An Additional Report on the Estimation of the Effective Population Size for Changes in the Number of Individuals Between the Parental and the Progeny Populations*

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The effective size of a population is expressed by the number of adults in an ideal population in which a random gamete has an equal probability of coming from any parent. This number is obtained mathematically from the number of individuals and the breeding structure of the population. The author derived a formula which made it possible to calculate the effective population size \( N \) based on the conditions that (i) the number of breeding males is much smaller than that of breeding females and that (ii) the number of progeny per parent is variable, namely

\[
N = \frac{4N_m}{V_{bm} + 1}
\]

In this formula, \( N_m \) is the number of breeding males in the parental population and \( V_{bm} \) the coefficient of variability of the number of gametes contributed by the members of the male parents (Nozawa, 1957; Nozawa and Kondo, 1957). This formula, however, was derived with an assumption that the difference in the number of breeding adults between the parental and the progeny population was not large. In actual populations of farm animals such an assumption is generally not satisfied. So, it is desirable to remove this assumption and generalize the formula mentioned above.

Let us take into consideration the autosomal genes. We start with a parental population of \( N_m \) breeding males and \( N_f \) breeding females, and assume that the progeny population is composed of \( s_m N_m \) males and \( s_f N_f \) females; both \( s_m \) and \( s_f \) do not take any negative value and the number of individuals decreases in the case of \( 0 < s < 1 \), and increases in the case of \( s > 1 \). Now, let us consider the distribution of number of gametes contributed by the individual members of parental population: in the male parents, the \( j \)-th male contributes \( k_{mj} \) gametes to the next generation, the mean and variance of \( k_{mi} \) being

\[
\bar{k}_{mj} = \frac{\sum_{i=1}^{n_m} k_{mi}/N_m = s_m N_m + s_f N_f}{N_m}
\]

and

\[
\sigma^2_{bm} = \frac{\sum_{i=1}^{n_m} (k_{mi} - \bar{k}_{bm})^2 / N_m}{N_m}
\]

respectively; in the female parents, the \( j \)-th female contributes \( k_{fj} \) gametes to the next generation, of which the mean and variance are

\[
\bar{k}_{fj} = \frac{\sum_{i=1}^{n_f} k_{fj}/N_f = s_m N_m + s_f N_f}{N_f}
\]

and

\[
\sigma^2_{bf} = \frac{\sum_{i=1}^{n_f} (k_{fj} - \bar{k}_{fj})^2 / N_f}{N_f}
\]

respectively. In the case where two gametes are taken randomly from the set of gametes coming from the parental population, the probability that these two gametes are contributed by the same parent is

\[
\frac{N_m}{\sum_{i=1}^{n_m} k_{mi}(k_{mi} - 1) + \sum_{j=1}^{n_f} k_{fj}(k_{fj} - 1)} \cdot \frac{2(\hat{s}_m N_m + \hat{s}_f N_f)}{\{2(s_m N_m + s_f N_f) - 1\}}
\]

\[
= \frac{N_m}{\sum_{i=1}^{n_m} k_{mi}(k_{mi} - 1) + \sum_{j=1}^{n_f} k_{fj}(k_{fj} - 1)} \cdot \frac{2(\hat{s}_m N_m + \hat{s}_f N_f)}{\{2(s_m N_m + s_f N_f) - 1\}}
\]

\[
\text{if the effective population size equals } N, \text{ it is clear from the definition of this measure that the above probability is nearly } 1/N. \text{ Hence,}
\]

\[
N = \frac{\sum_{i=1}^{n_m} k_{mi}(k_{mi} - 1) + \sum_{j=1}^{n_f} k_{fj}(k_{fj} - 1)}{N_m \sigma^2_{bm} + N_f \sigma^2_{bf} + 2(\bar{k}_m + \bar{k}_f - 1)}
\]

\[
\frac{\sum_{i=1}^{n_m} k_{mi}(k_{mi} - 1) + \sum_{j=1}^{n_f} k_{fj}(k_{fj} - 1)}{N_m \sigma^2_{bm} + N_f \sigma^2_{bf} + 2(\bar{k}_m + \bar{k}_f - 1)}
\]

\[
\text{This is a general formula to estimate the effective population size, in which (i) the inequality of the number of breeding males and females, (ii) the variability in the number of progeny of each parent and (iii) the change in the number of breeding individuals between the parental and the progeny populations are all taken into consideration.}
\]

Now, let us simplify formula (1) by considering the breeding structure of some farm animals. If the number of breeding males is much smaller than that of breeding females, as seen in the populations of dairy cattle, namely

\[
\hat{s}_m N_m + \hat{s}_f N_f = s_f N_f
\]

\[
\hat{k}_m = s_f N_f, N_m
\]

\[
\hat{k}_f = s_f
\]

and if the coefficients of variability of the number of gametes contributed by the member of the male and female parents are \( V_{bm} = \sigma_{bm}/\hat{k}_m \) and \( V_{bf} = \sigma_{bf}/\hat{k}_f \), respectively, the formula (1) will be transformed into

\[
1
\]

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This is a generalization of WRIGHT'S formula, because if \( s = 1 \) it reduces to the following formula devised by him (1938):

\[
N = \frac{4N_s - 2}{\sigma_s^2[2] + 2}
\]

where \( \sigma_s^2[2] = \sum_{i=1}^{N} (k_i - 2)^2 / N_0 \). It may be expected that the numerator of formula (3) and, accordingly, the effective population size \( N \) have negative values when \( 0 < s < \frac{1}{2N_0} \). Such a situation, however, can never be realized, because \( 2N_0 \) is the total number of gametes contributed by the parental population to the progeny population. Thus, \( s \) cannot be smaller than \( \frac{1}{2N_0} \), and \( N = 0 \) when \( s = \frac{1}{2N_0} \).

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References

