THE CONSTRUCTION OF A BI-MODAL PROPOSITIONAL LOGIC S2-S2 AND ITS DECISION METHOD

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The following paper concerns (1) the construction of a bi-modal propositional logic S2-S2, which is an extension of Lewis S2 in the sense it is provided with two necessities or possibilities. The paper also gives (2) the solution of the decision problem of the proposed logic. (For bi-modal logics refer to such works as Rubin[14], Fitting[1], Ishimoto-Fujikawa [5], Ishimoto-Watanabe[6], Uchida[17], Ono[10], Rennie[13], Šehtman[16], [17], Yamamoto[18] and Ishimoto[4]. Philosophically bi-modal logic goes back to Hintikka[2].)

§ 1. The system S2-S2 The class of well-formed formulas (wff) is defined in the well-known way on the basis of a countable collection of propositional variables $p, q, r, \dots$ and four logical symbols, namely, $\wedge$ (conjunction), $\sim$ (negation), $\Diamond_1$ (possibility 1) and $\Diamond_2$ (possibility 2). The wffs will be denoted by such meta-logical variables as $A, B, C, \dots$. We also use such abbreviations as the following; $A \vee B = \sim(\sim A \wedge \sim B)$, $A \supset B = \sim(A \wedge \sim B)$, $A \equiv B = (A \supset B) \wedge (B \supset A)$, $A \exists_1 B = \sim \Diamond_1(A \wedge \sim B)$, $A \exists_2 B = \sim \Diamond_2(A \wedge \sim B)$, $A \equiv_1 B = (A \exists_1 B) \wedge (B \exists_1 A)$, $A \equiv_2 B = (A \exists_2 B) \wedge (B \exists_2 A)$, $\Box_1 A = \sim \Diamond_1 \sim A$, $\Box_2 A = \sim \Diamond_2 \sim A$.

The axioms consists of the following wffs:

- AS22.1 $\vdash_{S2-S2} (p \wedge q) \rightarrow_1 (q \wedge p),$
 AS22.2 $\vdash_{S2-S2} (p \wedge q) \rightarrow_1 p,$
 AS22.3 $\vdash_{S2-S2} p \rightarrow_1 (p \wedge p),$
 AS22.4 $\vdash_{S2-S2} ((p \wedge q) \wedge r) \rightarrow_1 (p \wedge (q \wedge r)),$
 AS22.5 $\vdash_{S2-S2} ((p \rightarrow_1 q) \wedge (q \rightarrow_1 r)) \rightarrow_1 (p \rightarrow_1 r),$
 AS22.6 $\vdash_{S2-S2} ((p \rightarrow_2 q) \wedge (q \rightarrow_2 r)) \rightarrow_2 (p \rightarrow_2 r),$
 AS22.7 $\vdash_{S2-S2} (p \wedge (p \rightarrow_2 q)) \rightarrow_2 q,$
 AS22.8 $\vdash_{S2-S2} \Diamond_1(p \wedge q) \rightarrow_1 \Diamond_1 p,$
 AS22.9 $\vdash_{S2-S2} \Diamond_2(p \wedge q) \rightarrow_2 \Diamond_2 p,$
 AS22.10 $\vdash_{S2-S2} \sim \Diamond_1 p \rightarrow_1 \sim \Diamond_2 p.$

Transformation rules:

- TR1. Uniform substitution,
 TR2. Adjunction: $\vdash_{S2-S2} A$ and $\vdash_{S2-S2} B \Rightarrow \vdash_{S2-S2} A \wedge B,$
 TR3. Substitution of strict equivalents:
 $\vdash_{S2-S2} \dots A \dots$ and $\vdash_{S2-S2} A \equiv_1 B \Rightarrow \vdash_{S2-S2} \dots B \dots ,$
 TR4. Strict detachment:
 $\vdash_{S2-S2} A$ and $\vdash_{S2-S2} A \rightarrow_1 B \Rightarrow \vdash_{S2-S2} B,$
 TR5. $\vdash_{S2-S2} \sim \Diamond_2 A \Rightarrow \vdash_{S2-S2} \sim \Diamond_1 A.$

In this system we can establish the following derived rules, meta-theorems and theorems, on account of the lack of space the proofs of which will be omitted.

- DR1. $\vdash_{S2-S2} A$ and $\vdash_{S2-S2} A \rightarrow_2 B \Rightarrow \vdash_{S2-S2} B,$
 DR2. $\vdash_{S2-S2} \dots A \dots$ and $\vdash_{S2-S2} A \equiv_2 B \Rightarrow \vdash_{S2-S2} \dots B \dots ,$

- MT1. $\vdash_{S2-S2} \sim \Diamond_1 A \Leftrightarrow \vdash_{S2-S2} \sim \Diamond_2 A$
 MT2. $\vdash_{S2-S2} A \rightarrow_1 B \Leftrightarrow \vdash_{S2-S2} A \rightarrow_2 B,$
 MT3. $\vdash_{PC} A \Rightarrow \vdash_{S2-S2} \Box_1 A,$
 MT4. $\vdash_{PC} A \Rightarrow \vdash_{S2-S2} \Box_2 A.$

(Here PC denotes the classical propositional calculus or logic. For the laborious proof of MT3 refer to Hughes-Cresswell [3], pp. 216-226.)

The following two meta-theorems are the S2-S2 versions of celebrated Becker's Rule;

- MT5. $\vdash_{S2-S2} A \rightarrow_1 B \Rightarrow \vdash_{S2-S2} \sim \Diamond_1 B \rightarrow_1 \sim \Diamond_1 A,$
 MT6. $\vdash_{S2-S2} A \rightarrow_2 B \Rightarrow \vdash_{S2-S2} \sim \Diamond_2 B \rightarrow_2 \sim \Diamond_2 A.$

It is remarked in passing that we have the following two meta-theorem in S2-S2, both of which are demonstrated by induction on the length of the proof.

MT7. $\vdash_{S2} A \Rightarrow \vdash_{S2-S2} A'$ (A' is identical with A excepting that $\Diamond$'s in A are replaced either by $\Diamond_1$ or by $\Diamond_2$.)

MT8. $\vdash_{S2-S2} A \Rightarrow \vdash_{S2} A^*$ (A^* is identical with A excepting that $\Diamond_1$'s and $\Diamond_2$'s in A are replaced by $\Diamond$.)

Another ten theorems are also listed for reference.

- TS22.1 $\vdash_{S2-S2} \sim \Diamond_1 (p \wedge \sim p),$
 TS22.2 $\vdash_{S2-S2} \sim \Diamond_2 (p \wedge \sim p),$
 TS22.3 $\vdash_{S2-S2} \sim \Diamond_1 p \rightarrow_1 (p \rightarrow_1 q),$
 TS22.4 $\vdash_{S2-S2} \sim \Diamond_2 p \rightarrow_2 (p \rightarrow_2 q),$

- TS22.5 $\vdash_{S2-S2} (p \wedge (p \rightarrow_1 q)) \rightarrow_1 q,$
 TS22.6 $\vdash_{S2-S2} p \rightarrow_1 \Diamond_1 p,$
 TS22.7 $\vdash_{S2-S2} p \rightarrow_2 \Diamond_2 p,$
 TS22.8 $\vdash_{S2-S2} (\Diamond_1 p \vee \Diamond_1 q) \equiv_1 \Diamond_1 (p \vee q),$
 TS22.9 $\vdash_{S2-S2} (\Diamond_2 p \vee \Diamond_2 q) \equiv_2 \Diamond_2 (p \vee q),$
 TS22.10 $\vdash_{S2-S2} (\Diamond_2 p \vee \Diamond_2 q) \equiv_1 \Diamond_2 (p \vee q).$

By the following meta-theorem S2-S2 is seen to be consistent with respect to $\sim$;

$$MT9. \vdash_{S2-S2} A \Rightarrow \not\vdash_{S2-S2} \sim A$$

This is proved by the well-known method of collapsing. (cf. Hughes-Cresswell [3], pp. 41-42.)

§ 2. The decision method of S2-S2 In this section we shall be concerned with the decision problem of S2-S2, which will be solved by the matrix method essentially following McKinsey [9], Lemmon [7], Rasiowa [11] and Rasiowa-Sikorski [12].

DEFINITION 1. A matrix $M = \langle K, D, -, \cap, P_1, P_2 \rangle$ is a normal S2-S2 matrix if the following conditions are satisfied:

1.1 K is a Boolean algebra with respect to $\cap$, $-$ with an equality relation defined for the elements of K ,

1.2 If x, y are elements of K , then $x < y$ (i.e., $x = x \cap y$) iff $x \rightarrow_1 y \in D$ (Here $x \rightarrow_1 y$ stands for $\neg P_1(x \cap y)$. Other abbreviations will be stated presently.),

1.3 D is an additive ideal of K (i.e., D is a non-empty pro-

per subset of K such that, if $x, y \in D$ then $x \cap y \in D$, and if $x \in D$ and $y \in K$, then $x \cup y \in D$),

- 1.4 $\neg P_1 0 \in D$,
- 1.5 If $x \in K$ and $\neg P_1 x \in D$, then $x = 0$,
- 1.6 If $x \in K$, then $x < P_1 x$,
- 1.7 If $x \in K$, then $x < P_2 x$,
- 1.8 If $x, y \in K$, then $P_1(x \vee y) = P_1 x \vee P_1 y$,
- 1.9 If $x, y \in K$, then $P_2(x \vee y) = P_2 x \vee P_2 y$,
- 1.10 If $x \in K$, then $\neg P_1 x < \neg P_2 x$,
- 1.11 If $x \in K$ and $\neg P_2 x \in D$, then $\neg P_1 x \in D$.

Here we mean by an S2-S2 matrix an ordered six-tuple $M = \langle K, D, \neg, \cap, P_1, P_2 \rangle$ such that K and D are respectively a set and its non-empty proper subset, while $\neg, \cap, P_1$ and P_2 are respectively functions defined on K and class-closing there. It is further stipulated that such letters as $x, y, z, \dots$ will be employed as variables ranging over the elements of K . In the following we shall employ such abbreviations as; $(x \cup y) = \neg(\neg x \cap \neg y)$, $(x \rightarrow_2 y) = \neg P_2(x \cap \neg y)$, $(x \leftrightarrow_1 y) = (x \rightarrow_1 y) \cap (y \rightarrow_1 x)$, $x \leftrightarrow_2 y = (x \rightarrow_2 y) \cap (y \rightarrow_2 x)$, $0 = x \cap \neg x$ and $1 = x \cup \neg x$.

LEMMA 1. If $M = \langle K, D, \neg, \cap, P_1, P_2 \rangle$ is a normal S2-S2 matrix, then

- (i) If $x, x \rightarrow_1 y \in D$, then $y \in D$,
- (ii) $x \leftrightarrow_1 y \in D$ iff $x = y$,

(iii) $x \leftrightarrow_2 y$ iff $x = y$,

(iv) $\neg P_1 x \in D$ iff $\neg P_2 x \in D$,

(v) $\neg P_2 0 \in D$,

(vi) If $x < y$, then $P_1 x < P_1 y$,

(vii) If $x < y$, then $P_2 x < P_2 y$.

Proof. (i) Suppose $x \in D$ and $x \rightarrow_1 y \in D$. Then by 1.2 $x < y$. Since $x < y$ iff $y = x \cup y$ by 1.1, we have by 1.3 $x \cup y \in D$ or $y \in D$.

(ii) Suppose $x \leftrightarrow_1 y \in D$. Then $(x \rightarrow_1 y) \cap (y \rightarrow_1 x) \in D$ by definition. On the other hand, if $z \cap w \in D$, then by 1.3 $(z \cap w) \cup z$ and $(z \cap w) \cup w \in D$. By 1.1 we have $(z \cap w) \cup z = z$ and $(z \cap w) \cup w = w$. Hence $z \in D$ and $w \in D$. We, therefore, have from $(x \rightarrow_1 y) \cap (y \rightarrow_1 x) \in D$ $x \rightarrow_1 y \in D$ and $y \rightarrow_1 x \in D$. By 1.2 we, then, obtain $x < y$ and $y < x$, namely, $x = y$. The sufficiency is forthcoming from 1.1-1.3 and the definition of $\leftrightarrow_1$. (iii) is proved analogously resorting to 1.10 and 1.11.

(iv) is easily taken care of by 1.3, 1.10 and 1.11, while (v) is taken care of by 1.3, 1.4 and 1.10.

For proving (vi) suppose $x < y$. Then by 1.1 $y = x \cup y$. Hence $P_1 x = P_1(x \cup y)$. By 1.8 we have $P_1 x = P_1 x \cup P_1 y$, namely, $P_1 x < P_1 y$. (vii) is proved analogously making use of 1.9.

THEOREM 1. (Consistency Theorem) Every A provable in S2-S2 is valid.

Here A is said to be valid in an S2-S2 matrix $M = \langle K, D, \sim, \cap, P_1, P_2 \rangle$ if $V(A) \in D$ for every valuation V that assigns an element of K to every variable, where V is recursively defined for every formula on the basis of its values for variables in the following way;

- (i) $V(\sim B) = -V(B)$,
- (ii) $V(B \cap C) = V(B) \cap V(C)$,
- (iii) $V(\Diamond_1 B) = P_1 V(B)$,
- (iv) $V(\Diamond_2 B) = P_2 V(B)$.

A is valid simpliciter if it is valid in every normal S2-S2 matrix.

Proof. We are demonstrating this by induction on the length of the proof for the given normal S2-S2 matrix and a valuation there defined.

If A is one of AS22.1-AS22.4, $V(A) \in D$, for $V(A) = -P_1 0$ in view of 1.1, and by 1.4 we have $V(A) \in D$. On the other hand, AS22.5 is taken care of by the validity of $-P_1(x \cap -y) \cap -P_1(y \cap -z) < -P_1(x \cap -z)$, which in turn is forthcoming from $P_1(x \cap -z) < P_1((x \cap -y) \cup (y \cap -z))$ by 1.8. And this easily obtains from 1.1 and (vi) of LEMMA 1. AS22.6 is treated similarly by 1.1, 1.9 and (vii) of LEMMA 1.

For taking care of AS22.7 we observe $x \cap -P_2(x \cap -y) < y$ which obtains from $x < y \cap P_2(x \cap -y)$, and this is forthcoming

from $x < y \cup (x \cap \neg y) = y \cup x$ by 1.7. We, thus, have $(x \cap (x \rightarrow_2 y)) \cap \neg y = 0$ by 1.1, from which follows $V(\text{AS22.7}) \in D$ by (v) of LEMMA 1.

AS22.8 and AS22.9 do not present any difficulties in view of 1.1 and respectively by (vi) and (vii) of LEMMA 1 together with 1.2. Lastly AS22.10 is taken care of by 1.2 and 1.10. Every axiom of S2-S2, therefore, is valid in the normal S2-S2 matrix M.

This completes the basis.

For treating TR1 we observe the following: If $V(A) \in D$ for the valuation V of the matrix but $V(A(B/p)) \notin D$ with p occurring in A, then $V'(A)$ would not belong to D for V' , which assigns $V(B)$ to p. This takes care of TR1.

For treating TR2 suppose that $V(A) \in D$ and $V(B) \in D$. (Here and in what follows V is again a valuation of the given matrix.) Then, by 1.3 $V(A) \cap V(B) \in D$, from which by the definition of valuation we have $V(A \wedge B) \in D$.

As for TR3 suppose that $V(A \equiv_1 B) \in D$ and $V(\dots A \dots) \in D$, the former of which gives rise to $V(A) = V(B)$ by (ii) of LEMMA 1. We, therefore, have $V(\dots B \dots) \in D$.

For TR4 suppose that $V(A) \in D$ and $V(A \rightarrow_1 B) \in D$, the latter of which amounts to $V(A) \rightarrow_1 V(B) \in D$. By (i) of LEMMA 1, we, then, have $V(B) \in D$.

As for TR5, suppose that $V(\sim \Diamond_2 A) \in D$, that is, $\neg P_2 V(A) \in D$.

Then by 1.11 we have $\neg P_1 V(A) \in D$.

This completes the proof of THEOREM 1.

THEOREM 2. There is an infinite normal S2-S2 matrix which verifies only theorems of S2-S2 (and hence is a characteristic S2-S2 matrix.)

For proving this theorem we first show how to construct a matrix $M^c = \langle K^c, D^c, -, \cap^c, P_1^c, P_2^c \rangle$, and then show (i) it constitutes a normal S2-S2 matrix, and (ii) it falsifies every wff which is not a thesis of S2-S2.

With this in view we first construct the Lindenbaum matrix $M = \langle K, D, -, \cap, P_1, P_2 \rangle$ of S2-S2. In other words, we shall define K and D respectively as the set of all the wffs of S2-S2 and that of all the theses of S2-S2, and define $-, \cap, P_1, P_2$ respectively as follows; $\neg A = \sim A$, $A \cap B = A \wedge B$, $P_1 A = \Diamond_1 A$ and $P_2 A = \Diamond_2 A$ with the equality understood as the identity of formulas.

Since the matrix thus constructed does not satisfy (ii) of LEMMA 1 being, therefore, not normal, we define anew $K^c, D^c, -, \cap^c, P_1^c$ and P_2^c respectively in the following way in terms of M as above constructed;

$$K^c = \{ E(A) \mid A \in K \}, D^c = \{ E(A) \mid A \in D \}, \neg^c E(A) = E(\neg A),$$

$$E(A) \cap^c E(B) = E(A \wedge B), P_1^c E(A) = E(P_1 A), P_2^c E(A) = E(P_2 A),$$

where $E(A)$ denotes the congruence class generated by A , namely, the set of all the wffs strictly equivalent 1 to A in S2-S2,

i.e., $E(A) = \{B \mid \vdash_{S2-S2} B \equiv_1 A \text{ or } B \leftrightarrow_1 A \in D\}$. On the other hand, two congruence classes are equal (in M^C) if they are identical set-theoretically. From this easily follows $E(A) = E(B)$ iff $\vdash_{S2-S2} A \equiv_1 B$. In view of MT1, $\{B \mid \vdash_{S2-S2} B \equiv_1 A\} = \{B \mid \vdash_{S2-S2} B \equiv_2 A \text{ or } B \leftrightarrow_2 A \in D\}$.

We now prove that the matrix $M^C = \langle K^C, D^C, \neg^C, \cap^C, P_1^C, P_2^C \rangle$ defined as above constitutes a normal S2-S2 matrix.

Proof: It is sufficient to prove that M^C satisfies DEFINITION 1.1-1.11. As for 1.1 and 1.2 it is easily seen that M^C satisfies the conditions, and does not need any elaboration.

For proving 1.3 suppose $E(A), E(B) \in D^C$. Then by the definition of D and the congruence classes, $A, B \in D$, i.e., $\vdash_{S2-S2} A$ and $\vdash_{S2-S2} B$. In view of TR2 $\vdash_{S2-S2} A \wedge B$, namely, $A \cap B \in D$. Therefore, $E(A \cap B) = E(A) \cap^C E(B) \in D^C$. Suppose $E(A) \in D^C$ and $E(B) \in K^C$. Then, by the definition of D^C , K^C and the congruence classes, $A \in D$, i.e., $\vdash_{S2-S2} A$, and since $\vdash_{S2-S2} A \rightarrow_1 (A \vee B)$, $\vdash_{S2-S2} A \vee B$, namely, $A \cup B \in D$. Therefore, $E(A \cup B) = E(A \vee B) = E(A) \cup^C E(B) \in D^C$. Hence M^C satisfies 1.3.

As for 1.4 we have as a thesis of S2-S2 TS22.1, namely, $\vdash_{S2-S2} \sim \Diamond_1 (p \wedge \sim p)$. Therefore, $\neg P_1(A \cap \neg A) \in D$, namely, $\neg P_1 0 \in D$. Hence $E(\neg P_1 0) = \neg^C P_1^C E(0) \in D^C$. Therefore M^C satisfies 1.4.

For proving 1.5 suppose $\neg^C P_1^C E(A) = E(\neg P_1 A) \in D^C$. Then $\vdash_{S2-S2} \sim \Diamond_1 A$. Now we have TS22.3, namely, $\vdash_{S2-S2} \sim \Diamond_1 A \rightarrow_1 A \rightarrow_1 (A \wedge \sim A)$.

Hence $\vdash_{S2-S2} A \supset_1 (A \wedge \sim A)$. On the other hand, by $\vdash_{PC} (A \wedge \sim A) \supset A$ and MT3 we have $\vdash_{S2-S2} (A \wedge \sim A) \supset_1 A$. Hence $\vdash_{S2-S2} A \equiv_1 (A \wedge \sim A)$, which in turn give rise to $E(A) = E(A \wedge \sim A) = E(A) \cap^{C-C} E(A) = 0^C$. M^C , therefore, satisfies 1.5.

Let us prove 1.6. Since we have as a thesis TS22.3, i.e., $\vdash_{S2-S2} p \supset_1 \Diamond_1 p$, $A < P_1 A$ by the definition of D and 1.2. Hence by the definition of congruence classes and of D^C , $E(A) < P_1^C E(A)$. Therefore, M^C satisfies 1.6.

As for 1.7, 1.8 and 1.9, we have as theses of S2-S2 TS22.7, TS22.8 and TS22.9, which respectively take care of 1.7, 1.8 and 1.9.

For treating 1.10 and 1.11 let us observe AS22.10 and TR5, which respectively take care of 1.10 and 1.11. Therefore M^C satisfies both 1.10 and 1.11.

This completes the proof that M^C is a normal S2-S2 matrix, which happens to be infinite.

What remains to be proved is that M^C falsifies all the non-theorems of S2-S2. A wff A is falsified by the matrix iff there exists an assignment defined on K to the variables occurring in A , for which $V(A) \notin D$. Suppose A is not a thesis of S2-S2. Since M is a Lindenbaum matrix, there is an assignment V such that $V(A) = A$. Hence it is sufficient to prove that $A \notin D$. For proving this let us suppose, if possible, $A \in D$. Then by the defi-

dition of $D \vdash_{S2-S2} A$, which is contrary to the assumption. Therefore, $A \notin D$. In view of the definition of D^c , $A \notin D$ gives rise to $E(A) \notin D^c$. Hence M^c falsifies all the non-theorems of S2-S2.

This completes the proof of THEOREM 2.

THEOREM 3. Let $M = \langle K, D, -, \cap, P_1, P_2 \rangle$ be a normal S2-S2 matrix, and let $a_1, a_2, \dots, a_n$ be a finite sequence of elements of K . Then, there exist a finite normal S2-S2 matrix $M' = \langle K', D', -, \cap', P'_1, P'_2 \rangle$, with at most 2^{n+1} elements such that K' contains a sequence $b_1, b_2, \dots, b_n$ of elements satisfying the following conditions:

- (1) $a_i \in D$ iff $b_i \in D'$,
- (2) $\neg a_i = a_j$ holds, if and only if $\neg' b_i = b_j$ holds,
- (3) $a_i \cap a_j = a_k$ holds, if and only if $b_i \cap' b_j = b_k$ holds,
- (4) $P_1 a_i = a_j$ holds, if and only if $P'_1 b_i = b_j$,
- (5) $P_2 a_i = a_j$ holds, if and only if $P'_2 b_i = b_j$,

where the equality relation remains the same in M' .

For proving THEOREM 3 we first construct an algebra, which satisfies the above conditions and then show that it is a normal S2-S2 matrix.

Let K' be a Boolean subalgebra of K generated by $P_1 0, x_1, x_2, \dots, x_n$ by applying a finite number of times the operations $\cap$ and $-$ to these elements of K . Then K' contains at most 2^{n+1} elements. Further we let the operations $\cap'$ and $\neg'$ be respective-

ly the same as $\cap$ and $-$ — excepting that they are defined on K' instead of on K . D' , on the other hand, is defined to be the intersection of D and K' . The above conditions (1), (2) and (3) are then easily seen to be satisfied.

For taking care of the remaining conditions we define cover₁ and cover₂ as follows. x belonging to K' is covered₁ by y belonging to K' if $x < y$ and $P_1 y \in K'$. x belonging to K' is covered₂ by y belonging to K' if $x < y$ and $P_2 y \in K'$. In terms of the notion of 'cover' thus defined we now define the operations P'_1 and P'_2 so that these operations may satisfy the conditions (4) and (5): if x is an element of K' , which is covered₁ by $x_1, x_2, \dots, x_n$ ($\in K'$) (only a finite number of which are existent in view of the definition of K'), then $P'_1 x = P_1 x_1 \cap P_1 x_2 \cap \dots \cap P_1 x_n$, and if x is an element of K' , which is covered₂ by $x_1, x_2, \dots, x_n$, then $P'_2 x = P_2 x_1 \cap P_2 x_2 \cap \dots \cap P_2 x_n$. Since 1 (which is certainly a member of K') covers₁₍₂₎ every element of K' , $P'_1 x$ and $P'_2 x$ are defined for any $x \in K'$.

Before proceeding to the proof that these operations satisfy the conditions we must prove the following: if x is an element of K' , then $P_1 x < P'_1 x$ and $P_2 x < P'_2 x$. Let x be covered₁ by $x_1, x_2, \dots, x_n$. We, then, have $P'_1 x = P_1 x_1 \cap P_1 x_2 \cap \dots \cap P_1 x_n$. Since $x < x_1, x < x_2, \dots, x < x_n$, we obtain $P_1 x < (P_1 x_1 \cap P_1 x_2 \cap \dots \cap P_1 x_n)$ from the property of the normal matrix, and hence by the

definition of P_1^i , we have $P_1x < P_1^ix$. In the same way we can prove $P_2x < P_2^ix$.

We now show that the above conditions (4) and (5) are also satisfied. For this it suffices to prove that if x and y are elements of K' such that $P_1x = y$, then $P_1^ix = y$, and if x and y are elements of K' such that $P_2x = y$, then $P_2^ix = y$. Since $P_1x = y$ and $x < x$, x is covered₁ by x itself. If x is the only element ($\in K'$) covering₁ x , then we are done. Suppose, therefore, that x is also covered₁ by $x_1, x_2, \dots, x_n$. Then, by the definition of P_1^i , we see that $P_1^ix = P_1x_1 \cap \dots \cap y \cap \dots \cap P_1x_n$. We, thus, have $P_1^ix = P_1x_1 \cap \dots \cap y \cap \dots \cap P_1x_n < y = P_1x$. In the same way, $P_2^ix < y$. On the other hand, we have $P_1x < P_1^ix$. In view of the hypothesis $P_1x = y$, we, therefore, have $y < P_1^ix$. In the same way, $y < P_2^ix$. Hence $P_1^ix = y$ and $P_2^ix = y$, i.e., $P_1x = P_1^ix$ and $P_2x = P_2^ix$ for $x \in K'$ hold. The matrix M' , thus, satisfies (4) and (5).

We next prove that the matrix M' is a normal S2-S2 matrix satisfying 1.1-1.11.

As for 1.1 K' is obviously a Boolean algebra with respect to $\cap$ and $-$ by the definition of K' . 1.2 is also obvious from the definition of K' and D' . For proving that D' is non-empty proper subset of K' it is sufficient to show that $1 \in D'$ and $0 \notin D'$. By the definition of K' $0 \in K'$ and $1 \in K'$. On the other hand, $1 \in D$ and $0 \notin D$ since M is a normal S2-S2 matrix. ($0 \in D$ would

give rise to $D = K$ against 1.3.) Hence $1 \in D' = K' \cap D$ and $0 \notin D' = K' \cap D$. If $x \in D'$ and $y \in D'$, then $x \in D$, $x \in K'$, $y \in D$ and $y \in K'$. Since M is a normal S2-S2 matrix, this gives rise to $x \cap y \in D$. We also have $x \cap y \in K'$. Hence $x \cap y \in D'$. If $x \in D'$ and $y \in K'$, then $x \in D$ and $y \in K$, and, therefore, $x \cup y \in D$. But clearly $x \cup y \in K'$. We, thus, have $x \cup y \in D'$. Therefore D' constitutes an additive ideal of K' .

For proving 1.4, we first observe, $P_1 0 \in K'$ and by the definition of K' , $0 = -(x \cap -x) \in K'$ for $x \in K'$. On the other hand, as was proved above, $P_1 0 = P_1 0$. Hence $-P_1 0 = -P_1 0$. Since M is a normal S2-S2 matrix, $-P_1 0 \in D$. Hence $-P_1 0 \in D$ and $-P_1 0 \in K'$. We, thus, have $-P_1 0 \in D'$ by the definition of D' .

For taking care of 1.5 it is sufficient to prove that if $-P_1 x \in D'$ for $x \in K'$, then $x = 0$. Suppose $P_1 x = P_1 x$, we have $-P_1 x = -P_1 x$. Since M is a normal S2-S2 matrix, we have $-P_1 x \in D$. Hence $x = 0$.

We are now proving 1.6. Since M is a normal S2-S2 matrix, $x < P_1 x$. We also have $P_1 x = P_1 x$ for $x \in K'$ as was proved earlier. Hence $x < P_1 x$. In the same way, 1.7 is satisfied.

For proving 1.8 suppose that $x(\in K')$ is covered₁ by $x_1, x_2, \dots, x_m$ and $y(\in K')$ by $y_1, y_2, \dots, y_n$. It is also assumed that $x \cup y(\in K')$ is covered₁ by $z_1, z_2, \dots, z_s$. Then $P_1 x \cup P_1 y = (P_1 x_1 \cap \dots \cap P_1 x_m) \cup (P_1 y_1 \cap \dots \cap P_1 y_n) = (P_1 x_1 \cup P_1 y_1) \cap \dots \cap$

$(P_1x_m \cup P_1y_n) = P_1(x_1 \cup y_1) \cap \dots \cap P_1(x_m \cup y_n) > P_1z_1 \cap \dots \cap P_1z_s = P_1^i(x \cup y)$, since every $x_i \cup y_j$ ($1 \leq i \leq n$, $1 \leq j \leq m$) is among the elements $z_1, \dots, z_s$ which respectively cover₁ $x \cup y$. The converse inclusion is forthcoming from the definition of P_1^i . Hence 1.8 is satisfied. Analogously it is shown that 1.9 is satisfied.

Turning to the proof of 1.10 we have $-P_1x < -P_2x$. On the other hand, $P_1x = P_1^i x \in K'$ and $P_2x = P_2^j x \in K'$, which respectively give rise to $-P_1x = -P_1^i x$ and $-P_2x = -P_2^j x$. Therefore, $-P_1^i x < -P_2^j x$. Hence 1.10 is satisfied.

For proving 1.11 suppose $-P_2^j x \in D'$. Then $-P_2^j x = -P_2x \in D$. Since M is a normal S2-S2 matrix, if $-P_2x \in D$, then $-P_1x \in D$. Hence $-P_1x = -P_1^i x \in K'$. Therefore, $-P_1^i x \in D'$ by the definition of D' . Hence 1.11 was proved.

This completes the proof of the theorem.

THEOREM 4. (Completeness theorem) If A is a wff of S2-S2 containing n subformulas, then A is a thesis of S2-S2 iff it is valid in every normal S2-S2 matrix containing at most 2^{n+1} elements.

Proof: The necessity is straightforward by THEOREM 1, namely, Consistency theorem.

For proving the sufficiency we let A be a wff not provable in S2-S2. Then, $E(A)$ does not belong to D^C in view of the defi-

nition of the normal S2-S2 characteristic matrix $M^C = \langle K^C, D^C, -^C, \cap^C, P_1^C, P_2^C \rangle$, whose existence is guaranteed by THEOREM 2. From the matrix M^C and $E(A_1), E(A_2), \dots, E(A_n)$ ($0 < n$, and $A_1, A_2, \dots, A_n$ is an enumeration of all the distinct subformulas of A), we can construct the (finite) matrix $M' = \langle K', D', -, \cap', P_1', P_2' \rangle$ as indicated in THEOREM 3. Now it is easily seen that $E(A)$ belongs to K' , since A itself constitutes a subformula of A . Thus, $E(A)$, though it is an element of K' , does not belong to D' . Since we have an assignment V such that $V(A_i) = E(A_i)$ for every propositional variables contained in A and $V(A) = E(A)$ does not belong to D' , the given A is not valid in the (finite) normal S2-S2 Matrix M' .

This completes the proof of THEOREM 4.

Since we can effectively construct all the normal S2-S2 matrices containing not more than 2^{n+1} elements ($0 < n$), THEOREM 4 provides us with a decision method for the proposed S2-S2.

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