependence of the Isostatic Depth on the Horizontal Scale of the Topographies to be Compensated. Geophysical Notes, 3, No. 6.