THE INITIAL CRACKING MOMENTS OF THE BRACKETED PARTS OF CONCRETE BEAMS

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SYNOPSIS

The beams with brackets at the ends, have the tendency to crack from the reentrant corners of the bracketed parts, the strength of this cracking of reinforced and prestressed concrete beams is investigated.

An equation is presented for predicting the initial cracking moments, the calculated results are checked by the results of experiments which are tested under both moving load and two points load. To simplify the design work, design charts are also presented.

INTRODUCTION

The beams with brackets at the ends are often used as structural members not only in bridges but also in building frames. The typical example of such beams are the suspended beams of the cantilever bridges. Owing to the abrupt change in the depth of the cross section at the brackets, this type of beams have the tendency to cracking from the reentrant corners of the bracketed parts¹⁾. For convenience, these cracks are termed "corner crack". This corner cracks are formed at approximately 50 % of design load on reinforced concrete beams²⁾ and approximately 60% of design load on prestressed concrete beams³⁾ respectively.

The causes of these corner cracks may be considered to be the great increase of shearing stress owing to the great decrease of the beam depth, and sometimes it may occur by the tensile stress due to the nonuniform drying shrinkage, nonuniform temperature distribution during the hardening of concrete, eccentric loading or torsional loading, inability of end rotation at supports and horizontal forces

introduced by the imperfect mechanism of expansion bearings.

There are several papers which deal with the bracketed parts of concrete beams. S. Kamiyama⁷⁾ reported the investigation of the strain distribution of the bracketed parts of concrete beams and presented an equation for calculation of shearing stress and the method of reinforcement of this parts. S. Okamoto and T. Ishiwata⁸⁾ reported their experimental study on the stress distribution of the reinforced concrete beams, S. Onishi, M. Manabe, and M. Hisamatsu¹⁰⁾ reported the experimental study of prestressed concrete beams. M. Fujii¹¹⁾ reported the analytical result of bracketed parts by the finite element method. Y. Nojiri and H. Akiyama¹²⁾ reported the study on the stresses of bracketed parts by the photoelastic test. All of these papers present excellent data.

The occurrence of corner cracks may result in excessive deflections, corrosion of reinforcements, and other critical situation. An investigation of corner cracking strength is then an important problem. The purpose of this paper is to deal with the initial corner cracking moment from the standpoint of diagonal tension, and furthermore, to make the charts of the initial corner cracking moments for practical use.

TEST SPECIMENS

The data of nine series of seventeen specimens are adopted for comparison with the theoretical values which are calculated by the equation discussed in the following chapters. The specimens identified by the letter RC are reinforced without prestressing, the specimens identified by the letter PC are reinforced with prestressing, the letter *T* denotes tee section specimens, the letter *B* and *N* denote the specimens which were tested by the two points load, the letter *A* denotes the specimens which were

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Fig. 1 Specimens and loading arrangement of reinforced concrete rectangular beams

Fig. 2 Specimens and loading arrangement of reinforced concrete tee beams

Fig. 3 (a) Specimens and loading arrangement of prestressed concrete rectangular beams

Fig. 3 (b) Specimens and loading arrangement of prestressed concrete rectangular beams

Fig. 4 Specimens and loading arrangement of prestressed concrete tee beams

Table 1 Calculated and experimental analysis of results (Reinforced concrete beams)

Specimens	Experimental									Calculated				Calculated Experimental
	h_s cm	d_s cm	b_0 cm	b cm	A_{SH} cm ²	A_{SD} cm ²	σ_{tu} ky/cm ²	a/h_s	M_c t-m	x cm	α_c	Z cm ³	M_c t-m	
RCN	20	17	—	12	4.02	4.02	28.5	3.00	0.225	11.17	0.51	1 128	0.164	0.73
RCA	20	17	—	12	4.02	4.02	37.7	0.75	0.225	11.68	0.51	1 096	0.210	0.94
RCBT	15	12	12	40	4.02	4.02	33.6	4.00	0.190	6.17	0.51	825	0.143	0.75
RCAT	15	12	12	40	4.02	4.02	37.2	2.00	0.200	6.14	0.51	837	0.157	0.79

Table 2 Experimental and calculated analysis of results (Prestressed concrete beams)

Specimens	Experimental									Calculated				Calculated Experimental
	h_s cm	d_s cm	b_0 cm	b cm	A_{SH} cm ²	A_{SD} cm ²	σ_{tu} ky/cm ²	a/h_s	M_c t-m	x cm	α_c	Z cm ³	M_c t-m	
PCN	20	13.3	---	12	0.97	---	29.0	4.00	0.355	10.00	0.87	800	0.247	0.70
PCA	20	13.3	--	15	1.93	4.02	35.3	0.75	0.418	9.28	0.87	957	0.363	0.87
PCB	20	13.3	---	15	1.93	4.02	34.0	4.00	0.450	9.36	0.87	960	0.352	0.78
PCBT	15	9.0	12	40	—	4.02	43.6	5.33	0.263	5.72	0.87	625	0.264	1.00
PCAT	15	9.0	12	40	—	4.02	35.6	2.00	0.245	5.77	0.87	665	0.236	0.96

tested by the moving load, VIZ., the load was applied from *A* load position to *G* load position at each increment of loading until failure occurred to examine the effect of shear span depth ratio on the corner cracking. The details of these specimens are given in Fig. 1, 2, 3, 4 and Table 1, 2.

DEFINITION OF INITIAL CORNER CRACKING MOMENTS

Before the discussion of initial corner cracking moments, we should decide the definition of cracking moments. From the standpoint of maintenance or durability of the structure, the cracking moment should be defined as the moment when the crack of the structure reaches to the limit of the allowable width. But there are many difficulties to determine the width of cracks accurately. Irregularity of crack width, location of measurement, and kind of apparatus for measurement will be the problems at the time measurement.

In this paper, for experimental purpose, the corner cracking moment is defined as the moment at which the first corner crack discovered by the naked eyes and which is acting on the cross section cutting the corner. For theoretical purpose, the initial corner cracking moment is defined as the moment at which the principal tensile stress at the reentrant corner of bracket reaches the value of the limiting stress, σ_{tu} .

Fig. 5 Typical strain distribution by the load

INITIAL CORNER CRACKING MOMENT OF REINFORCED CONCRETE BEAMS

In Fig. 6 the inclined line $a \sim f$ at an angle θ_i shows the initial corner crack. This corner crack will be formed under any condition of loading if the principal tensile stress, σ_1 , at the reentrant corner of bracket "*a*" reaches the value of the limiting stress σ_{tu} .

Fig. 6 The stresses just before corner cracking

Let there be given an elementary triangular prism, cut from "*a*" which there are acting the unit stresses as shown in Fig. 6. When the principal tensile stress, σ_1 reaches σ_{tu} the normal stress of this elementary triangular prism become σ_x . Put the relation between σ_x and σ_{tu} as equation (1)

$$\sigma_x = \alpha_c \sigma_{tu} \quad (1)$$

Consider section *a-a* of Fig. 5 and assume that the stress distribution as Fig. 7. We can calculate

Fig. 7 The stress distribution of bracket base

the resisting moment, M , by the following equation

$$M = Z \sigma_x \dots\dots\dots (2)$$

Combining Eq. (1) and (2) we obtain;

$$M_c = \alpha_c Z \sigma_{tu} \dots\dots\dots (3)$$

INITIAL CORNER CRACKING MOMENTS OF PRESTRESSED CONCRETE BEAMS

Fig. 8 shows the elementary triangular prism which is cut from the reentrant corner of bracket of the prestressed concrete beam. The unit stresses acting on the elementary triangular prism are occurred only from the prestressing force, the relationship of these stresses being given by

Fig. 8 The stresses due to the prestress

$$\sigma_{np} = \sigma_{xp} \cos^2 \phi + \sigma_{yp} \sin^2 \phi + \tau_{xyp} \sin 2\phi \dots\dots (4)$$

where

$$\phi = (90^\circ - \theta_i)$$

then we may rewrite as

$$\sigma_{np} = \sigma_{xp} \sin^2 \theta_i + \sigma_{yp} \cos^2 \theta_i + \tau_{xyp} \sin 2\theta_i \dots\dots (5)$$

now, if the beam is subjected to the load, normal stress, σ_{np} , will occur on plane making an angle θ_i with the beam axis and will change its quantity and approach the limiting stress, σ_{tu} , as shown in Fig. 6. At this stage the initial corner cracking moment will be written as

$$M_c = \alpha_c Z (\sigma_{tu} + \sigma_{np}) \dots\dots\dots (6)$$

where σ_{tu} , σ_{np} represent the absolute value. Substituting equation (5) in (6), it will be

$$M_c = \alpha_c Z (\sigma_{tu} + \sigma_{xp} \sin^2 \theta_i + \sigma_{yp} \cos^2 \theta_i + \tau_{xyp} \sin 2\theta_i) \dots\dots\dots (7)$$

If it can be assumed that

$$\sigma_{yp} = 0$$

$$\tau_{xyp} = 0$$

then

$$M_c = \alpha_c Z (\sigma_{tu} + \sigma_{xp} \sin^2 \theta_i) \dots\dots\dots (8)$$

COEFFICIENT, α_c

For evaluation of the coefficient, α_c , the following assumptions are made:

(1) The initial inclination angle of the corner crack with respect to the horizontal beam axis is assumed to be known.

(2) At section $a-a$ of Fig. 5, the flow of stresses may be assumed to be not yet irregular and the

strain distribution follows the Bernoulli's hypothesis of the linear strain distribution.

(3) The crack will be developed when the principal tensile stress at the reentrant corner reached the tensile strength of the concrete.

Referring to Fig. 6, the following relationship can be obtained

$$\sigma_{tu} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \dots\dots\dots (9)$$

$$\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \tan 2\phi \dots\dots\dots (10)$$

where

$$\tan 2\phi = \tan 2(90^\circ - \theta_i) = -\tan 2\theta_i$$

and assume

$$\sigma_y = 0$$

then

$$\tau_{xy} = -\frac{\sigma_x}{2} \tan 2\theta_i \dots\dots\dots (11)$$

from equations (9) and (11), we have

$$\sigma_x = \frac{2}{1 + \sqrt{1 + \tan^2 2\theta_i}} \sigma_{tu} \dots\dots\dots (12)$$

compare equations (1) and (12), it can put as follows;

$$\alpha_c = \frac{2 \cos 2\theta_i}{\cos 2\theta_i + 1} \dots\dots\dots (13)$$

$$0 < \alpha_c < 1$$

SECTION MODULUS, Z

(1) Rectangular Section

In Fig. 7. We assume approximately

$$k_d \div k_H$$

$$\sigma_{SD} \div \frac{\sigma_s}{\cos \theta_1} \quad \theta_1 < 90^\circ$$

For equilibrium, $T = C$

$$C = \frac{x^2 b}{2(h_s - x)} \sigma_x \dots\dots\dots (14)$$

$$T_c = \frac{(h_s - x) b \sigma_x}{2} \dots\dots\dots (15)$$

$$T_s = \frac{n(d_s - x)(A_{SH} + A_{SD})}{(h_s - x)} \sigma_x \dots\dots\dots (16)$$

equating right hand sides of equations (14), (15) and (16) we have

$$x = \frac{1}{2} \frac{b h_s^2 + 2 n d_s (A_{SH} + A_{SD})}{b h_s + n (A_{SH} + A_{SD})} \dots\dots\dots (17)$$

Taking moments about the neutral axis, the resisting moment at section $a-a$ can be written as

$$M = \left[\frac{b x^3}{3(h_s - x)} + \frac{b(h_s - x)^2}{3} + \frac{n(A_{SH} + A_{SD})(d_s - x)^2}{(h_s - x)} \right] \sigma_x \dots\dots\dots (18)$$

the section modulus is given by

$$\left. \begin{aligned} Z &= \frac{bx^3}{3(h_s-x)} + \frac{b(h_s-x)^2}{3} \\ &+ \frac{n(A_{SH}+A_{SD})(d_s-x)^2}{(h_s-x)} \\ x &= \frac{1}{2} \frac{bh_s^2 + 2nd_s(A_{SH}+A_{SD})}{bh_s + n(A_{SH}+A_{SD})} \end{aligned} \right\} \dots\dots (19)$$

(2) Tee Section

The section modulus for tee section beams can be obtained by the similar procedure as derived in the rectangular section, and it can be written as

$$\left. \begin{aligned} Z &= \frac{bx^3 + b_0(h_s-x)^3 - (b-b_0)(x-t)^3}{3(h_s-x)} \\ &+ \frac{n(A_{SH}+A_{SD})(d_s-x)^2}{(h_s-x)} \\ X &= \frac{1}{2} \frac{(b-b_0)t^2 + b_0h_s^2 + 2nd_s(A_{SH}+A_{SD})}{(b-b_0)t + b_0h_s + n(A_{SH}+A_{SD})} \end{aligned} \right\} \dots\dots\dots (20)$$

For the prestressed concrete beams, the section modulus of bonded beams should be computed on the basis of the transformed section and the net concrete section is used for unbonded beams (gross concrete section may be conveniently used).

DISCUSSION

Fig. 9 and 10 show the principal stress at each load stage calculated from the average strain values of eight points on both side of corner of brackets of RCN and PCN series. These Figures show the principal stress at the corner change both quantity and direction according to the increase of load. In such case, it is more rational to determine the direction of principal stress which is just before the corner cracking according to the inclination angle of the corner crack measured by the crack pattern of tested beams than to determine it by means of

Fig. 9 Principal stress at the corner of bracket of the reinforced concrete specimen

Fig. 10 Principal stress at the corner of bracket of the prestressed concrete specimen

Fig. 11 Crack pattern of reinforced concrete beam

Fig. 12 Crack pattern of prestressed concrete beam

Fig. 13 Idealized pattern of corner crack

calculation from the measured strain data.

Fig. 11 and 12 show the sketch of crack patterns of the tested specimens. When the stress on the concrete reaches the limiting tensile stress, the corner cracks form diagonally from the reentrant corner of the bracket. After the corner cracks stretch for a short distance and approach reinforcement, they change inclination upward. When the cracks reach a point somewhat above the neutral axis in the compression zone, the corner cracks propagate gradually towards the load point with a flatter slope. An simplified corner crack is shown in Fig. 13. The inclination angle of corner crack, θ_c , changes

Table 3 Inclination angles of corner cracks

Locations		θ_i		θ_c		Locations		θ_i		θ_c	
		Degrees	Means	Degrees	Means			Degrees	Means	Degrees	Means
RCN-1	NW	34	33	36	37	PCN-1	NW	10	11	21 16	17
	NE	32		38			NE	13			
	SW	40					SW	11			
	SE	25					SE	10			
RCN-2	NW	34	35		38	PCN-2	NW	10	17	15 16	24
	NE	38					NE	11			
	SW	32					SW	10			
	SE	32					SE	10			
RCA-1	NW	35	36	37	38	PCB-1	NW	12	22	27 20	38
	NE	33		38			NE	26			
	SW	37					SW	13			
	SE	40					SE	18			
RCA-2	NW	35	34		38	PCA-1	NW	22	28	40 36	23
	NE	37					NE	18			
	SW	37					SW	27			
	SE	25					SE	28			
RCAT-1	NW	29	36		38	PCA-2	NW	21	23	26	24
	NE	40					NE	23			
	SW	40					SW	21			
	SE	40					SE	14			
RCAT-2	NW	35	34		38	PCAT-1	NW	20	28	40 36	23
	NE	40					NE	20			
	SW	40					SW	15			
	SE	25					SE	20			
RCBT-1	NW	32	34		38	PCAT-2	NW	40	23	26	24
	NE	33					NE	32			
	SW	38					SW	38			
	SE	26					SE	40			
RCBT-2	NW	35	34		38	PCBT-1	NW	16	23	26	24
	NE	39					NE	10			
	SW	37					SW	32			
	SE	35					SE	30			
Average		35°		38°		Average		20°		26°	

with the prestressing forces and prestressing directions, while the initial inclination angle θ_i changes approximately between 10° and 40° as showed in Table 3.

It has been found by the tests that the initial inclination angle on the corner cracks has a tendency to inclinate with the beam axis about 35° in case of reinforced concrete beams and about 20° in case of prestressed concrete beams.

In the above expression, if $\theta_i=35^\circ$ is taken for reinforced concrete, it will be;

$$\cos 2 \theta_i = \cos 70^\circ = 0.342$$

$$\alpha_c = \frac{0.684}{1.342} = 0.51$$

and if $\theta_i=20^\circ$ for prestressed concrete beams

$$\cos 2 \theta_i = \cos 40^\circ = 0.766$$

$$\alpha_c = \frac{1.532}{1.766} = 0.87$$

From another viewpoint, the engineering meaning of reciprocal of α_c may be considered as if it were the factor of stress concentration. In other words, the coefficient, α_c , indicates the degree of irregularities in stress distribution to be given by the abrupt change in depth of cross section.

The test results obtained are given in Tables 1 and 2 together with the results calculated from equations (3) and (8) using $\alpha_c=0.51$ for reinforced concrete specimens and $\alpha_c=0.87$ for prestressed concrete specimens respectively. It is evident from Tables 1 and 2 that the agreement between the experimental corner cracking moments and those calculated by equations (3) and (8) is generally good.

DESIGN CHARTS

The initial corner cracking moment of rectangular beams with brackets is given by the equations (19) and (3) or (8). It may be rewritten as follows:

$$\left. \begin{aligned} \frac{M_c}{bh_s^2 \alpha_c \sigma_{tu}} &= \frac{k^3}{3(1-k)} + \frac{(1-k)^2}{3} \\ &+ \frac{n(p_H + p_D)(k_H - k)^2}{(1-k)} \end{aligned} \right\} \dots (21)$$

$$k = \frac{1}{2} \frac{1 + 2n(p_H + p_D)k_H}{1 + n(p_H + p_D)}$$

the solution of this formula is given in Charts 1-7.

The initial corner cracking moment of tee section beams with brackets is given by the equations (20) and (3) or (8) and it may be rewritten as follows:

$$\left. \begin{aligned} \frac{M_c}{b_0 h_s^2 \alpha_c \sigma_{tu}} &= \frac{(1 - 3k + 3k^2)}{3(1-k)} \\ &+ \frac{(\varphi_1 - 1)(\varphi_2^3 - 3k\varphi_2^2 + 3k^2\varphi_2)}{3(1-k)} \\ &+ \frac{n(p_H + p_D)(k_H - k)^2}{(1-k)} \end{aligned} \right\}$$

$$k = \frac{1}{2} \left[\frac{1 + 2n(p_H + p_D)k_H + (\varphi_1 - 1)\varphi_2^2}{1 + n(p_H + p_D) + (\varphi_1 - 1)\varphi_2} \right] \dots (22)$$

The solution of his formula is given also in Charts 8-9.

CONCLUSIONS

(1) The method given in this paper for evaluation of the initial corner cracking moment of reinforced and prestressed concrete beams can be applied with confidence to the design of bracketed parts of the beams of all rectangular, tee sections used in practice.

(2) The initial inclination angles of corner cracks vary between 25° and 40° in reinforced concrete beams and between 10° and 40° in prestressed concrete beams. For the beams which were ordinary reinforced or prestressed and serve under design load, $\theta_i = 35^\circ$ for reinforced and $\theta_i = 20^\circ$ for prestressed concrete beams are suggested for the computation.

(3) The corner cracking coefficient $\alpha_c = 0.51$ for reinforced concrete beams and $\alpha_c = 0.87$ for prestressed concrete beams are capable of predicting within a reasonable limit.

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NOTATION

- a : shear span, distance between load point and nearest support
- l : span of beam
- l_s : length of bracket base to the support
- b : width of compression face of beam
- b_0 : width of web in tee section
- h : over-all depth of beam
- h_s : over-all depth of bracket
- t : thickness of flange in tee section
- d : distance from extreme compression fiber to centroid of tension reinforcement of beam
- d_s : distance from extreme compression fiber to centroid of tension reinforcement of bracket
- M_c : initial corner cracking moment
- Z : section modulus
- n : ratio of elasticity of steel to that of concrete
- σ_{tu} : tensile strength of concrete
- α_c : coefficient
- θ_c : inclination angle of corner crack at shear failure (see Figure 9)
- θ_i : initial inclination angle of corner crack
- θ_1 : angle between inclined steel and longitudinal axis of member
- A_{SH} : total area of longitudinal steel
- A_{SV} : total area of steel placed perpendicular to the longitudinal reinforcement
- A_{SD} : total area of inclined steel
- p_H : A_{SH}/bh_s
- p_V : A_{SV}/bh_s
- p_D : A_{SD}/bh_s
- k_H : d_s/h_s
- φ_1 : b/b_0
- φ_2 : t/h_s

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Chart 1 Initial corner cracking moment for rectangular beams ($n=5$)

Chart 2 Initial corner cracking moment for rectangular beams ($n=6$)

Chart 3 Initial corner cracking moment for rectangular beams ($n=7$)

Chart 4 Initial corner cracking moment for rectangular beams ($n=8$)

Chart 5 Initial corner cracking moment for rectangular beams ($n=9$)

Chart 6 Initial corner cracking moment for rectangular beams ($n=10$)

Chart 7 Initial corner cracking moment for rectangular beams ($n=15$)

Chart 8 Nomography for location of neutral axis

Chart 9 Nomography for initial corner cracking moment