Love Waves in case the Surface Layer is Variable in Thickness

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§ 1. Introduction
Recently, the seismic model study has made a remarkable progress and wave characteristics of various complicated structures have been studied experimentally by many authors. F. Press (1957) investigated flexural waves, which are easily excited and propagated in plates having various shapes (thickness change, lithology change, fault and scarp). A. Takagi (1959) studied Rayleigh waves along the surface of a layered structure of variable thickness and reported many interesting properties of the waves, for example, that when Rayleigh waves are incident from the side of thicker surface layer to that of thinner surface layer, the transmitted Rayleigh waves characteristic for the thinner layer can be easily observed even at points close to the transitional zone, but that when Rayleigh waves are incident from the thinner layer side, the transmitted Rayleigh waves maintain characteristics for the thinner layer even at points distant from the transitional zone.

It is important to treat such problems theoretically. In this paper, Love waves propagated in layered media of which the surface layer is of variable thickness are investigated by means of the method developed by D. S. Jones (1952) for some diffraction problems.

§ 2. The case that Love waves are incident from the side of thicker surface layer to that of thinner surface layer

We consider a layered structure with a surface layer of which the thickness is $H$ for the range $x > 0$ and $h = H - \delta$ for the range $x < 0$ with the origin of coordinates taken along the interface between two media, with the $z$-axis vertically downward (Fig. 1).

\[ v_{n,1} = A \cos \beta_{1,N} H \cdot \exp \left\{ - \beta_{1,N} z - i \kappa_{1,N} (x - x_0) \right\}, \quad z \geq 0, \quad 0 \geq x \geq -H, \]

\[ v_{n,2} = A \cos \beta_{2,N} (x + H) \cdot \exp \left\{ - i \kappa_{1,N} (x - x_0) \right\}, \quad 0 \geq x \geq -H, \]

where

\[ \beta_{1,N} = \sqrt{\kappa_{1,N}^2 - k_1^2}, \quad \beta_{2,N} = \sqrt{k_2^2 - \kappa_{1,N}^2}, \quad k_1 = \omega / V_1, \quad k_2 = \omega / V_2, \]

and $\kappa_{1,N}$ is a root of the equation

\[ \tan \beta_{1,N} H = \gamma \frac{\beta_{1,N}}{\beta_{2,N}}, \quad \gamma = \frac{\mu_1}{\mu_2}, \]

and if we write $\kappa_{1,N} = \omega / C_{1,N}$, $C_{1,N}$ represents the phase velocity of Love waves of the $N$-th mode in a layered structure with a surface layer having the thickness $H$. 


Fig. 1. Velocities of shear waves and rigidities are taken $V_1, \mu_1$ in the substratum and $V_2 (< V_1), \mu_2 (< \mu_1)$ in the surface layer, respectively.

Let us express incident Love waves, omitting the time factor $e^{-i\omega t}$, as follows;
Denote the total displacements by

\[ v = v_0 + v_1, \quad z \geq 0, \quad \infty > x > -\infty, \]  
\[ v = v_0 + v_2, \quad 0 \geq z \geq -h, \quad \infty > x > -\infty, \]  
\[ v = v_0 + v_3, \quad -h \geq z \geq -H, \quad x \geq 0. \]  

Then boundary conditions are given by

(i) \( z = 0, \infty > x > -\infty; \)

\[ v_1 = v_2, \quad \frac{\partial v_1}{\partial z} = \frac{\partial v_2}{\partial z}, \]  

(ii) \( z = -h, x \geq 0; \)

\[ v_2 = v_3, \quad \frac{\partial v_2}{\partial z} = \frac{\partial v_3}{\partial z}, \]  

(iii) \( z = -h, x \leq 0; \)

\[ \frac{\partial v_3}{\partial z} = \frac{\partial v_3}{\partial z} = A \beta_{2,2} \sin \beta_{2,2} \cdot \exp \left\{ -ik_1(x-x_0) \right\}, \]  

(iv) \( z = -H, x \geq 0; \)

\[ \frac{\partial v_2}{\partial z} = 0, \]  

(v) \( x = 0, -h \geq z \geq -H; \)

\[ \frac{\partial v_3}{\partial z} = \frac{\partial v_3}{\partial z} = i A \kappa_{1,1} \cos \beta_{1,1} \cdot \exp \left\{ ik_{1,1}x \right\}. \]

\( v_1 \) satisfies the equation of motion

\[ \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} = \frac{1}{V_1^2} \frac{\partial^2 v_1}{\partial t^2} \]  

and \( v_j \) (\( j = 2, 3 \)) the equation

\[ \frac{\partial^2 v_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial z^2} = \frac{1}{V_j^2} \frac{\partial^2 v_j}{\partial t^2} . \]

If we introduce a virtual resistance \( \varepsilon (\partial v_i / \partial t) \) (\( \varepsilon > 0 \)) to (6) and assume the time factor \( e^{-\iota \omega t} \), equation (6) is reduced to

\[ \frac{\partial^2 v_i}{\partial x^2} + \frac{\partial^2 v_i}{\partial z^2} + k_i^2 v_i = 0, \]

where \( k_i = (\omega^2 + i \omega) / V_i^2 \), that is, \( k_i \) has a positive imaginary part. Hereafter, we consider that \( \mathcal{R} k_i > 0 \) and \( k_i = \omega / V_i \) as the limiting case of \( \varepsilon \rightarrow 0 \). \( v_1 \) and \( v_3 \) satisfy the equation similar to (8), substituting \( k_2 \) in place of \( k_1 \), and \( \mathcal{R} k_2 \) should be positive and \( |k_1| < |k_2| \).

If we write

\[ V_i(\zeta, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v_1 e^{i\eta \zeta} d\eta \quad \text{and} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v\eta e^{i\eta \xi} d\eta \equiv V_1(\zeta, z), \quad \zeta = \zeta + i \eta, \]

then, according to a generalized Fourier's theorem, if \( |v_1| < A e^{v(\gamma, x)} \) as \( x \rightarrow +\infty \) and \( |v_1| < B e^{v(\gamma, x)} \) as \( x \rightarrow -\infty \), then \( V_1(\zeta, z) \) is analytic for \( \gamma > \gamma_- \), \( V_1(\zeta, z) \) is analytic for \( \gamma < \gamma_+ \) and therefore \( V_i(\zeta, z) \) is analytic in the strip \( \gamma_+ > \gamma > \gamma_- \). In our case, since the displacements
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consist of reflected waves at \( x=0 \), transmitted waves and diffracted waves, we can take \( \eta_+ = \mathcal{F} k_1 \) and \( \eta_- = -\mathcal{F} k_1 \).

Applying the Fourier transform in \( x \) to the partial differential equations for \( v_1 \) and \( v_2 \) (cf. (8)), we find that

\[
\frac{d^2}{dz^2} V_1(\zeta, z) - \beta_1^2 V_1(\zeta, z) = 0, \quad \beta_1 = \sqrt{\zeta^2 - k_1^2},
\]

(10)

\[
\frac{d^2}{dz^2} V_2(\zeta, z) - \beta_2^2 V_2(\zeta, z) = 0, \quad \beta_2 = \sqrt{\zeta^2 - k_2^2}.
\]

(11)

These equations have solutions

\[
V_{1+}(\zeta, z) + V_{1-}(\zeta, z) = Be^{-\beta_1 z},
\]

(12)

\[
V_{2+}(\zeta, z) + V_{2-}(\zeta, z) = Ce^{-\beta_2 z} + De^{\beta_2 z},
\]

(13)

where \( B, C \) and \( D \) are functions of \( \zeta \) only. From the boundary condition (5a), we can write

\[
C = \frac{\beta_2 - \gamma \beta_1}{2\beta_2} B, \quad D = \frac{\beta_2 - \gamma \beta_1}{2\beta_2}.
\]

(14)

Hence,

\[
V_{2+}(\zeta, z) + V_{2-}(\zeta, z) = \frac{1}{\beta_2} [\beta_2 \cosh \beta_2 z - \gamma \beta_1 \sinh \beta_2 z] B.
\]

(15)

For brevity, we shall write \( V_{1+}(\zeta), V_{2+}(\zeta), \) etc. instead of \( V_{1+}(\zeta, -h), V_{2+}(\zeta, -h), \) etc. . . .

Differentiate (15) with respect to \( z \) and put \( z = -h \). This gives

\[
V_{2+}'(\zeta) + V_{2-}'(\zeta) = -[\beta_2 \sinh \beta_2 h + \gamma \beta_1 \cosh \beta_2 h] B.
\]

(16)

Eliminating \( B \) from (15) and (16), we obtain

\[
V_{2+}(\zeta, z) + V_{2-}(\zeta, z) = -\frac{1}{\beta_2} \beta_2 \cosh \beta_2 z - \gamma \beta_1 \sinh \beta_2 z \left[ V_{2+}'(\zeta) + V_{2-}'(\zeta) \right],
\]

(17)

\[
F_2(\zeta) = \beta_2 \sinh \beta_2 h + \gamma \beta_1 \cosh \beta_2 h,
\]

(18)

or

\[
V_{2+}(\zeta) + V_{2-}(\zeta) = -\frac{1}{\beta_2} \beta_2 \cosh \beta_2 h + \gamma \beta_1 \sinh \beta_2 h \left[ V_{2+}'(\zeta) + V_{2-}'(\zeta) \right].
\]

(19)

Next, we apply an operator \( \frac{1}{\sqrt{2\pi}} \int_0^\infty dx e^{ix} \) to the partial differential equation for \( v_3 \). We get

\[
\frac{d^2}{dz^2} V_3(\zeta, z) - \beta_2^2 V_3(\zeta, z) = \frac{1}{\sqrt{2\pi}} \left( \frac{\partial v_3}{\partial x} \right)_{x=0} - \frac{i\zeta}{\sqrt{2\pi}} (v_3)_{x=0},
\]

(20)

where \( (\partial v_3/\partial x)_{x=0} \) is given by the boundary condition (5e) but \( (v_3)_{x=0} \) is still an unknown function. In order to eliminate the unknown \( (v_3)_{x=0} \), changing the sign of \( \zeta \) and adding the resulting equation to (20), we find

\[
\frac{d^2}{dz^2} [V_{3+}(\zeta, z) + V_{3-}(-\zeta, z)] - \beta_2^2 [V_{3+}(\zeta, z) + V_{3-}(-\zeta, z)] = \frac{2iA_{k_1,\nu}}{\sqrt{2\pi}} \cos \beta_2 \nu (z + H) \cdot \exp \{i\epsilon_1 \nu x_0 \},
\]

(21)
the solution of which is given, taking into account the boundary condition (5d), by

\[ V_{s+}(\zeta, z) + V_{s-}(-\zeta, z) = E \cosh \beta s (z+H) - \frac{2iA_{k_1, N}}{\sqrt{2\pi}} \frac{\cos \beta s_{z, N}(z+H)}{\zeta^2 - \kappa_1, N^2} \exp \{i\kappa, N x_0\} \]  

(22)

As before, we differentiate (22) with respect to \( z \) and put \( z = -h \). Then we have

\[ V_{s+}'(\zeta) + V_{s-}'(-\zeta) = E \beta s \sinh \beta s \delta + \frac{2iA_{k_1, N} \beta s_{z, N}}{\sqrt{2\pi}} \frac{\sin \beta s_{z, N} \delta}{\zeta^2 - \kappa_1, N^2} \exp \{i\kappa, N x_0\} \]  

(23)

Elimination of \( E \) between (22) and (23) gives

\[ V_{s+}(\zeta, z) + V_{s-}(-\zeta, z) = \frac{1}{\beta s} \cosh \beta s (z+H) \left[ V_{s+}'(\zeta) + V_{s-}'(-\zeta) - \frac{2iA_{k_1, N}}{\sqrt{2\pi}} \frac{\sin \beta s_{z, N} \delta}{\zeta^2 - \kappa_1, N^2} \exp \{i\kappa, N x_0\} \right] \]

- \frac{2iA_{k_1, N}}{\sqrt{2\pi}} \frac{\cos \beta s_{z, N} \delta (z+H)}{\zeta^2 - \kappa_1, N^2} \exp \{i\kappa, N x_0\} ,

(24)

hence, for \( z = -h \),

\[ V_{s+}(\zeta) + V_{s-}(-\zeta) = \frac{1}{\beta s} \coth \beta s \left[ V_{s+}'(\zeta) + V_{s-}'(-\zeta) - \frac{2iA_{k_1, N}}{\sqrt{2\pi}} \frac{\sin \beta s_{z, N} \delta}{\zeta^2 - \kappa_1, N^2} \exp \{i\kappa, N x_0\} \right] \]

- \frac{2iA_{k_1, N}}{\sqrt{2\pi}} \frac{\cos \beta s_{z, N} \delta}{\zeta^2 - \kappa_1, N^2} \exp \{i\kappa, N x_0\}.

(25)

Now, from the boundary condition (5b), we can take

\[ V_{s+}(\zeta) = V_{s+}(-\zeta) \] , \[ V_{s+}'(\zeta) = V_{s+}'(-\zeta) \]

and from the application of an operator \( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \) to (5c), we can get

\[ V_{s-}'(\zeta) = \frac{iA_{k_2, N}}{\sqrt{2\pi}} \frac{\sin \beta s_{z, N} \delta}{\zeta - \kappa_1, N} \exp \{i\kappa, N x_0\} , \] \( (\zeta - \kappa_1, N) < 0 \).

(27)

Using (19) and (25) to (27) and eliminating \( V_{s+}(\zeta) \) and \( V_{s+}'(\zeta) \), we can obtain the following equation of modified Wiener-Hopf form;

\[ \frac{1}{\beta s} \frac{1}{\sinh \beta s \delta} \frac{F_1(\zeta)}{F_1(\zeta)} [V_{s+}'(\zeta) + V_{s-}'(\zeta)] + \frac{1}{\beta s} \coth \beta s \delta \cdot [V_{s+}'(-\zeta) + V_{s-}'(-\zeta)]

+ [V_{s+}(\zeta) - V_{s+}(-\zeta)] - \frac{2iA_{k_1, N}}{\sqrt{2\pi}} \frac{\cos \beta s_{z, N} \delta}{\zeta^2 - \kappa_1, N^2} \exp \{i\kappa, N x_0\} = 0 ,

(28)

where

\[ F_1(\zeta) = \beta s \sinh \beta s H + \gamma \beta s \cosh \beta s H \]

(29)

and \( F_1(\zeta) \) is given by (18).

The factorization is made as follows;

\[ \frac{\beta s \delta}{\sinh \beta s \delta} \frac{F_1(\zeta)}{F_1(\zeta)} = K_+(\zeta) K_-(\zeta) , \]

(30)

where

\[ K_+(\zeta) = K_-(\zeta) = \frac{L_+(\zeta)}{H_+(\zeta)} \prod_{n=1}^{\infty} \frac{(\zeta + \kappa_1, n)}{(\zeta + \kappa_2, n)} , \]

(31)
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\[ H_n(\zeta) = H_n(-\zeta) \] is given by (A6), \( L_n(\zeta) = L_n(-\zeta) \) is given by (A13) in Appendix and \( \kappa_{1,n} \) and \( \kappa_{2,n} \) are zeros of \( F_1(\zeta) \) and \( F_2(\zeta) \), respectively. \( K+(\zeta) \) is regular in \( S(\zeta + k_1) > 0 \) and \( K-(\zeta) \) is regular in \( S(\zeta - k_1) < 0 \). Furthermore, we decompose \( \cosh \beta_2 \delta = f_+(\zeta) + f_-(\zeta) \),

\[ \frac{1}{\beta_2} \cosh \beta_2 \delta = f_+(\zeta) + f_-(\zeta) , \quad (32) \]

where

\[ f_+(\zeta) = f_-(\zeta) = \frac{1}{2k_2 \delta} \sum_{n} \frac{i}{\zeta + k_2} \frac{1}{\zeta + ip_n} \quad (33) \]

and \( p_n \) is given by (A2) in Appendix.

Insert (30) and (32) in (28) and rearrange to find

\[ \frac{K_+ (\zeta)}{\zeta + k_2} V_2' (\zeta) + \frac{i A \beta_{2,n} \sin \beta_{2,n} \delta}{\sqrt{2\pi}} \frac{K_+ (\kappa_{1,n} \zeta)}{\zeta - \kappa_{1,n} \zeta} \exp \{ ik_{1,n} \zeta \} \exp (ip_n \zeta) \]

\[ + \frac{1}{\zeta + k_2} \sum_{n} \frac{i}{p_n} k_2 + ip_n \frac{V_2'(ip_n)}{K_+(ip_n)} = P_-(\zeta) , \quad (34) \]

in which \( V_{2,\cdot}'(\zeta) \) \( [V_{2,\cdot}'(\zeta) = V_{2,\cdot}'(\zeta) + V_{2,\cdot}'(-\zeta)] \) is, of course, a known function and is given by (27). Then, the left-hand side of (34) is regular in the domain \( S(\zeta + k_1) > 0 \) (\( \zeta = \kappa_{1,n} \) is not an irregular point since the left-hand side is zero at this point) and the right-hand side is regular in the domain \( S(\zeta - k_1) < 0 \) and therefore a function, defined by the left- or right-hand side, is regular in the whole of the \( \zeta \)-plane, since the above-mentioned two half-planes overlap. Hence, if the left- and right-hand sides tend to zero as \( \zeta \) tends to infinity in appropriate half-planes, both are zero from Louville's theorem. If we assume \( (\partial V_2 / \partial x) \sim \text{constant} \times x^{-1/2} \) as \( x \to +0 \) on \( z = -h \), then \( |V_{2,\cdot}'(\zeta)| < \text{constant} \times |\zeta|^{-1/2} \). From this edge condition, we can take that each side of equation (34) is identically zero. This edge condition, which is given by the physics of the problem, plays an important rôle in the solution of this problem since it is concerned with the uniqueness of the result.

Thus we can obtain

\[ V_2'(\zeta) = - W_\frac{\zeta + k_2}{K_+(\zeta)} \left[ \frac{i}{\zeta + k_2} \zeta - \sum_{n} \frac{i}{p_n} k_2 + ip_n \chi_n \right] , \quad (35) \]

where

\[ W = A \beta_{2,n} \sqrt{2\pi} \frac{K_+ (\kappa_{1,n} \zeta)}{\kappa_{1,n} \zeta} \sin \beta_{2,n} \delta \exp \{ ik_{1,n} \zeta \} \exp (ip_n \zeta) , \quad (36) \]

\[ \chi_0 = \frac{1}{W} V_2'(k_2) , \quad \chi_n = \frac{1}{W} \frac{V_2'(ip_n)}{K_+(ip_n)} . \quad (37) \]

An infinite set of simultaneous linear algebraic equations for \( \chi_n \) is obtained by setting \( \zeta = ip_m \) \((m = 0, 1, 2, \cdots)\) in (35):

\[ [K_+(k_2)]^{x_0} = - \frac{2k_2}{k_1,n - k_2} - \chi_0 + i \sum_{n} \frac{2k_2}{p_n} \chi_n , \quad (38) \]

\[ [K_+(ip_m)]^{x_m} = - \frac{k_2 + ip_m}{k_1,n - ip_m} - \chi_0 + i \sum_{n} \frac{k_2 + ip_m}{p_n} \chi_n . \quad (39) \]

Furthermore, if we set \( \zeta = -k_2 \) or \( -ip_n \), the following relation is easily found

\[ V_2'(-k_2) = - V_2'(k_2) , \quad V_2'(-ip_n) = - V_2'(ip_n) . \quad (40) \]
We use the Fourier inversion formula to find
\[
v_2 = \frac{1}{\sqrt{2\pi}} \int_{i\epsilon - \infty}^{i\epsilon + \infty} V_2(\zeta, z) e^{-i\zeta z} d\zeta, \quad \mathscr{F} k_1 > c > -\mathscr{F} k_1. \tag{41}
\]

Therefore, from (17), we have
\[
v_2 = -\frac{1}{\sqrt{2\pi}} \int_{i\epsilon - \infty}^{i\epsilon + \infty} \frac{\beta_2 \cosh \beta_2 z - \gamma \beta_1 \sinh \beta_2 z}{F_2(\zeta)} V_2'(\zeta) e^{-i\zeta z} d\zeta. \tag{42}
\]

Next, \(v_3\) in \(x > 0\) is given by
\[
v_3 = \frac{1}{\sqrt{2\pi}} \int_{i\epsilon - \infty}^{i\epsilon + \infty} V_3(\zeta, z) e^{-i\zeta z} d\zeta, \tag{43}
\]
but, since \(V_3(\zeta, z)\) is regular in the lower half-plane,
\[
\int_{i\epsilon - \infty}^{i\epsilon + \infty} V_3(\zeta, z) e^{-i\zeta z} d\zeta = 0, \quad x > 0,
\]
hence, we can also write as
\[
v_3 = \frac{1}{\sqrt{2\pi}} \int_{i\epsilon - \infty}^{i\epsilon + \infty} [V_3(\zeta, z) + V_3(-\zeta, z)] e^{-i\zeta z} d\zeta, \quad x > 0. \tag{44}
\]

From (24) and (27), we find
\[
V_3(\zeta, z) + V_3(-\zeta, z) = \frac{1}{\beta_2} \frac{\cosh \beta_2(z + H)}{\sinh \beta_2 \delta} [V_3'(\zeta) + V_3'(-\zeta)]
\]
\[
- \frac{iA}{\sqrt{2\pi}} \cos \beta_2 \cdot \left( \frac{1}{\zeta - \kappa_1, \kappa} - \frac{1}{\zeta + \kappa_1, \kappa} \right) \exp \{i\kappa_1, \kappa x_0\}. \tag{45}
\]

Then \(v_3\) in \(x > 0\) is given by
\[
v_3 = v_3 + A \cos \beta_2 \cdot \left( \frac{1}{\zeta - \kappa_1, \kappa} - \frac{1}{\zeta + \kappa_1, \kappa} \right) \exp \{i\kappa_1, \kappa x_0\}, \quad x > 0, \tag{46}
\]
\[
\tilde{v}_3 = \frac{1}{\sqrt{2\pi}} \int_{i\epsilon - \infty}^{i\epsilon + \infty} \frac{\cosh \beta_2(z + H)}{\sinh \beta_2 \delta} [V_3'(\zeta) + V_3'(-\zeta)] e^{-i\zeta z} d\zeta. \tag{47}
\]

First, we will investigate the transmitted Love waves, i.e. \(v_2\) in \(x < 0\). In the region \(0 > z > -h\) and \(x\) is large negative, we can close the contour in the upper half-plane. \(V_3'(\zeta)\) has only a pole at \(\zeta = \kappa_1, \kappa\) in the upper half-plane. From (35),
\[
[V_3'(\zeta) - \kappa_1, \kappa] \zeta = \kappa_1, \kappa = -\frac{iA}{\sqrt{2\pi}} \frac{\sin \beta_2 \cdot \exp \{i\kappa_1, \kappa x_0\}}{\beta_2 \cdot \delta}, \tag{48}
\]
then, using the relation (3), it is found that the contribution \(v_{2,1}\) from this pole gives
\[
v_{2,1} = A \sin \beta_2 \cdot \exp \{i\kappa_1, \kappa(x - x_0)\} \exp \{-i\kappa_1, \kappa(x - x_0)\}, \quad x < 0, \tag{49}
\]
which exactly cancels the incident Love wave \(v_{0,2}\).

Denoting the zeros of \(F_2(\zeta)\) in the upper half-plane by \(\kappa_{2, m} (m=1, 2, \cdots)\), these poles give
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\[ v_{2,z} = -\sqrt{\frac{2\pi}{i}} \sum_{m=1}^{\infty} \frac{1}{\beta_{2,m}^2} \beta_{2,m} \cos \beta_{2,m} z - i \beta_{1,m} \sin \beta_{2,m} z V_2'(\kappa_{2,m}) \cdot \exp \{ -i\kappa_{2,m} z \} \]

\[ = -\sqrt{\frac{2\pi}{i}} \sum_{m=1}^{\infty} \frac{C_{2,m}}{U_{2,m}} \frac{1}{\beta_{2,m}} \left( \cos \frac{\beta_{2,m}(x+h)}{\beta_{2,m}} \right) V_2'(\kappa_{2,m}) \cdot \exp \{ -i\kappa_{2,m} x \}, \quad x < 0, \quad (50) \]

where

\[ \beta_{1,m} = \sqrt{\kappa_{2,m}^2 - k_1^2}, \quad \beta_{2,m} = \sqrt{\kappa_{2,m}^2 - \kappa_{3,m}^2}, \]

\[ C_{2,m} = \omega / \kappa_{2,m} \]: phase-velocity of Love waves of the \( m \)-th mode propagated in

the layered structure with a surface layer having thickness \( h \),

\[ U_{2,m} \]: group-velocity of Love waves corresponding to the phase-velocity \( \beta_{2,m} \).

These waves represent the transmitted Love waves.

Next, we will evaluate the reflected Love waves, i.e. \( v_3 \) in \( x > 0 \) (\( v_2 \) for large \( x \) is identical to \( v_3 \)). For the large positive \( x \), we can close the contour in the lower half-plane. In (47), zeros of \( \beta_2 \sin \beta_2 \delta \) and \( \sinh \beta_2 \delta \) are given by \( \zeta = -k_2 \) and \( \zeta = -ip_m \) (\( m = 1, 2, \ldots \)). At these points, however, \( V_2(\zeta) + V_2'(\zeta) = 0 \) as shown in (40). Therefore, these points are not poles. In

the upper half-plane, \( V_2'(\zeta) \) has poles at \( \zeta = -\kappa_{1,m} \) (\( m = 1, 2, \ldots \)) and \( V_2'(\zeta) \) has a pole \( \zeta = -\kappa_{1,N} \). At \( \zeta = -\kappa_{1,N} \),

\[ \left[ V_2'(\zeta)(\zeta + \kappa_{1,N}) \right]_{\zeta = -\kappa_{1,N}} = \frac{iA\beta_{2,N}}{\sqrt{2\pi}} \sin \beta_{2,N}\delta \cdot \exp \{ i\kappa_{1,N} x_0 \}, \quad (52) \]

then, the contribution \( \bar{v}_{3,1} \) from this pole is

\[ \bar{v}_{3,1} = -A \cos \beta_{2,N}(x+H) \cdot \exp \{ i\kappa_{1,N}(x+x_0) \}, \quad x > 0, \quad (53) \]

which cancels the second term in the right-hand side of (46). The contribution \( \bar{v}_{3,2} \) from

the poles \( \zeta = -\kappa_{1,m} \) is easily obtained as

\[ \bar{v}_{3,2} = \sqrt{2\pi} i \sum_{m=1}^{\infty} \cos \beta_{2,m}(x+H) \left( \frac{V_2'((\zeta + \kappa_{1,m}))}{\beta_{2,m}} \right) \cdot \exp \{ i\kappa_{1,m} x \}, \quad x > 0, \quad (54) \]

which represents the reflected Love waves.

\section{3. The case that Love waves are incident from the side of thinner surface layer to that

of thicker surface layer}

Write the incident Love waves as

\[ v_{0,1} = A \cos \beta_{2,m} h \cdot \exp \{ -\beta_{1,m} z + i\kappa_{2,m}(x+x_0) \}, \quad z > 0, \quad (55) \]

\[ v_{0,2} = A \cos \beta_{2,m}(x+h) \cdot \exp \{ i\kappa_{2,m}(x+x_0) \}, \quad 0 \geq z \geq -h, \]

where

\[ \beta_{1,N} = \sqrt{\kappa_{2,N}^2 - k_1^2}, \quad \beta_{2,N} = \sqrt{\kappa_{2,N}^2 - \kappa_{3,N}^2}, \quad (56) \]

and \( \kappa_{2,\nu} \) is a root of the equation

\[ \tan \beta_{2,\nu} h = \frac{\beta_{1,N}}{\beta_{2,\nu}}, \quad (57) \]

Displacements are taken as
then, the given boundary conditions are

(i) \( z = 0, \, \infty > x > -\infty \);

\[ v_1 = v_2, \quad \gamma \frac{\partial v_1}{\partial z} = \frac{\partial v_2}{\partial z}, \]  

(59a)

(ii) \( z = -h, \, x \geq 0 \);

\[ v_3 = A \cdot \exp \{ ik_2 x \}, \quad \frac{\partial v_3}{\partial z} = \frac{\partial v_2}{\partial z}, \]  

(59b)

(iii) \( z = -h, \, x \leq 0 \);

\[ \frac{\partial v_3}{\partial z} = 0, \]  

(59c)

(iv) \( z = -H, \, x \geq 0 \);

\[ \frac{\partial v_3}{\partial z} = 0, \]  

(59d)

(v) \( z = 0, \, -h \geq z \geq -H \);

\[ \frac{\partial v_3}{\partial z} = 0. \]  

(59e)

Proceed in the same way as before up to equation (19). For \( V_3(\zeta, z) \), since \( \frac{\partial v_3/\partial x}{\alpha} = 0 \) in this case, we have, instead of (24),

\[ V_{3+}(\zeta, z) + V_{3+}'(\zeta, z) = \frac{1}{\beta_2} \cosh \frac{\beta_2 (z + H)}{\sinh \beta_2 \delta} \left[ V_{3+}'(\zeta) + V_{3+}'(\zeta) \right]. \]  

(60)

Hence, at \( z = -h \), it is reduced to

\[ V_{3+}(\zeta) + V_{3+}'(\zeta) = \frac{1}{\beta_2} \coth \beta_2 \delta \left[ V_{3+}(\zeta) + V_{3+}'(\zeta) \right]. \]  

(61)

Use the boundary conditions (59a, c) to find

\[ v_{3+}(\zeta, z) - V_{3+}(\zeta) = \frac{i A}{\sqrt{2\pi}} \frac{1}{\zeta + k_2 x} \exp \{ ik_2 x \}, \quad \mathcal{F}(\zeta + k_2 x) > 0, \]  

\[ V_{3-}(\zeta) = 0. \]  

(62)

Thus, from (19) and (61), we obtain the equation, corresponding to (28),

\[ \frac{1}{\beta_2} \frac{1}{\sinh \beta_2 \delta} \frac{F_2(\zeta)}{F_2(\zeta)} V_{3+}(\zeta) + \frac{1}{\beta_2} \coth \beta_2 \delta \cdot V_{3+}'(\zeta) + [V_{3+}(\zeta) - V_{3+}(\zeta)] \]  

\[ = -\frac{i A}{\sqrt{2\pi}} \frac{1}{\zeta + k_2 x} \exp \{ ik_2 x \} = 0. \]  

(63)

According to the same factorization as before, it is found that

\[ K_+(\zeta) V_{3+}(\zeta) + \frac{i A \delta}{\sqrt{2\pi}} \frac{k_2 + k_2 x}{K_+(\zeta + k_2 x)} \frac{1}{\zeta + k_2 x} \exp \{ ik_2 x \} \]  

\[ + \frac{1}{\zeta + k_2} \frac{V_{3+}(\zeta)}{K_+(\zeta) - \sum \frac{i}{\zeta + i p_n} \frac{V_{3+}(i p_n)}{K_+(i p_n)} = P_-(\zeta),} \]  

(64)
and this corresponds to the equation (34). Thus, applying Louville’s theorem, we have

\[ V\beta + (\zeta) = -\frac{W_\beta + h_2}{K_\beta (\zeta)} \left( i \frac{\zeta + h_2}{\zeta + h_2, x} + \frac{1}{\zeta + h_2 - \sum_{n=1}^{\infty} \frac{i}{\zeta + h_2 - ip_n} \zeta + ip_n} \right), \]  

(65)

where, in this case,

\[
W = \frac{A \delta}{\sqrt{2 \pi}} \frac{k_\beta + h_2, x}{K_\beta (k_\beta, x)} \exp \{ i k_\beta x x_0 \}, \\
\lambda_0 = \frac{V_{\beta +} (h_2)}{K_\beta (h_2)}, \\
\lambda_n = \frac{V_{\beta +} (ip_n)}{K_\beta (ip_n)}. 
\]

(66)

Besides, the relations similar to (40) are also obtained as

\[ V_{\beta +} (-h_2) = -V_{\beta +} (h_2), \quad V_{\beta +} (-ip_n) = -V_{\beta +} (ip_n). \]  

(67)

Now, from (17) and (62), we can write

\[ V_{\beta} (\zeta, z) = -\frac{1}{\beta_2} \frac{\beta_2 \cosh \beta_2 z \gamma \beta_1 \sinh \beta_2 z}{F_2 (\zeta)} V_{\beta +} (\zeta), \]

(68)

hence, the displacement \( v_2 \) in \( 0 \leq z \leq -h \) is given by the following integral;

\[ v_2 = \frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \frac{e^{i z \zeta}}{\beta_2} \frac{\cosh \beta_2 (z + H)}{\sinh \beta_2 \theta} \left[ V_{\beta +} (\zeta) + V_{\beta +} (\zeta) e^{-i \zeta \theta} \right] d \zeta, \quad x > 0. \]  

(69)

\( v_3 \) in \( x > 0 \) is also given by (44). Using the relation (60), we have

\[ v_3 = \frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \frac{e^{i z \zeta}}{\beta_2} \frac{\cos \beta_2 (z + H)}{\sin \beta_2 \theta} \left[ V_{\beta +} (\zeta) + V_{\beta +} (\zeta) e^{-i \zeta \theta} \right] d \zeta, \quad x > 0. \]

(70)

The transmitted Love waves can be obtained from the residue of integral in (70). For large positive \( x \), we can complete the contour in the lower half-plane. \( \zeta = -h_2 \) and \( \zeta = -ip_\theta (m=1,2,\cdots) \) are not poles of the integrand according to the relations given by (67). Poles in the lower half-plane are only \( -\kappa_1, m (m=1,2,\cdots) \), which are zeros of \( K_\beta (\zeta) \). Thus, the transmitted Love waves are given by

\[ v_3 = \sqrt{2 \pi} i \sum_{m=1}^{\infty} \frac{\cos \beta_2, m (z + H)}{\beta_2, m \sin \beta_2, m \theta} \left[ V_{\beta +} (\zeta) + V_{\beta +} (\zeta) e^{-i \zeta \theta} \right] \exp \{ i \kappa_1, m x \}, \quad x > 0, \]

(71)

which is of the same form as (54). Poles of integrand in (69) are given by \( \kappa_2, m (m=1,2,\cdots) \), which are zeros of \( F_2 (\zeta) \), in the upper half-plane. Thus, closing the contour in the upper half-plane, we finally obtain

\[ v_3 = -\sqrt{2 \pi} i \sum_{m=1}^{\infty} \frac{1}{\beta_2, m} \frac{\beta_2, m \cos \beta_2, m z - \gamma \beta_1, m \sin \beta_2, m z}{[dF_2 (\zeta)/d\zeta]_{\zeta=\kappa_2, m}} V_{\beta +} (\kappa_2, m) \exp \{ -i \kappa_2, m x \}, \quad x < 0, \]

(72)

as reflected Love waves.
\[ § 4. \textbf{Approximate solutions for the case of small } k_{2}\delta \]

In order to evaluate the reflected and the transmitted Love waves numerically, we must solve the infinite set of simultaneous linear equations, for example as (38) and (39), and even if we could obtain the solutions of the above linear equations, we must also calculate \( K_+(k_{2},m) \) in \( V_2'(k_{2},m) \) (see (50)). Besides, to obtain the exact values of \( K_+(k_{2},m) \), we must get all zeros of \( F_1(\zeta) \) and \( F_2(\zeta) \), defined by (29) and (18), for all frequencies \( \omega \).

Here, we assume that \( k_{2}\delta=k_{0}(H-h) \) is very small, i.e., the difference in thickness is very small compared with the wave-length of waves under consideration.

(a) The case when Love waves are incident from the side of thicker surface layer to that of thinner surface layer (Case I).

When \( \zeta \) is real, \( H_+(\zeta) \), defined by (A3), can be written as

\[
H_+(\zeta) = \left[ \frac{\sinh \beta_2\delta}{\beta_2\delta} \right]^{1/2} \exp \left[ \frac{-\chi(\zeta) + i \sum_{n=1}^{\infty} (\zeta \delta_n - \phi_n)}{\pi} \right] , \quad \phi_n = \tan^{-1} \left( \frac{\zeta}{p_n} \right) .
\] (73)

Hence, if we omit the term of \( O(k_{2}\delta) \) compared with 1, we find

\[
H_+(k_{2}) \approx 1 ,
\] (74)

then

\[
K_+(k_{2}) \approx 1 ,
\] (75)

to the approximation of the same order.

Next, when \( \zeta = ip_m=i/m\pi/\delta \), since

\[
\chi(ip_m) = \frac{p_m\delta}{\pi} \left( 1 - \frac{1}{C - \ln \frac{ip_m\delta}{\pi}} \right) + \frac{p_m\delta}{2} = m(1 - C - \ln m)
\] (76)

(see A7)), we have

\[
H_+(ip_m) = \prod_{n=1}^{\infty} \left( p_n\delta_n + p_m\delta_m \right) \exp \left( -p_m\delta_n - \chi(ip_m) \right) \approx \prod_{n=1}^{\infty} \left( 1 + \frac{m}{n} \right) e^{-m/n} \approx \frac{m^m e^{-m}}{\Gamma(1+m)} .
\] (77)

The remaining terms of \( K_+(ip_m) \) are \( 1 + O(k_{2}\delta) \). Thus we can write

\[
K_+(ip_m) \approx \frac{m^m e^{-m}}{\Gamma(1+m)} .
\] (78)

When \( m \) is very large, \( K_+(ip_m) \approx 1/\sqrt{2\pi m} \). Omit the term of \( O(k_{2}\delta) \) in (38), (39) to find

\[
\left[ 1 + \left( K_+(k_{2}) \right)^2 \right] \chi_0 = -\frac{2ik_{2}}{k_{2} - \kappa_{1,N}} , \quad \left( K_+(ip_m) \right)^2 \chi_m = -i - \chi_0 - \sum_{n=1}^{\infty} \frac{m}{m+n} \chi_n .
\] (79)

Hence, from the above approximations,

\[
\chi_0 \approx \frac{ik_{2}}{k_{2} - \kappa_{1,N}} , \quad \frac{m^m e^{-m}}{\Gamma(1+m)^2} \chi_m \approx \frac{k_{1,N}}{k_{2} - \kappa_{1,N}} - \sum_{n=1}^{\infty} \frac{m}{m+n} \chi_n .
\] (80)
In order to obtain \( V_2'(\zeta) \) given by (35), we must know the value of \( K_+(\kappa_1,\kappa) \). We can easily get \( K_+(\kappa_1,\kappa) \approx 1 \) from the same approximation as \( K_+(\kappa_2) \), taking into account that \( \kappa_1,\kappa_2 \approx \kappa_1,\kappa \approx \kappa_2,\kappa_2 \approx k_2 \). \( K_+(\kappa_3,\kappa_2) \) nearly equals to 1, too.

Thus, \( V_2'(\kappa_2,\kappa) \) in the right-hand side of equation (50) is given by

\[
V_2'(\kappa_2,\kappa) \approx \frac{A_0^2 \beta_{2,\kappa}}{\sqrt{2\pi}} \frac{k_2 + k_2,\kappa}{k_1 + k_1,\kappa} \sin \beta_{2,\kappa} \cdot \exp \left\{ i(k_1,\kappa z_0) \left[ \frac{i}{k_2,\kappa - k_1,\kappa} + \frac{1}{k_2 + k_2,\kappa} \sum_{n=1}^{\infty} \frac{\delta}{n\pi} \right] \right\},
\]

which is of the order \((k_2 \delta)\) for \( m \neq N \). When \( m = N \), i.e., the modes of incident and transmitted Love waves are equal, we can obtain the solution as follows:

Differentiate the equation

\[
\tan \sqrt{k^2 - k_1^2} = \gamma \frac{\sqrt{k^2 - k_1^2}}{\sqrt{k_2^2 - k_2^2}}
\]

with respect to \( l \), keeping \( k_1 \) and \( k_2 \) constant. This gives

\[
\frac{dk}{dl} = \frac{C}{U} - 1
\]

where \( C = \omega / \kappa \) and \( U = d\omega / dk \). Hence, for the same mode and the same frequency,

\[
\kappa_1,\kappa - \kappa_2,\kappa \approx \frac{k_2 \kappa}{(C_{2,\kappa} - 1)} \delta.
\]

Inserting this relation to the expression of \( V_2'(\kappa_2,\kappa) \), we find

\[
V_2'(\kappa_2,\kappa) \approx \frac{IA_0^2 \beta_{2,\kappa}}{\sqrt{2\pi}} \frac{k_2}{k_2,\kappa} \frac{1}{(C_{2,\kappa} - 1)} \sin \beta_{2,\kappa} \cdot \exp \left\{ i(k_1,\kappa z_0) \left[ 1 + O(k_2 \delta) \right] \right\}.
\]

Thus, (50) can be finally written as

\[
v_{2,2} = A \cos \beta_{2,\kappa}(z + \delta) \exp \left\{ -i(k_2,\kappa z + i(k_1,\kappa z_0) \left[ 1 + O(k_2 \delta) \right] \right\}.
\]

This is the transmitted Love waves for small \((k_2 \delta)\). It must be noted that all the waves with different modes from the incident Love waves are of the order \((k_2 \delta)\).

Next, we will estimate the reflected Love waves to the same approximation. From (A17),

\[
\frac{K_+(\zeta)}{(\zeta + k_1, \kappa)} \approx \frac{L_+(\zeta)}{H_+ (\zeta + k_2, \kappa)} \sum_{n=1}^{\infty} \frac{(\zeta + k_1, \kappa)}{(C_{1,\kappa} - 1)} \delta.
\]

where \( H' \) means an infinite product except \( n = m \). According to the same approximate procedure, we have

\[
\left[ \frac{K_+(\zeta)}{(\zeta + k_1, \kappa)} \right]_{\zeta = -k_1, \kappa} \approx \frac{1}{k_2,\kappa - k_1, \kappa} \approx \frac{1}{H} \frac{1}{(C_{1,\kappa} - 1)} \delta.
\]

\[
[V_2'(\zeta + k_1, \kappa)]_{\zeta = -k_1, \kappa} \approx -i \frac{A}{\sqrt{2\pi}} \frac{k_1, \kappa}{\beta_{2,\kappa} H} \frac{k_2 + k_1, \kappa k_1, \kappa}{k_1, \kappa + k_1, \kappa} \left( C_{1, \kappa} - \frac{1}{U_{1, \kappa}} \right) \delta \sin \beta_{2,\kappa} \cdot \exp \left\{ i(k_1,\kappa z_0) \right\}.
\]
Hence, we can obtain, as the reflected Love waves,

\[ \vec{v}_{3,a} = A \sum_{m=1}^{\infty} \cos \beta_{2,m}(x + H) \frac{\kappa_{1,m}}{\beta_{2,m} \beta_{3,m} H} \frac{\kappa_{1,m}}{\kappa_{1,m} + \kappa_{1,m}} \left( \frac{C_{1,m}}{U_{1,m}} - 1 \right) \frac{\sin \beta_{3,m}}{\sin \beta_{2,m}} \exp \{ i\kappa_{1,m}x - i\kappa_{1,m}x_0 \} . \]  

(90)

When \( m = N \), it is reduced to

\[ \vec{v}_{3,a} = A \cos \beta_{2,N}(x + H) \cdot \frac{\delta}{2H} \frac{\kappa_{2,N}^{2} - \kappa_{1,N}^{2}}{\kappa_{2,N}^{2} - \kappa_{1,N}^{2}} \left( \frac{C_{1,N}}{U_{1,N}} - 1 \right) \exp \{ i\kappa_{1,N}(x + x_0) \} . \]  

(91)

(b) The case when Love waves are incident from the side of thinner surface layer to that of thicker surface layer (Case II).

We can evaluate the waves for this case by the same procedure. From (65), the equations corresponding to (80) are

\[ x_0 = \frac{i\kappa}{k_2 + \kappa_2, \omega} , \]

\[ x_n = x_0 \sum_{m=1}^{\infty} \frac{m}{m + 2} \frac{m + 2}{m + 2} \bigg( \frac{1}{m + 2} \bigg)^2 x_m \]  

(92)

Hence,

\[ [V_{3+}(\zeta)(\zeta + \kappa_1, \omega)] = -\frac{iA\delta}{2\pi} \frac{k_2^{2} - \kappa_2, \omega \kappa_2, \omega \kappa_1, \omega}{\kappa_1, \omega - \kappa_2, \omega} \frac{C_1, \omega}{U_1, \omega} \exp \{ i\kappa_2, \omega x_0 \} , \]  

(93)

which is of the order \( (k_2 \delta) \) only when \( m = N \). For \( m = N \), (93) can be reduced to

\[ [V_{3+}(\zeta)(\zeta + \kappa_1, \omega)] = -\frac{iA\delta}{2\pi} \beta_{3, \omega} \exp \{ i\kappa_3, \omega x_0 \} . \]  

(94)

Then, we have, from (71),

\[ v_3 = A \cos \beta_3, \omega(x + H) \cdot \exp \{ i\kappa_3, \omega x + i\kappa_3, \omega x_0 \} . \]  

(95)

All the waves of other modes are of the order \( (k_2 \delta) \).

To the same approximation,

\[ V_{3+}(\zeta, \omega) = -\frac{iA\delta}{2\pi} \frac{k_2^{2} + \kappa_2, \omega \kappa_2, \omega}{\kappa_2, \omega + \kappa_2, \omega} \exp \{ i\kappa_2, \omega x_0 \} . \]  

(96)

Putting this into (72), we finally obtain

\[ v_3 = -A \sum_{m=1}^{\infty} \cos \beta_{2,m}(x + H) \frac{\kappa_{2,m}}{\beta_{2,m} \beta_{3,m} h} \frac{\kappa_{2,m}}{\kappa_{2,m} + \kappa_{2,m}} \left( \frac{C_{2,m}}{U_{2,m}} - 1 \right) \exp \{ -i\kappa_{2,m}x + i\kappa_{2,m}x_0 \} . \]

(97)

When \( m = N \),

\[ v_3 = -A \cos \beta_{2,N}(x + H) \cdot \frac{\delta}{2h} \frac{k_2^{2} + \kappa_2, N \kappa_2, N}{k_2, N + \kappa_2, N} \left( \frac{C_{2,N}}{U_{2,N}} - 1 \right) \exp \{ -i\kappa_{2,N}(x - x_0) \} . \]

(98)

§ 5. Ratio of energy flux of transmitted or reflected Love waves to that of incident Love waves

We could obtain the transmitted and reflected waves for small \( (k_2 \delta) \). These waves are
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summarized as follows:

(a) Case I.

Incident waves;

\[ v_{i,1} = A \cos \beta_{2,1} \cdot H \cdot \exp \left( -\beta_{1,1} z \right) \cos \left[ \omega t + \kappa_{1,1} (x - x_0) \right], \quad z \geq 0, \quad x > 0. \]  
\[ v_{i,2} = A \cos \beta_{2,1} \cdot (z + H) \cdot \cos \left[ \omega t + \kappa_{1,1} (x - x_0) \right], \quad 0 \geq z \geq -H, \quad x > 0. \]  

Transmitted waves;

\[ v_{t,1} = A \cos \beta_{2,1} \cdot H \cdot \exp \left( -\beta_{1,1} z \right) \cos \left[ \omega t + \kappa_{2,1} x - \kappa_{1,1} x_0 \right], \quad z \geq 0, \quad x < 0. \]  
\[ v_{t,2} = A \cos \beta_{2,1} \cdot (z + H) \cdot \cos \left[ \omega t + \kappa_{2,1} x - \kappa_{1,1} x_0 \right], \quad 0 \geq z \geq -H, \quad x < 0. \]  

Reflected waves;

\[ v_{r,1} = A \sum_{m=1}^{\infty} a_m \cos \beta_{2,m} \cdot H \cdot \exp \left( -\beta_{1,m} z \right) \cos \left[ \omega t - \kappa_{1,m} x - \kappa_{1,1} x_0 \right], \quad z \geq 0, \quad x > 0, \]  
\[ v_{r,2} = A \sum_{m=1}^{\infty} a_m \cos \beta_{2,m} \cdot (z + H) \cdot \cos \left[ \omega t - \kappa_{1,m} x - \kappa_{1,1} x_0 \right], \quad 0 \geq z \geq -H, \quad x > 0, \]  

\[ a_m = \frac{k_{2,1} \kappa_{1,m}}{\beta_{2,1} \kappa_{1,m} H} \frac{C_{1,m}}{U_{1,m}} \left( \frac{C_{1,m}}{U_{1,m}} - 1 \right) \sin \beta_{2,m} \cdot \delta. \]  

(b) Case II.

Incident waves;

\[ v_{i,1} = A' \cos \beta_{2,1} \cdot H \cdot \exp \left( -\beta_{1,1} z \right) \cos \left[ \omega t - \kappa_{2,1} x - \kappa_{1,1} x_0 \right], \quad z \geq 0, \quad x < 0. \]  
\[ v_{i,2} = A' \cos \beta_{2,1} \cdot (z + H) \cdot \cos \left[ \omega t - \kappa_{2,1} x - \kappa_{1,1} x_0 \right], \quad 0 \geq z \geq -H, \quad x < 0. \]  

Transmitted waves;

\[ v_{t,1} = A' \cos \beta_{2,1} \cdot H \cdot \exp \left( -\beta_{1,1} z \right) \cos \left[ \omega t - \kappa_{2,1} x - \kappa_{2,1} x_0 \right], \quad z \geq 0, \quad x > 0. \]  
\[ v_{t,2} = A' \cos \beta_{2,1} \cdot (z + H) \cdot \cos \left[ \omega t - \kappa_{2,1} x - \kappa_{2,1} x_0 \right], \quad 0 \geq z \geq -H, \quad x > 0. \]  

Reflected waves;

\[ v_{r,1} = -A' \sum_{m=1}^{\infty} a_m' \cos \beta_{2,m} \cdot H \cdot \exp \left( -\beta_{1,m} z \right) \cos \left[ \omega t + \kappa_{2,m} x - \kappa_{2,1} x_0 \right], \quad z \geq 0, \quad x < 0, \]  
\[ v_{r,2} = -A' \sum_{m=1}^{\infty} a_m' \cos \beta_{2,m} \cdot (z + H) \cdot \cos \left[ \omega t + \kappa_{2,m} x - \kappa_{2,1} x_0 \right], \quad 0 \geq z \geq -H, \quad x < 0, \]  

\[ a_m' = \frac{k_{2,m} \delta}{\beta_{2,m} \kappa_{2,m} H} \frac{k_{2,1} \delta + \kappa_{2,1} \kappa_{2,m}}{\kappa_{2,m} + \kappa_{2,1}} \left( \frac{C_{2,m}}{U_{2,m}} - 1 \right). \]  

Since waves in cases I and II are of the same form as seen from the above equations, we will proceed as follows only for waves in case I.

The rate of energy, which passes the plane perpendicular to \( x \)-axis, per unit time and per unit area, is given by

\[ F = \mp \frac{\partial u}{\partial t} \frac{\partial v}{\partial t}, \]  

\( - \) being taken when waves are propagated to the direction of positive \( x \) and \( + \) to the opposite direction. Hence, we obtain
The energy flow for a certain period $T=2\pi/\omega$ is, from $\bar{F} = \int_0^T F \, dt$, found to be

$$
\begin{align*}
\bar{F}_{0,1} &= \mu_0 A^2 \kappa_{1, \nu} \cos^2 \beta_{2, \nu} H \cdot \exp \{-2\beta_{1, \nu} x\} \sin^2 \left[ \omega t + \kappa_{1, \nu} (x - \xi_0) \right], \quad x > 0, \\
\bar{F}_{0,2} &= \mu_0 A^2 \kappa_{1, \nu} \cos^2 \beta_{2, \nu} H \cdot \exp \{-2\beta_{1, \nu} x\} \sin^2 \left[ \omega t + \kappa_{1, \nu} (x - \xi_0) \right], \quad 0 > x > -H, \\
\bar{F}_{0,3} &= \mu_0 A^2 \kappa_{1, \nu} \cos^2 \beta_{2, \nu} H \cdot \exp \{-2\beta_{1, \nu} x\} \sin^2 \left[ \omega t + \kappa_{1, \nu} (x - \xi_0) \right], \quad x < 0.
\end{align*}
$$

Thus, the total flux of energy across the vertical plane of unit breadth is

$$
\begin{align*}
[T.F.]_{\text{inc}} &= \int_0^\infty \bar{F}_{0,1} \, dz + \int_{-H}^0 \bar{F}_{0,2} \, dz + \frac{1}{2} \mu_0 A^2 \kappa_{1, \nu} \tan \beta_{2, \nu} H \left[ 1 + \frac{2}{\beta_{1, \nu}^2} \right] \\
&= \frac{1}{2} \pi \mu_0 A^2 \kappa_{1, \nu} \frac{\beta_{2, \nu} H}{C_{1, \nu}} \frac{\beta_{2, \nu} H}{U_{1, \nu}} \quad \text{for incident Love waves,} \\
[T.F.]_{\text{trans}} &= \frac{1}{2} \pi \mu_0 A^2 \kappa_{1, \nu} \frac{\beta_{2, \nu} H}{C_{1, \nu}} \frac{\beta_{2, \nu} H}{U_{1, \nu}} \quad \text{for transmitted Love waves,} \quad (106a)
\end{align*}
$$

and

$$
\begin{align*}
[T.F.]_{\text{refl}} &= \frac{1}{2} \pi \mu_0 A^2 \kappa_{1, \nu} \sum_{m=N}^{\infty} a_m^2 \frac{\beta_{2, \nu} H}{C_{1, \nu}} \frac{\beta_{2, \nu} H}{U_{1, \nu}} \quad \text{for reflected Love waves,} \quad (106b)
\end{align*}
$$

Hence,

$$
\begin{align*}
[T.F.]_{\text{trans}} &= \frac{\kappa_{1, \nu} \beta_{2, \nu}^2 h}{\kappa_{1, \nu} \beta_{2, \nu}^2 H} \frac{C_{1, \nu}}{U_{1, \nu}} - 1 \quad \text{for transmitted Love waves,} \quad (107a)
\end{align*}
$$

and

$$
\begin{align*}
[T.F.]_{\text{refl}} &= \sum_{m=N}^{\infty} a_m^2 \frac{C_{1, \nu}}{V_{1, \nu}} - 1 \quad \text{for reflected Love waves,} \quad (107b)
\end{align*}
$$

When $m=N$, i.e., the mode of reflected Love wave is equal to that of incident wave, it is
§ 6. Numerical examples

Fig. 2 shows phase- and group-velocities of Love waves (1st mode) in the layered structure of $\mu_1/\mu_2=2$ and $V_1/V_2=4/3$, and $\frac{[T.F.]_{\text{inc}}}{\frac{1}{2}\pi \mu_2 A^2}$ for $k_2H$ is shown in Fig. 3. We must remember the assumption that $k_2\delta=\omega(H-h)/V_2$ is very small. If we take $\delta/H=\alpha$, then $k_2H=k_2\alpha/\alpha$ and $k_2h=k_2H(h/H)$. Hence, when $\alpha$ is not small, we can discuss only the waves for very small $k_2H$. For an example, if $h/H=0.8$ or $\delta/H=0.2$ and $k_2\delta\leq0.01$, only the waves in the range $k_2H\leq0.05$ or $k_2h\leq0.04$ can be discussed. The smaller the value of $h/H$, the smaller the range that we can discuss for. And for the small values of $\delta/H$ or for the small values of $k_2H$, the value of $\frac{[T.F.]_{\text{trans.}}}{[T.F.]_{\text{inc.}}}$ is, of course, nearly 1.

Fig. 4 shows $\frac{[T.F.]_{\text{refl.}}}{[T.F.]_{\text{inc.}}} \times \{1-(h/H)^2\}$ for the reflected Love waves of the first mode, i.e., $a^2\times(1-(h/H))^2$. From this figure, we can estimate $\frac{[T.F.]_{\text{trans.}}}{[T.F.]_{\text{inc.}}}$ to some extent.

§ 7. Conclusion

We studied the problem of propagation of Love waves in the layered media with a surface layer of variable thickness. It is difficult to see the exact behaviours of Love waves in such media, because we must solve an infinite set of simultaneous linear equations and calculate phase- and group-velocities of Love waves for infinite numbers of modes,
as stated at the beginning of § 4.

We derived approximate solutions for the case that the thickness difference of surface layer is small compared with the wave-length under consideration, leaving the solutions for larger thickness difference in the future.

If the incident Love wave is of the N-th mode, only the transmitted Love wave of the N-th mode predominates and the waves of other modes are small. But the amplitudes of reflected Love waves are of the order (thickness-difference/wave-length) for all modes.

The mean group-velocity \( U_m \) for the total epicentral distance \( \Delta \) is, of course, given by

\[
\Delta/\Delta_U = \Delta_1/U_1 + \Delta_2/U_2,
\]

for longer periods, as illustrated in Fig. 5. The period for Airy phase, of course, differs from that for a uniformly layered structure.

The method used in this paper can be applied to solving the problem of propagation of Love waves in layered structures in which the boundary has a step form, as shown in Fig. 6, with some modifications.

\[\begin{array}{c}
\text{\textbf{\textit{\textbf{\texttt{Fig. 5}}}}}
\end{array}\]

\[\begin{array}{c}
\text{\textbf{\textit{\texttt{Fig. 6}}}}
\end{array}\]

§ 8. Acknowledgements

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APPENDIX. Decomposition of \( \frac{\beta_\delta}{\sinh \beta_\delta} \frac{F_+(\zeta)}{F_-(\zeta)} = K_+(\zeta)K_-(\zeta) \) [cf. (30)]

According to the infinite product theorem (E. C. Titchmarsh (1939), p. 113), we can write

\[
\frac{\sinh \beta_\delta}{\beta_\delta} = \prod_{n=1}^{\infty} \left( p_n \delta + i\zeta \delta_n \right) = H(\zeta),
\]

where

\[
p_n \delta_n = (1 - k_n^2 \delta_n^2)^{1/2} = -i(k_n^2 \delta_n^2 - 1)^{1/2},
\]

\[
\delta_n = \delta/n\pi.
\]

We decompose \( H(\zeta) \) as follows;

\[
H(\zeta) = H_+(\zeta)H_-(\zeta), \quad H_+(\zeta) = \prod_{n=1}^{\infty} \left( p_n \delta_n = i\zeta \delta_n \right) \exp \{ \pm i\zeta \delta_n = \chi(\zeta) \},
\]

taking \( \chi(\zeta) \) as an arbitrary function. When \( \zeta \) is very large,
where $C$ is the Euler's constant and $\Gamma(z)$ the Gamma function. From Stirling's formula, we have

\[ \Gamma\left(1 - i\frac{\zeta_0}{\pi}\right) \sim \sqrt{2\pi} \exp\left\{-\frac{\pi}{4}i + \frac{\zeta_0}{\pi}\left(1 - \ln\frac{\zeta_0}{\pi}\right) - \frac{\zeta_0}{2}\right\}, \]  

\[ (A5) \]

hence,

\[ H_+(\zeta) \sim \frac{1}{\sqrt{2\zeta_0}} \exp\left\{\frac{\pi}{4}i\right\} \exp\left\{-i\frac{\zeta_0}{\pi}\left(1 - C - \ln\frac{\zeta_0}{\pi}\right) + \frac{\zeta_0}{2} - \chi(\zeta)\right\}. \]

\[ (A6) \]

Thus, if we choose

\[ \chi(\zeta) = -i\frac{\zeta_0}{\pi}\left(1 - C - \ln\frac{\zeta_0}{\pi}\right) + \frac{\zeta_0}{2}, \]

\[ (A7) \]

then, $|H_+(\zeta)| \sim |\zeta|^{-1/2}$ as $|\zeta| \to \infty$. Since

\[ H_-(\zeta) = H_+(\zeta), \]

\[ (A8) \]

$H_-(\zeta)$ has the same behaviour for $|\zeta| \to \infty$ as $H_+(\zeta)$.

If we write all zeros of $F_1(\zeta)$ and $F_2(\zeta)$, given by (29) and (18), as $\pm\kappa_{1,n}$ and $\pm\kappa_{2,n}$ $(n=1, 2, 3, \cdots)$, we can take

\[ F_1(\zeta) = \prod_{n=1}^{\infty} (\zeta^2 - \kappa_{1,n}^2) G_1(\zeta), \quad F_2(\zeta) = \prod_{n=1}^{\infty} (\zeta^2 - \kappa_{2,n}^2) G_2(\zeta), \]

\[ (A9) \]

where

\[ G_1(\zeta) = F_1(\zeta)/\prod_{n=1}^{\infty} (\zeta^2 - \kappa_{1,n}^2), \quad G_2(\zeta) = F_2(\zeta)/\prod_{n=1}^{\infty} (\zeta^2 - \kappa_{2,n}^2), \]

\[ (A10) \]

and both functions have no zero. Furthermore, we decompose $L(\zeta) = G_1(\zeta)/G_2(\zeta)$ as follows;

\[ L(\zeta) = \frac{G_1(\zeta)}{G_2(\zeta)} = L_+(\zeta)L_-(\zeta), \]

\[ (A11) \]

then, $L_+(\zeta)$ is given by the integral

\[ \ln L_+(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln L(w)}{w - \zeta} dw, \]

\[ (A12) \]

where we take the contour $\Gamma$ as shown in Fig. Al, according to the definition of the upper sheet of the Riemann-plane ($\Re \beta_1 > 0, \Re \beta_2 > 0$). Hence,

\[ \ln L_+(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln F_1(w)}{w - \zeta} dw - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln F_2(w)}{w - \zeta} dw \]

\[ = \frac{1}{\pi} \int_{0}^{\infty} \frac{\phi_1 - \phi_2}{u - i\zeta} du \left[ k_1 \phi_1 - \phi_2 \right] du, \]

\[ (A13) \]

where

\[ \tan \phi_1 = \frac{\gamma \sqrt{\mu^2 + k_1^2} \cos \sqrt{\mu^2 + k_1^2} H}{\sqrt{\mu^2 + k_1^2} \sin \sqrt{\mu^2 + k_1^2} H}, \]

\[ \tan \phi_2 = \frac{\gamma \sqrt{k_1^2 - \mu^2} \cos \sqrt{k_1^2 - \mu^2} H}{\sqrt{k_1^2 - \mu^2} \sin \sqrt{k_1^2 - \mu^2} H}, \]

\[ (A14) \]
and $\phi_0$, $\phi_\pi$ are given by the equations similar to (A14), taking $\hbar$ in place of $H$. And we have

$$L_-(\zeta) = L_+(\zeta).$$  \hspace{1cm} \text{(A15)}

From (A1) and (A9), if we write

$$\frac{\beta_0}{\sinh \beta_0} \frac{F_1(\zeta)}{F_2(\zeta)} = K_+(\zeta)K_-(\zeta) = \frac{L_+(\zeta)L_-(\zeta)}{H_+(\zeta)H_-(\zeta)} \prod_{n=1}^{\infty} \frac{(\zeta^2 - \kappa_1^2)}{(\zeta^2 - \kappa_2^2)},$$  \hspace{1cm} \text{(A16)}

we can finally obtain

$$K_+(\zeta) = K_-(\zeta) = \frac{L_+(\zeta)}{H_+(\zeta)} \prod_{n=1}^{\infty} \frac{(\zeta + \kappa_1)}{(\zeta + \kappa_2)},$$  \hspace{1cm} \text{(A17)}

and $|K_+(\zeta)|$, $|K_-(\zeta)| \sim |\zeta|^{1/2}$ as $|\zeta| \to \infty$.

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