Formulations of the Complex Variable Boundary Element Method for Potential Flow with Sink and Source Singularities

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The complex variable boundary element method (CVBEM) owes its elegance, the computational accuracy and efficiency, to the Cauchy's integral formula, and it can be interpreted as a semi-analytical scheme. Despite its sound mathematical foundations, the real performance of the CVBEM may be affected by its computational foundations. This study focuses on two of such fundamental aspects: the formulations of the CVBEM and the treatment of sink and source singularities. Here proposed are three types of formulations for the CVBEM, and their performances are assessed. Formulation I using known-variable equivalence may accumulate errors on unscheduled boundaries resulting in severe oscillation in error profiles. Formulation II using unknown-variable equivalence yields error profiles that are smoother and smaller than those of Formulation I. Formulation III using dual-variable equivalence yields the best accuracy among the three. By virtue of the semi-analytic nature of the CVBEM, the singularity programming, which separates singular and non-singular solutions, can be applied to the CVBEM. Non-singular flow behaves smoothly, and the CVBEM yields accurate non-singular solutions. A complete solution is obtained by adding an analytic singular solution to the non-singular solution. With the singularity programming, the CVBEM can handle the sink and source singularities without a loss of accuracy.

Keywords
Potential flow, Simulation, Boundary element method, Singularity, Streamline

1. Introduction

The boundary element methods (BEM) can be categorized into two distinct classes: the real variable BEM (RVBEM) and the complex variable BEM (CVBEM). The first class has been used in many engineering disciplines, such as electrostatics, elastostatics, elastodynamics, and flow mechanics. Although the second class has drawn attention only recently, its mathematical backbone, the complex variable theory, has long been a vital tool particularly in flow mechanics. It appears fair to consider that Hunt and Isaacs were the first to present a complete idea of the CVBEM. They utilized the Cauchy's integral formula to formulate the integral equation applicable to two-dimensional steady-state flow. The main advantages of this method are its compactness and the fact that the solution yields both the velocity potential and the stream function. A notable successor was Hromadka, who investigated several aspects of the CVBEM, including (but not limited to) higher-order basis functions, error analysis, multiply connected domains, regional heterogeneity, advective transport, and approximative boundary, and compiled those results in a series of textbooks.

From an application point of view, the CVBEM has been applied to many flow problems in the realm of petroleum engineering. The method is suitable to track streamlines because of its ability to evaluate the stream function directly. Sato developed a CVBEM code for tracking streamlines and compared its performance with the RVBEM. The superiority of the CVBEM over the RVBEM was confirmed in computational accuracy and efficiency. In addition, the CVBEM can handle discrete heterogeneities, such as high-conductivity fractures and no-flow barriers. Sato et al. applied the CVBEM to examine the consistency between fracture and anisotropic systems. Sutopo et al. utilized the CVBEM to compute benchmark performances of displacement problems in fractured reservoirs. The applicability of the CVBEM is being broadened to many flow problems, and the application aspects have frequently been reported in recent years.

The fundamental aspects of the CVBEM, in contrast, have not yet been well investigated. Since the CVBEM is still a developing technique, it is important...
to examine its computational foundations for further development. The current study focuses on two fundamental aspects of the CVBEM: the formulations for potential flow and the treatment of sink and source singularities. First, mathematical preliminaries of the CVBEM are reviewed, after which three types of formulations are described. Next, the treatment of sinks and/or sources located at boundaries is considered. Through application problems, performances of the CVBEM are assessed for potential flow with sink and source singularities.

2. Mathematical Preliminaries of the CVBEM

For incompressible and irrotational fluid flow through a two-dimensional homogeneous domain, the components of velocity of the fluid \((v_x, v_y)\) can be given by

\[
\begin{align*}
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0 \\
\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} &= 0
\end{align*}
\] (1)

where \(\Phi\) is called the velocity potential and is analytic, that is, its derivative exists at all points in a flow domain. Since the fluid is incompressible, the equation of continuity can be written as

\[
\nabla^2 \Phi = 0
\] (2)

Combining Eqs. (1) and (2) yields the Laplace equation.

\[
\nabla^2 \Phi = 0
\] (3)

2.1. Cauchy’s Integral Formula

Let us consider another analytic function \(\Psi\) which satisfies the Cauchy–Riemann equations

\[
\begin{align*}
\frac{\partial \Phi}{\partial x} &= \frac{\partial \Psi}{\partial y} \\
\frac{\partial \Phi}{\partial y} &= -\frac{\partial \Psi}{\partial x}
\end{align*}
\] (4)

\(\Psi\) is called the stream function. With the Cauchy–Riemann equations, it can be easily shown that the stream function also satisfies the Laplace equation.

\[
\nabla^2 \Psi = 0
\] (5)

2.2. Discretization

Let us associate the complex number \(z = x + iy\) with a point \((x, y)\) in the \(xy\)-plane and consider a complex function \(\Phi\) that has \((v_x, v_y)\) in its real part and \(\Psi\) in its imaginary part

\[
\Phi = \Psi
\] (6)

\(\Omega\) is called the complex velocity potential and satisfies the Laplace equation.

\[
\nabla^2 \Omega = 0
\] (7)

A necessary condition\(^{21}\) that \(\Omega\) be analytic in a flow domain is that in the same domain \(\Phi\) and \(\Psi\) satisfy the Cauchy–Riemann equations, which are met in Eq. (4). For \(\Omega\) analytic inside and along a simple closed boundary \(\Gamma\), the Cauchy’s integral formula\(^{10}\) is satisfied

\[
\Omega(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\Omega(\zeta)}{\zeta - z} d\zeta
\] (8)

where \(\zeta\) is on \(\Gamma\) and the contour integral is calculated by traversing \(\Gamma\) so that the domain of interest lies on the left (Fig. 1).

This formula is the basis of the CVBEM and states that the value of \(\Omega\) at a point \(z\) interior to \(\Gamma\) is completely determined by knowing the values of \(\Omega\) on \(\Gamma\). Although a theoretical extension to three-dimensional problems is being attempted\(^{11}\), the basic CVBEM is applicable only to two-dimensional problems due to the use of complex variables.

For the Laplace operator \(\nabla^2\), the RVBEM has been widely and successfully used. The RVBEM is formulated in terms of \(\Phi\) and its normal gradient \(\frac{\partial \Phi}{\partial n}\), based on an integral formula known as the Green’s second identity\(^{1}\). \(\Psi\) is not included in the formulation. The resultant boundary integral equation can be interpreted as only the real part of the Cauchy’s integral formula; thus, the RVBEM does not fully utilize the information that the Cauchy’s integral can provide. Including the imaginary part of the integral in the formulation is achieved in the CVBEM technique.

2.2. Discretization

The contour integral appeared in the Cauchy’s integral formula Eq. (8) cannot always be evaluated analytically. To overcome this difficulty, the contour integral is approximated by the summation of line integrals. The boundary \(\Gamma\) is discretized into \(n_b\) straight-line elements, and the approximate expression for Eq. (8) is obtained as

\[
\Omega(z) = \frac{1}{2\pi i} \sum_{j=1}^{n_b} \oint_{\Gamma_j} \frac{\Omega(\zeta)}{\zeta - z} d\zeta
\] (9)

where \(\Delta \Gamma_j\) is the \(j\)-th boundary element and \(\Gamma = \Delta \Gamma_1 \cup \Delta \Gamma_2 \cup \cdots \cup \Delta \Gamma_{n_b}\).

Along the \(n_b\) boundary elements, interpolation functions must be given to define variations of the complex velocity potential. The interpolation scheme used in this study is the linear elements. The nodal points,
where the boundary values are set, are allocated at the edges of the elements and the boundary values are linearly interpolated in between. The linear elements require \( n_b \) nodal points and provide a continuous profile of \( \Omega \) at intersection of the elements. While the node-numbering direction is arbitrary, the positive direction is chosen in accordance with the Cauchy’s integral formula. In the case of a filled domain, for instance, the positive direction is the counterclockwise direction.

**Figure 2** shows a sample discretization of the boundary \( \Gamma \) (dashed line) into \( n_b \) boundary elements (solid lines). The \( j \)-th element \( \Gamma_j \) is defined by the nodal points \( \zeta_j \) and \( \zeta_{j+1} \), and the complex velocity potential along \( \Gamma_j \) is interpolated as

\[
\Omega(\zeta) = \frac{(\zeta - \zeta_j)\Omega_{j+1} + (\zeta_{j+1} - \zeta)\Omega_j}{\zeta_{j+1} - \zeta_j}, \quad \zeta \in \Gamma_j \tag{10}
\]

where \( \Omega_j \) and \( \Omega_{j+1} \) are the complex velocity potential at \( \zeta_j \) and \( \zeta_{j+1} \), respectively.

Substituting Eq. (10) into the line integral in Eq. (9) yields

\[
\int_{\Delta \Gamma_j} \frac{\partial \Omega}{\partial \nu} d\Gamma = \int_{\Delta \Gamma_j} \frac{d\zeta}{\zeta - z} \Omega(\zeta) \tag{11}
\]

The first integral can be evaluated as below and is denoted as \( h_j \).

\[
h_j = \int_{\Delta \Gamma_j} \frac{d\zeta}{\zeta - z} = \ln \frac{\zeta_{j+1} - z}{\zeta_j - z} + i\alpha_j(z) \tag{12}
\]

where \( \alpha_j(z) \) is the angle at an internal point \( z \) (Fig. 2). In a similar way, the second integral is evaluated as

\[
\int_{\Delta \Gamma_j} \frac{d\zeta}{\zeta - z} = \int_{\Delta \Gamma_j} \left(1 + \frac{z}{\zeta - z}\right) d\zeta = \zeta_{j+1} - \zeta_j + zh_j \tag{13}
\]

Thus, Eq. (9) becomes

\[
\Omega(z) = \frac{1}{2\pi i} \sum_{j=1}^{n_b} (\Omega_{j+1} - \Omega_j) \left(\frac{1}{\zeta_j - z} - \frac{1}{\zeta_{j+1} - z}\right) + \frac{1}{2\pi i} \sum_{j=1}^{n_b} \left(\frac{z - \zeta_j}{\zeta_{j+1} - \zeta_j} - \frac{z - \zeta_{j+1}}{\zeta_j - \zeta_{j+1}}\right) \Omega_j h_j \tag{14}
\]

Since the first summation becomes zero, the discretized form of the Cauchy’s integral formula with linear elements can be written as

\[
\Omega(z) = \frac{1}{2\pi i} \sum_{j=1}^{n_b} \left(\frac{z - \zeta_j}{\zeta_{j+1} - \zeta_j} - \frac{z - \zeta_{j+1}}{\zeta_j - \zeta_{j+1}}\right) \Omega_j h_j \tag{15}
\]

### 2.3. Nodal Equation

For well-posed problems, only half of the boundary conditions are prescribed satisfactorily because \( \Phi \) and \( \Psi \) are related to each other. For instance, Fig. 1 shows that the boundary values \( \Phi \) are given on the boundary \( \Gamma_F \) and the boundary values \( \Psi \) are given on \( \Gamma_S \), where \( \Gamma_F \cup \Gamma_S = \Gamma \) and \( \Gamma_F \cap \Gamma_S = \emptyset \). The boundary values \( \Psi \) on \( \Gamma_F \) and \( \Phi \) on \( \Gamma_S \) are not known.

When \( z \) is located within the domain, Eq. (9) (or Eq. (15)) is of no use to compute \( \Omega \) due to the incompleteness of boundary values. Contrarily, if \( z \) is allocated on the boundary, Eq. (9) can be used for the purpose of obtaining the unprescribed boundary values.

Let \( \zeta \) be a nodal point and consider \( z \rightarrow \zeta \). Due to the term \( \zeta - z \) in the denominator of the integrand in Eq. (9), a difficulty arises for singularity. **Appendix A** shows how to avoid the singularity and derives a nodal equation Eq. (A.5) for \( \zeta \). For the velocity potential, the imaginary part of Eq. (A.5) gives

\[
2\pi i \Phi(\zeta) = \ln \frac{\zeta_{j+1} - \zeta_j}{\zeta_j - \zeta_{j+1}} \Psi_k + (2\pi - \theta_k) \Phi_k \tag{16}
\]

and for the stream function, the real part gives

\[
-2\pi i \Psi(\zeta) = \ln \frac{\zeta_{j+1} - \zeta_j}{\zeta_j - \zeta_{j+1}} \Phi_k - (2\pi - \theta_k) \Psi_k \tag{17}
\]

where

\[
\begin{align*}
H_j &= \frac{\zeta_{j+1} - \zeta_j}{\zeta_j - \zeta_{j+1}} \ln \frac{\zeta_{j+1} - \zeta_j}{\zeta_j - \zeta_{j+1}} \\
I_j &= \frac{\zeta_{j+1} - \zeta_j}{\zeta_j - \zeta_{j+1}} \ln \frac{\zeta_{j+1} - \zeta_j}{\zeta_j - \zeta_{j+1}} \tag{18}
\end{align*}
\]

### 3. Formulations of the CVBEM

Equations (16) and (17) are used along with the prescribed boundary values to evaluate unprescribed
boundary values. To this end, three kinds of formulations can be provided. Table 1 summarizes the equivalence equations and the corresponding nodal equations, where the following notations are used.

Φ(ζ_k) + iΨ(ζ_k): approximate boundary values at ζ_k based on Eqs. (16) and (17)

Φ_k + iΨ_k: boundary values at ζ_k used in Eqs. (16) and (17)
P_k + iS_k: prescribed boundary values at ζ_k

Although the basic idea of these formulations is given in Ref. 4), their performances have not been well examined. In this section, details of the individual formulations are first discussed, and then, their performances are assessed.

3.1. Formulation I: Known-variable Equivalence

When only the velocity potential P_k is prescribed at the nodal point ζ_k, Φ(ζ_k) and Φ_k are presumed to be equal to P_k:

Φ(ζ_k) = Φ_k = P_k (19)

Substituting Eq. (19) into Eq. (16) gives

(20)

and ĵ_k is left unknown. Equation (17) is not utilized since we assume that the approximate boundary value ĵ(ζ_k) may not be equal to ĵ_k.

In a similar way, when only the stream function S_k is prescribed at the nodal point ζ_k, Ψ(ζ_k) and Ψ_k are presumed to be equal to S_k:

Ψ(ζ_k) = Ψ_k = S_k (21)

Substituting Eq. (21) into Eq. (16) gives

(22)

and ĵ_k is left unknown. Equation (17) is not utilized since we assume that the approximate boundary value ĵ(ζ_k) may not be equal to ĵ_k.

3.2. Formulation II: Unknown-variable Equivalence

When only the velocity potential P_k is prescribed at the nodal point ζ_k, the counterpart nodal equation ĵ(ζ_k) is equated with ĵ_k. The approximate boundary value Φ(ζ_k) is not forced to be equal to P_k, and Φ_k is presumed to be equal to P_k:

Φ(ζ_k) = P_k

Substituting Eq. (23) into Eq. (17) gives

(24)

and Ψ_k is left unknown. Since the approximate boundary value Ψ(ζ_k) has no constraint, Eq. (16) is not utilized.

In a similar way, when only the stream function S_k is prescribed at the nodal point ζ_k, Ψ(ζ_k) is equated with Ψ_k. The approximate boundary value Φ(ζ_k) is not forced to be equal to S_k, and Ψ_k is presumed to be equal to S_k:

(25)

Substituting Eq. (25) into Eq. (16) gives

(26)

and Φ_k is left unknown. Since the approximate boundary value Ψ(ζ_k) has no constraint, Eq. (17) is not utilized.

### Table 1 Summary of Three Kinds of Formulations

<table>
<thead>
<tr>
<th>P_k is given</th>
<th>S_k is given</th>
</tr>
</thead>
<tbody>
<tr>
<td>equivalence</td>
<td>nodal equation</td>
</tr>
<tr>
<td>I</td>
<td>Φ(ζ_k) = Φ_k = P_k</td>
</tr>
<tr>
<td>II</td>
<td>Φ_k = P_k</td>
</tr>
<tr>
<td>III</td>
<td>Φ(ζ_k) = P_k</td>
</tr>
</tbody>
</table>
In either of the boundary-condition cases, one of the unprescribed boundary-value equations (Eq. (17) when $P_k$ is given or Eq. (16) when $S_k$ is given) is used. The result is a set of $n_b$ equations for $n_b$ nodal points, which yields unique solutions for $n_b$ unprescribed boundary values.

### 3.3. Formulation III: Dual-variable Equivalence

Formulation I and Formulation II use either Eq. (16) regarding the velocity potential or Eq. (17) regarding the stream function. Such formulation is sufficient to yield a set of $n_b$ equations for $n_b$ unprescribed boundary values.

The third approach is to use Eqs. (16) and (17) simultaneously. When only the velocity potential $P_k$ is prescribed at the nodal point $\zeta_k$, $\Phi(\zeta_k)$ is presumed to be equal to $P_k$. The boundary value $\Psi_k$ is not forced to be equal to $P_k$. As for the stream function, the approximate boundary value $\psi(\zeta_k)$ is equated with $\Psi_k$:

\[
\begin{align*}
\Phi(\zeta_k) &= P_k \\
\Psi'(\zeta_k) &= \Psi_k
\end{align*}
\]

Substituting Eq. (27) into Eqs. (16) and (17) gives

\[
\begin{align*}
2\pi P_k &= \ln \frac{\zeta_{k+1} - \zeta_k}{\zeta_{k-1} - \zeta_k} \Psi_k + (2\pi - \theta_k) \Phi_k \\
&+ \sum_{j=1}^{n_b} \left( \text{Re}[H_j \Psi_{j+1}] + \text{Im}[H_j \Phi_{j+1}] \right) \\
&- \text{Re}[\{I_j \Psi_{j-1} - \text{Im}[I_j \Phi_{j-1}]\}] \\
0 &= \ln \frac{\zeta_{k+1} - \zeta_k}{\zeta_{k-1} - \zeta_k} \Phi_k + \theta_k \Psi_k \\
&+ \sum_{j=1}^{n_b} \left( \text{Re}[H_j \Phi_{j+1}] - \text{Im}[H_j \Psi_{j+1}] \right) \\
&- \text{Re}[\{I_j \Phi_{j-1} + \text{Im}[I_j \Psi_{j-1}]\}]
\end{align*}
\]

In a similar way, when only the stream function $S_k$ is prescribed at the nodal point $\zeta_k$, $\Psi(\zeta_k)$ is presumed to be equal to $S_k$. The boundary value $\phi_k$ is not forced to be equal to $S_k$. As for the velocity potential, the approximate boundary value $\Phi(\zeta_k)$ is equated with $\Phi_k$:

\[
\begin{align*}
\Phi(\zeta_k) &= \Phi_k \\
\Psi'(\zeta_k) &= S_k
\end{align*}
\]

Substituting Eq. (29) into Eqs. (16) and (17) gives

\[
\begin{align*}
0 &= \ln \frac{\zeta_{k+1} - \zeta_k}{\zeta_{k-1} - \zeta_k} \Psi_k - \theta_k \Phi_k \\
&+ \sum_{j=1}^{n_b} \left( \text{Re}[H_j \Psi_{j+1}] + \text{Im}[H_j \Phi_{j+1}] \right) \\
&- \text{Re}[\{I_j \Psi_{j-1} - \text{Im}[I_j \Phi_{j-1}]\}] \\
-2\pi S_k &= \ln \frac{\zeta_{k+1} - \zeta_k}{\zeta_{k-1} - \zeta_k} \Phi_k - (2\pi - \theta_k) \Psi_k \\
&+ \sum_{j=1}^{n_b} \left( \text{Re}[H_j \Phi_{j+1}] - \text{Im}[H_j \Psi_{j+1}] \right) \\
&- \text{Re}[\{I_j \Phi_{j-1} + \text{Im}[I_j \Psi_{j-1}]\}]
\end{align*}
\]

In either of the boundary-condition cases, $\Phi_k$ and $\Psi_k$ are left as unknown values at $\zeta_k$ ($2n_b$ unknown boundary values in total). Unlike the two previous formulations, both boundary-value equations (Eqs. (16) and (17)) are used. The result is a set of $2n_b$ equations for $n_b$ nodal points, which yields unique solutions for $2n_b$ unknown boundary values.

### 3.4. Numerical Results

The problem considered here is the flow around a cylinder in a right-angled bend, as shown in Fig. 3. The velocity potential is varying smoothly along the boundary $DEF$, which causes flow from the boundary $FE$ to the boundary $ED$. There is no flow across the boundary $FABCD$. A possible solution that satisfies these conditions is

\[
\begin{align*}
\Phi(\zeta_k) &= \Phi_k \\
\Psi'(\zeta_k) &= S_k
\end{align*}
\]

(31)

It is assumed that the stream function along the boundary $FABCD$ and the potential function along the boundary $DEF$ are prescribed. To solve this problem, the CVBEM with various $n_b$ values (Table 2) is applied.

#### 3.4.1. Errors in Boundary Solutions

To assess the accuracy of the CVBEM, computational errors along the boundary are examined. Using Formulation I with $n_b = 16$, the boundary solutions are obtained. Figure 4 shows the CVBEM results ($\Phi(\zeta_k)$ and $\Psi(\zeta_k)$) and errors. Since Formulation I equates $\Phi(\zeta_k)$ with $P_k$ along the boundary $DEF$, the CVBEM yields no error (Fig. 4 (left)). As for $\Psi(\zeta_k)$ along the boundary $DEF$, on the contrary, a large amount of error is observed (Fig. 4 (right)). The reverse is true along the boundary $FABCD$, no error in $\Psi(\zeta_k)$ and a large amount of error in $\Phi(\zeta_k)$. The sum of the absolute values of errors in the velocity potential ($E_{\Phi}$) and the stream function ($E_{\psi}$) are 2.026 and 1.497, respectively.

It is expected that the results approach the analytical solution as $n_b$ increases. Figure 5 shows the solutions and errors with $n_b = 80$. Note that the scale for
errors is different from Fig. 4. The solution with 80 nodes is 10 times more accurate than the one with 16 nodes; the total errors $E_{\Phi}$ and $E_{\Psi}$ are reduced to 0.228 and 0.156, respectively.

In either case ($n_b=16$ or 80), severe oscillation is observed in the error profiles. Forcing the approximate boundary values to be equal to the prescribed boundary values yields such errors in unprescribed boundary values. The most difficult variables to reproduce are the velocity potentials at the nodal points $D$ and $F$. To reduce errors further, finer boundary elements must be arranged around these nodal points.

The dashed lines in Fig. 6 summarize the effect of $n_b$ on errors $E_{\Phi}$ and $E_{\Psi}$. As $n_b$ increases, the total errors $E_{\Phi}$ and $E_{\Psi}$ decrease. The influence of $n_b$ on $E_{\Phi}$ and $E_{\Psi}$ is not linear; the degree of error reduction becomes smaller for larger values of $n_b$.

Figure 7 shows the solutions and errors for Formulation II with $n_b=80$. Since Formulation II does not equate approximate boundary values ($\Phi(\zeta_k)$ or $\Psi(\zeta_k)$) with prescribed boundary values ($P_k$ or $S_k$), the
CVBEM yields errors at all the nodal points. However, the total errors $E_\Phi$ and $E_\Psi$ are 0.095 and 0.110, respectively, which are less than those of Formulation I.

The error profiles are relatively smooth; the severe oscillation observed with Formulation I (Fig. 5) is not present. Errors are more evenly distributed because Formulation II does not force the approximate boundary values to be equal to the prescribed boundary values. The most difficult variable to reproduce is the stream function at the nodal point $E$.

The effect of $n_b$ on errors is summarized by the dotted lines in Fig. 6. Compared with Formulation I, Formulation II exhibits improvement in accuracy. As is the case with Formulation I, however, the degree of error reduction becomes smaller for larger values of $n_b$.

Figure 8 shows the solutions and errors for Formulation III with $n_b = 80$. Since Formulation III equates $\Phi(\xi)$ with $P_i$ along the boundary DEF, the CVBEM yields no error (Fig. 8 (left)), while a small amount of error is observed in $\Psi(\xi)$ (Fig. 8 (right)). The reverse is true along the boundary FABCD; no error in $\Psi(\xi)$ and a small amount of error in $\Phi(\xi)$. The total errors $E_\Phi$ and $E_\Psi$ are 0.063 and 0.038, respectively.

The effect of $n_b$ on errors is summarized by the solid lines in Fig. 6. Formulation III yields further improvement in accuracy from Formulation II. It can be concluded that Formulation III is the most accurate and thus the most efficient one among the three formulations for reducing boundary-solution errors.

3.4.2. Errors in Internal Solutions

The accuracy of the CVBEM must be assessed for internal solutions as well as for boundary solutions.

With the CVBEM, $\Omega(\zeta)$ are computed at 8198 evenly-spaced points within the flow domain. Figure 9...
shows the errors in $\Phi$ (left) and $\Psi$ (right) for Formulation I with $n_b = 80$. Larger errors are observed near the boundary. This can be explained according to the maximum modulus theorem\(^2\), implying that the maximum error necessarily occurs for $\zeta$ on the boundary. Away from the boundary by the length of a single element, errors tend to vanish rapidly.

It should be recognized that the errors in $\Phi$ near the boundary $DEF$ are not zero despite the fact that the errors in boundary solutions $\Phi(\zeta_k)$ along the boundary $DEF$ are zero. This is because the internal solutions $\Phi(z)$ are affected by the stream function $\Psi_k$ as well as by the velocity potential $\Phi_k$. The same is true for $\Psi(z)$ near the boundary $FABCD$.

**Figure 10** shows the errors for Formulation II with $n_b = 80$. Unlike the result with Formulation I, the errors are not concentrated near the boundary but are smeared over the whole flow domain.

**Figure 11** shows the errors for Formulation III with $n_b = 80$. Considering the small magnitude of errors and the overall error profiles (no error concentration near the boundary), Formulation III appears to be the best among the three for computing internal solutions.

**Figure 12** shows the contour plots of $\Phi(z)$ (left) and $\Psi(z)$ (right) computed by Formulation III with $n_b = 80$, which is accurate enough for practical purposes. Similar plots can be obtained with the analytical solution Eq. (31), which cannot be distinguished from **Fig. 12** because of the high accuracy of the CVBEM.

### 4. Treatment of Sink and Source Singularities

The sink and source singularity damages computational accuracy, in particular, when the singularity lies so close to the boundary that the linear interpolation along the boundary is no longer appropriate. To overcome this problem, the singularity is deducted from the complete solution and the remaining solution is
Fig. 10 Internal Errors Using Formulation II with \( n_b = 80 \): (left) \( \Phi \) and (right) \( \Psi \)

Fig. 11 Internal Errors Using Formulation III with \( n_b = 80 \): (left) \( \Phi \) and (right) \( \Psi \)

Fig. 12 Contour Plots of (left) \( \Phi \) and (right) \( \Psi \)
obtained by the CVBEM. This process can be carried out because the analytic nature of the solution is preserved through the Cauchy’s integral formula in the CVBEM.

4.1. Singular Solutions of Sinks and Sources

The singular solution due to a sink located at \( z_{w_j} \) with a flow rate \( q_{w_j} (>0) \) can be written as

\[
\Omega_{w_j}(z) = \frac{q_{w_j}}{2\pi h} \ln(z - z_{w_j})
\]  

(32)

where \( h \) is the thickness of the flow medium. For the source singularity, the same equation can be used with a negative value of \( q_{w_j} \).

When multiple sinks and/or sources are involved in the problem, the singular solution can be obtained by use of superposition. For a \( n_w \)-sink/source system, the singular solution becomes

\[
\Omega^* = \sum_{j=1}^{n_w} \Omega_{w_j}(z) = \sum_{j=1}^{n_w} \frac{q_{w_j}}{2\pi h} \ln(z - z_{w_j})
\]  

(33)

4.2. Singularity Programming

According to the principle of superposition, a complete solution \( \Omega \) can be interpreted as a sum of singular and non-singular solutions, \( \Omega^* \) and \( \Omega^{**} \), respectively, that is

\[
\Omega = \Omega^{**} + \Omega^*
\]  

(34)

The so-called singularity programming approach lets us achieve the separation of sink and source singularities and non-singular solutions. The procedure is a sequence of desuperposition and superposition of singularities.

By use of desuperposition, the boundary values for the non-singular solution can be obtained. When the velocity potential \( \Phi_k \) is specified at the nodal point \( \zeta_k \), the non-singular boundary value \( \Phi^{**}_k \) becomes

\[
\Phi^{**}_k = \Phi_k - \Phi^*_k
\]  

(35)

where \( \Phi^*_k \) is given as a real part of Eq. (33):

\[
\Phi^*_k = \sum_{j=1}^{n_w} \text{Re} \left[ \Omega_{w_j}(\zeta_k) \right] = \sum_{j=1}^{n_w} \frac{q_{w_j}}{2\pi h} \ln|\zeta_k - z_{w_j}|
\]  

(36)

Similarly, when the stream function \( S_k \) is specified, the non-singular boundary value \( \Psi^{**}_k \) becomes

\[
\Psi^{**}_k = S_k - \Psi^*_k
\]  

(37)

where \( \Psi^*_k \) is given as an imaginary part of Eq. (33):

\[
\Psi^*_k = \sum_{j=1}^{n_w} \text{Im} \left[ \Omega_{w_j}(\zeta_k) \right] = \sum_{j=1}^{n_w} \frac{q_{w_j}}{2\pi h} \theta(\zeta_k, z_{w_j})
\]  

(38)

where \( \theta(\zeta_k, z_{w_j}) \) is the angle enclosed between the vector \( \zeta_k - z_{w_j} \) and the branch cut.

The non-singular solution \( \Omega^{**} \) under these boundary conditions Eq. (35) and/or Eq. (37) can be determined with the CVBEM, then, by superposition, the complete solution is obtained as

\[
\Omega = \Omega^{**} + \sum_{j=1}^{n_w} \Omega_{w_j}
\]  

(39)

4.3. Treatment of Logarithmic Singularity

When the sink (or source) \( z_{w_j} \) is located at a nodal point \( \zeta_k \), there arises a problem of logarithmic singularity since \( \zeta_k - z_{w_j} \) becomes 0. This can be dealt with by considering the actual sink geometry. The sink has a finite length of radius \( r_{w_j} \), and the interior of the sink must be excluded from the flow domain.

Figure 13 shows how to avoid the singularity problem. The nodal point \( \zeta_k \) cannot be inside of the sink, because the interior of the sink is not included in the flow domain. In this figure, \( \zeta_k \) is tentatively shifted to \( \zeta'_k \); the intersection between the sink circumference and the boundary element \( \Gamma_k \). This makes \( |\zeta'_k - z_{w_j}| = r_{w_j} \) and the singularity problem in Eq. (36) can be avoided. In addition, taking a branch cut out of the flow domain, the angle in Eq. (38) takes its principal value \( 0 \leq \theta(\zeta_k, z_{w_j}) < 2\pi \).

4.4. Numerical Results

As a commonly encountered sink and source arrangement, a quarter of a repeated five-spot pattern (Fig. 14) is frequently used in studying source-to-sink displacement performances. Let us consider a flow domain \( ABCD \) \((1 \times 1)\) with a constant matrix permeability. A source is located at \( z_{w_1} = 0 + 0i \) and a sink is located at \( z_{w_2} = 1 + 1i \). The source and sink radii, \( r_{w_1} \) and \( r_{w_2} \), are both 0.001. Injection and production rates are considered constant and are -1 and 1, respectively, and, hence, steady state will be reached when the system is totally equalized. Since a quarter of the pattern is considered, the flow rate from the source to the sink is 0.25. Therefore, the stream function along the boundary \( ABC \) can be set as 0.25, while the stream function along the boundary \( CDA \) is 0.0.

4.4.1. Boundary Solutions

The boundary \( ABCD \) is represented by arranging 8 equally-spaced boundary elements. To avoid the logarithmic singularity, the nodal point \( \zeta_k \) that coincides with \( z_{w_1} \) is shifted to a tentative node \( \zeta'_1 = r_{w_1} + 0i = 0.001 + 0i \) and the nodal point \( \zeta_k \) that coincides with \( z_{w_2} \) is shifted to a tentative node \( \zeta'_2 = (1 - r_{w_2}) + i = 0.999 \).
+ 1i. The branch cut on \( zw_1 \) is taken in the direction of \((-1, 0)\) and that on \( zw_2 \) is taken in the direction of \((1, 0)\), so that the arguments \( \theta'(\zeta', zw_1) \) and \( \theta'(\zeta', zw_2) \) can take their principal values.

Figure 15 compares the CVBEM (Formulation III) results with the exact solutions. Error profiles are also shown in Fig. 15. Average errors are \( E_{\Phi/nb} = 0.00177 \) for the velocity potential and \( E_{\Psi/nb} = 0.00255 \) for the stream function. The true \( \Phi \) varies within \(-1.0563 \leq \Phi \leq 1.0563\) and the variation range is 2.1126. The true \( \Psi \) varies within \( 0 \leq \Psi \leq 0.25\) and the variation range is 0.25. Thus, the relative errors (average error/variation range) are 0.1% and 1% for \( \Phi \) and \( \Psi \), respectively. The difference in the velocity potential between the sink and the source is computed as 2.1042. The exact value is known to be 2.1126, and the error is only 0.4%.

Even with the small number of \( nb \), the computational accuracy is satisfactory. This means that the singularity programming functions effectively. Figure 16 shows desuperposed singular and non-singular solutions. At the points \( A \) and \( C \), the singular solutions exhibit drastic variation due to the sink and source singularities. In contrast, the non-singular solutions behave smoothly at these points. In the singularity programming, only the non-singular part is dealt with by a numerical scheme and the singular part is handled analytically. Thus, the CVBEM with \( nb = 8 \) yields the results accurate enough for practical purposes.

Increasing \( nb \) results in further reduction in error, as shown in Fig. 17 with \( nb = 40 \). The relative errors are 0.003% for \( \Phi \) and 0.02% for \( \Psi \). The difference in the velocity potential between the sink and the source is computed as 2.1121, which agrees well with the analytic solution for this problem.

4.4.2. Internal Solutions

Figure 18 shows (left) the equi-potential lines and (right) the streamlines. The equi-potential lines are drawn contouring \( \Phi \) computed at 101 x 101 points evenly spaced within the flow domain. The streamlines are tracked from the source point \( zw_1 \) by means of the stream-function method. The displacing fluid (tracer) is indicated by solid lines and the displaced fluid by broken lines. Flow in the vicinity of the sink and source points are computed without any difficulty.

The tracer effluent concentration \( C(t) \) at the sink after a certain injection time \( t \) can be evaluated by comparing the injection time with the time \( t_{ib} \) required for the \( i \)-th streamline to reach the sink. If \( t_{ib} \) is smaller than \( t \), the flow rate \( q_{i} \) assigned to this streamline contributes to the effluent concentration. When \( ns \) streamlines are tracked, this can be written as

\[
C(t) = \frac{1}{q} \sum_{i=1}^{ns} q_{i} H(t - t_{ib})
\]

where, \( H(x) \) is the Heaviside step function, defined as
Fig. 16  Singular and Non-singular Solutions: (left) $\Phi$ and (right) $\Psi$

Fig. 17  Boundary Solutions and Errors with $n_b=40$: (left) $\Phi$ and (right) $\Psi$

Fig. 18  Internal Solutions: (left) $\Phi$ and (right) $\Psi$

Figure 19 (left) compares the CVBEM result with the exact solution. In this figure, a dimensionless quantity, pore volumes injected \( V_p \), is used instead of time. The agreement is excellent. In particular, the breakthrough time (0.718 pore volumes injected) is accurately computed by the model.

These results demonstrate the accuracy of the CVBEM and the proper functionality of the singularity programming. For comparison purposes, the results obtained by a finite difference model with \( 10 \times 10 \) and \( 30 \times 30 \) grid systems are shown in Fig. 19 (right). Due to numerical dispersion, the breakthrough time is estimated earlier than the actual.

5. Summary

Three types of CVBEM formulations are presented and their computational accuracy is assessed. Formulation I using known-variable equivalence yields exact boundary solutions for prescribed boundary values, but may accumulate errors on unprescribed boundaries, resulting in severe oscillation in error profiles. Formulation II using unknown-variable equivalence does not yield exact boundary solutions even for prescribed boundary values. However, error profiles are relatively smooth and the magnitude of errors is less than that of Formulation I. Formulation III using dual-variable equivalence yields the best accuracy among the three.

As for the internal solutions, errors are maximal in the vicinity of the boundary and tend to vanish away from the boundary. Such a profile is manifested with Formulation I. Formulation II yields rather smeared error profiles. Among the three formulations, Formulation III, again, is the most efficient for reducing errors of internal solutions.

The treatment of sink and source singularities in the CVBEM is presented and its computational accuracy is assessed. By virtue of the semi-analytic nature of the CVBEM, the singularity programming can be utilized to separate singular solutions from non-singular solutions. Non-singular flow behaves smoothly, and the CVBEM yields the non-singular solution with a high degree of accuracy. The complete solution can be obtained by adding analytic solutions for singularities and the non-singular solutions. The treatment of logarithmic singularity is also presented and it is shown that sinks and sources located on the boundary can be properly modelled in the CVBEM.

Appendix: Nodal Equation

To deal with the singularity problems associated with the nodal equation, rewrite Eq. (9) as

\[
2\pi i \Omega(z) = \int_{\partial \Gamma_1} \frac{\Omega(\zeta)}{\zeta - z} d\zeta + \int_{\partial \Gamma_2} \frac{\Omega(\zeta)}{\zeta - z} d\zeta + \sum_{j=1}^{\infty} \int_{\partial \Gamma_j} \frac{\Omega(\zeta)}{\zeta - z} d\zeta 
\]

(A.1)

Applying Eqs. (11), (12), and (13) to the first integral and let \( z \to \zeta_k \), it becomes

\[
\lim_{z \to \zeta_k} \int_{\partial \Gamma_j} \frac{\Omega(\zeta)}{\zeta - z} d\zeta = \lim_{z \to \zeta_k} \left[ \Omega_k - \Omega_{k-1} + \frac{(z - \zeta_{k-1}) \Omega_k - (z - \zeta_k) \Omega_{k-1}}{\zeta_k - \zeta_{k-1}} h_{k-1} \right]
\]
In a similar way, the second integral becomes

\[
\frac{\Omega_k - \Omega_{k-1} + \Omega_k \lim_{z \to \zeta_k} \ln \frac{\zeta_k - z}{\zeta_{k-1} - z}}{z - \zeta_k} \tag{A.2}
\]

The summation of the first and the second integrals cancels out the \(\lim_{z \to \zeta_k} \) terms

\[
\lim_{z \to \zeta_k} \left[ \int_{\Delta z_{k-1}} \frac{\Omega(\zeta)}{\zeta - z} \, d\zeta + \int_{\Delta z_k} \frac{\Omega(\zeta)}{\zeta - z} \, d\zeta \right] = \Omega_{k-1} - \Omega_k + \ln \frac{\zeta_k - z}{\zeta_{k-1} - z} \Omega_k \tag{A.3}
\]

\[
\Omega_{k+1} - \Omega_k = \ln \frac{\zeta_{k+1} - \zeta_k}{\zeta_k - \zeta_{k-1}} \Omega_k + i(2\pi - \theta_k) \Omega_k \tag{A.4}
\]

where \(\theta_k\) is the internal angle at the nodal point \(\zeta_k\).

Finally, when \(z \to \zeta_k\), Eq. (A.1) can be evaluated as

\[
2\pi i \Omega(\zeta_k) = \ln \frac{\zeta_{k+1} - \zeta_k}{\zeta_k - \zeta_{k-1}} \Omega_k + i(2\pi - \theta_k) \Omega_k
\]

\[
+ \sum_{j=1}^{k-1} \frac{(\zeta_k - \zeta_j) \Omega_{j+1} - (\zeta_k - \zeta_{j+1}) \Omega_j}{\zeta_{j+1} - \zeta_j} \ln \frac{\zeta_{j+1} - \zeta_k}{\zeta_j - \zeta_k} \tag{A.5}
\]

Equation (A.5) relates \(\Omega(\zeta_k)\) at the nodal point \(\zeta_k\) with the boundary values \(\Omega_j\).

References

要 旨

坑井特異点を含むポテンシャル流れに対する複素変数境界要素の定式化

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複素変数境界要素法（CVBEM）は、その計算精度と計算効率の高さをコーシーの積分公式に拠っており、準解析的な数値解法としらべることができる。そのような確固たる数値基礎を伴っているものの、CVBEM の実際の計算精度は計数基礎に左右され得るものである。本研究は、そのような基礎的側面のうち、CVBEM の定式化と坑井特異点の処理に焦点を据えたものである。

CVBEM の定式化には三種類考えられ、それぞれの計算精度を評価した。定式化 I は既知量を等置するものであり、未知境界値誤差を蓄積し、結果的に激しい誤差振動を招く危険性がある。これと比較して、未知量等置を用いる定式化 II は、誤差の蓄積と振動を小さく抑えることができる。定式化 III は未知量ならびに既知量を等置するものであり、三種類の定式化の中では最良の計算精度を示した。

特異点の処理に関しては、特異解と非特異解を区別する特異性プログラミングを CVBEM に適用することを考える。非特異な流れの数値は清らかであるから、CVBEM はその解を高精度で与え、完全解は解析的特異解を非特異解に重ね合わせることによって求められる。このように、CVBEM に特異性プログラミングを組み合わせることにより、坑井特異点を精度を損ねることなく取り扱うことができる。