ELASTO-PLASTIC ANALYSIS OF STEEL STRUCTURES CONSIDERING THE EFFECTS OF RESIDUAL STRESS AND FINITE DEFORMATION

By Toshiaki OHTA* and Tokuya YAMASAKI**

ABSTRACT

A general beam theory is developed capable of treating elasto-plastic bending behavior of steel structures subjected to (a) incrementary load and (b) repeated load combined with axial load; Attention is focused on the evaluation of the effects of residual stress and finite deformation.

Extended complementary energy method is employed throughout the analysis and numerical results are presented for stress, curvature and deflection of steel members with rectangular and I-sections of the structures.

Problems of plastic stability are also considered theoretically from the same energy viewpoint.

1. INTRODUCTION

The classical approach1),2),3),4),5) to the evaluation of inelastic behavior of steel member is generally based on the moment-curvature equation.

In this case, however, when a steel member having residual stress is subjected to variable cyclic load combined with axial load, serious difficulty is encountered in formulating exact representations5) for the inelastic behavior. For clarity, function of the moment-curvature equation changes extremely its form under the above mentioned load condition, and the analytical procedure due to such function can thus be much complicated.

The resolution of the foregoing difficulty may be given by using the extended complementary energy method of continuum mechanics together with the generalized moment-curvature formula which will be derived in this paper by the authors.

This can be accomplished by estimating directly the stress-strain relation for the given loads based on the numerical integration technique6),7),8),9) and trial and error method.

As a result, the cumbersome but otherwise rigorous solution of the elasto-plastic beam theory is replaced herein by much simpler generalized one with sufficient engineering accuracy. And the inelastic bending problems for steel structures with residual stress, subjected to combined loads such as combined bending moment and axial force, are solved by the numerical analysis and results are presented for the stress, the moment-curvature relation, the deflection and so on.

It is also shown that the elasto-plastic finite deformations of structures can be easily analyzed by the extended complementary energy method9).

Furthermore, in this investigation, problems of plastic stability, usually accompanied with the bending problems above mentioned, are considered by using matrix analysis from the same energy viewpoint.

2. ASSUMPTIONS

The method considered herein makes the following assumptions.

1) The stress-strain history curve for steel is idealized as shown in Fig. 1.
2) The plane sections of the member remain plane during bending.
3) The member is of uniform section.

* Assistant Professor, Department of Civil Engineering, Faculty of Engineering, Miyazaki University
** Assistant General Manager, Steel Structural Research Laboratories, KAWASAKI STEEL Coorporation

Fig. 1 Stress-Strain Diagram.
4) The member is subjected to bending moment and axial force.
5) Effect of shearing stress is neglected.
6) Possibility of local buckling is disregarded.

The coordinate systems are as follows: The coordinate x is parallel to the axis of the member and y is perpendicular to the neutral plane.

3. ELASTO-PLASTIC BENDING

(1) Stress-Strain Relation

Denoting normal stress and strain in x direction by \( \sigma \) and \( \varepsilon \), respectively, the stress-strain relation for the steel, as shown in Fig. 1, is generally expressed by the following non-dimensional form.

\[
\bar{\varepsilon} = \nu (\varepsilon - \varepsilon^*) \\
\sigma_y : \text{yield stress of steel,} \\
\varepsilon_y : \text{yield strain of steel.}
\]

\[\bar{\varepsilon} = \sigma/\sigma_y, \quad \varepsilon^* = \varepsilon^*/\varepsilon_y \]

Table 1 Values of \( \varepsilon_a, \varepsilon_b, \varepsilon_{ay}, \varepsilon_{by} \) and \( D_i \).

<table>
<thead>
<tr>
<th>Step</th>
<th>( \varepsilon_a )</th>
<th>( \varepsilon_{ay} )</th>
<th>( \varepsilon_b )</th>
<th>( \varepsilon_{by} )</th>
<th>( D_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \varepsilon_{ay} = \varepsilon_{ay}' - \varepsilon_{ay}^* )</td>
<td>( \varepsilon_a = \varepsilon_a' + \varepsilon_{ay}^* )</td>
<td>( \varepsilon_b = \varepsilon_b' - \varepsilon_{by}^* )</td>
<td>( D_i = D_i' )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( \varepsilon = \varepsilon_{ay}' - \varepsilon_{ay}^* + \varepsilon_{ay}^* )</td>
<td>( \varepsilon_a = \varepsilon_a' + \varepsilon_{ay}^* )</td>
<td>( \varepsilon_b = \varepsilon_b' - \varepsilon_{by}^* )</td>
<td>( D_i = D_i' )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( \varepsilon_{by} = \varepsilon_{by}' - \varepsilon_{by}^* + \varepsilon_{by}^* )</td>
<td>( \varepsilon_a = \varepsilon_a' + \varepsilon_{ay}^* )</td>
<td>( \varepsilon_b = \varepsilon_b' - \varepsilon_{by}^* )</td>
<td>( D_i = D_i' )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( \varepsilon_{b} = \varepsilon_{b}' + \varepsilon_{b}^* )</td>
<td>( \varepsilon_a = \varepsilon_a' + \varepsilon_{ay}^* )</td>
<td>( \varepsilon_b = \varepsilon_b' - \varepsilon_{by}^* )</td>
<td>( D_i = D_i' )</td>
<td></td>
</tr>
</tbody>
</table>

Because the procedure is programmed for an electronic computer such that the initial values of \( \varepsilon_a, \varepsilon_{ay}, \varepsilon_{ay}' \) are given by Eq. (2), the values of \( \nu, \varepsilon^* \) and \( \varepsilon_a, \varepsilon_{ay} \) for the given load are taken as follows;

If \( \varepsilon_a < \varepsilon_a^* \) and \( D_i = -1 \) then \( \nu = 1, \varepsilon^* = \varepsilon_a - \sigma_a/E_0 \) and \( \varepsilon_a, \varepsilon_{ay} \) are defined as shown at rank of Step 1 (loading at the elastic state) in Table 1.

If \( \varepsilon_a \geq \varepsilon_a^* \) then \( \nu = \mu_b, \varepsilon^* = \varepsilon_a - \sigma_a/E_b \) and \( \varepsilon_a, \varepsilon_{ay} \) are defined as shown at rank of Step 2 (loading at the elasto-plastic state).

If \( \varepsilon_{ay} < \varepsilon_a < \varepsilon_b \) and \( D_i = -1 \) then \( \nu = 1, \varepsilon^* = \varepsilon_b - \sigma_b/E_0 \) and \( \varepsilon_a, \varepsilon_{ay} \) are defined as shown at rank of Step 3 (unloading elastic state).

If \( \varepsilon_b < \varepsilon_a < \varepsilon_b \) and \( \varepsilon_{ay} < \varepsilon_a < \varepsilon_b \) then \( \nu = \mu_b, \varepsilon^* = \varepsilon_b - \sigma_b/E_b \) and \( \varepsilon_a, \varepsilon_{ay} \) are defined as shown at rank of Step 4 (unloading at the elasto-plastic state).

In Table 1, values of \( \varepsilon_a^*, \varepsilon_{ay}^*, \varepsilon_{ay}' \) are the ones of \( \varepsilon_a, \varepsilon_{ay}, \varepsilon_{ay}' \) at the former load-step, respectively.

While, the initial values of \( \varepsilon_a, \varepsilon_{ay}, \varepsilon_{ay}' \) are given as:

\[
\sigma_a = \sigma_b = - \sigma_y - \sigma_r, \quad \varepsilon_a = \varepsilon_{ay} = - \varepsilon_y - \varepsilon_r, \quad \varepsilon_b = \varepsilon_{by} = - \varepsilon_y - \varepsilon_r.
\]

\[
D_i = \begin{cases} 
1, & \text{if loading at the elastic state,} \\
-1, & \text{if unloading at the elastic state.}
\end{cases}
\]

(2) General Formula for the Moment-Curvature Relations

For equilibrium, external forces must equal to internal forces.

Therefore,

\[
\begin{bmatrix} M \\ N \end{bmatrix} = \int \sigma y dA_y - \int \sigma y dA_z \\
\begin{bmatrix} 0 & 1 \\ 1 & -a_i h \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} \]

where \( M \) : external bending moment, 
\( N \) : external axial force, 
\( M_j, N_j \) : elementary moment and force of cross section, 
\( a_i h \) : depth of centroid of cross section.

Substituting Eq. (4) into Eq. (5) and replacing \( M \) by \( M = M/M_j, N = N/N_j \), Eq. (5) will become

\[
\begin{bmatrix} 0 & 1 \\ 1 & -a_i h \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} \]

\[
\begin{bmatrix} 0 & 1 \\ 1 & -a_i h \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} \]
where  \( M_y = E_o I \phi_y \) : yield moment, \( N_y = \sigma_y A_o \),
\( a_o = N_y h / M_y \),
\( A_o \) : cross sectional area,
\( E_o \) : modulus of elasticity,
\( I \) : moment of inertia of cross section,
\( \phi_y \) : yield curvature.

On the other hand, curvature \( \phi \) and strain \( \varepsilon \) at the centroid of cross section are given as follows (See Fig. 2);

\[
\begin{bmatrix}
\phi \\
\varepsilon
\end{bmatrix} =
\begin{bmatrix}
\phi_y \\
\varepsilon_y
\end{bmatrix} =
\begin{bmatrix}
\alpha_2 & -\alpha_1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2
\end{bmatrix}
\]  

(7)

From Eqs. (6) and (7), we find the required equation as,

\[
\begin{bmatrix}
\phi \\
\varepsilon
\end{bmatrix} =
\begin{bmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2
\end{bmatrix} +
\begin{bmatrix}
G_1 \\
G_2
\end{bmatrix}
\]  

(8)

where

\[
\begin{bmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{bmatrix} =
\begin{bmatrix}
\alpha_2 & -\alpha_1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
D_1 \\
D_2
\end{bmatrix}^{-1}
\begin{bmatrix}
0 & 1 \\
1 & -\alpha_2
\end{bmatrix}
\]

\[
D_1 =
\begin{bmatrix}
\alpha_2 & \nu(1-\nu) \frac{dA_3}{A_0} \\
\alpha_1 & \nu(1-\nu) \frac{dA_3}{A_0}
\end{bmatrix}
\begin{bmatrix}
\nu \frac{dA_3}{A_0} \\
\nu \frac{dA_3}{A_0}
\end{bmatrix}
\]

\[
G_1 =
\begin{bmatrix}
\alpha_2 & -\alpha_1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
D_1 \\
D_2
\end{bmatrix}^{-1}
\begin{bmatrix}
E_1 \\
E_2
\end{bmatrix}
\]

\[
E_1 =
\begin{bmatrix}
\alpha_2 & \nu(1-\nu) \frac{dA_3}{A_0} \\
\alpha_1 & \nu(1-\nu) \frac{dA_3}{A_0}
\end{bmatrix}
\begin{bmatrix}
\nu \frac{dA_3}{A_0} \\
\nu \frac{dA_3}{A_0}
\end{bmatrix}
\]

\[
E_2 =
\begin{bmatrix}
\alpha_2 & \nu(1-\nu) \frac{dA_3}{A_0} \\
\alpha_1 & \nu(1-\nu) \frac{dA_3}{A_0}
\end{bmatrix}
\begin{bmatrix}
\nu \frac{dA_3}{A_0} \\
\nu \frac{dA_3}{A_0}
\end{bmatrix}
\]

\[
\alpha_2 = \frac{\tau \phi_y}{h \phi_y}
\]

a) Rectangular Section

Let consider a rectangular section shown in Fig. 3 as the first example, elements \( D_{ij} \) and \( E_{ij}(i, j = 1, 2) \) of matrices \([D_e]\) and \([E_e]\) in question are obtained by using so called Trapezoidal formula as,

\[
D_{11} = d \bar{h} \sum_{k=1}^{N_y} \left[ \nu_k (1 - \bar{y}_k) + \nu_{k+1} (1 - \bar{y}_{k+1}) \right]/2,
\]

\[
D_{12} = d \bar{h} \sum_{k=1}^{N_y} \left[ \nu_y \bar{y}_k + \nu_{y+1} \bar{y}_{k+1} \right]/2,
\]

\[
D_{22} = d \bar{h} \sum_{k=1}^{N_y} \left[ \nu_k \bar{y}_k + \nu_{k+1} \bar{y}_{k+1} \right]/6,
\]

\[
E_{11} = d \bar{h} \sum_{k=1}^{N_y} \left[ \nu_k \bar{y}_k + \nu_{k+1} \bar{y}_{k+1} \right]/2
\]

\[
E_{22} = d \bar{h} \sum_{k=1}^{N_y} \left[ \nu_k \bar{y}_k + \nu_{k+1} \bar{y}_{k+1} \right]/2,
\]

where \( d \bar{h} = h \phi_y \),

Fig. 3 Stress-Distribution Diagram for Rectangular Section.

Fig. 4 Residual Stress-Distribution Diagram for Rectangular Section.

Fig. 5 Moment-Curvature Relation for Rectangular Section under Incrementary Loading.

Fig. 6 Moment-Curvature Relation for Rectangular Section under Cycirc Loading.
\( \Delta h \): length of the divided element.

From the above result, moment-curvature relations are obtained for the rectangular section\(^{2}\) with residual stress of Fig. 4 by running computer program with \( \Delta h = h/10 \), \( \sigma_{ref} = -0.3 \sigma_{y}, \sigma_{a} = 2100 \) kg/cm\(^2\), \( E_{0} = 2.1 \times 10^{6} \) kg/cm\(^2\), \( \mu_{f} = 0.005 \) and \( N \) having the values 0 and 0.3, respectively, and plotted in Fig. 5.

On the other hand, moment-curvature hysteresis curve of the same section can be given as in Fig. 6 under cyclic bending combined with axial force \( N = 0.1 \).

**b) I-Section**

A wide-flange I-section shown in Fig. 7 is considered as the second illustrative example.

In this case, the elements \( D_{ij} \) and \( E_{ij} \) are

\[
D_{11} = \omega \Delta h \left[ \sum_{k=1}^{N} \{ v_{k} (1 - \bar{y}_{k}) + v_{k+1} (1 - \bar{y}_{k+1}) \} 
+ 2 \Delta h \left\{ \sum_{k=1}^{N} \bar{y}_{k} (v_{k} + v_{k+1}) + \sum_{k=1}^{N} \bar{y}_{k} (v_{k} + v_{k+1}) \right\} / 2 A_{0} \right]
\]

\[
D_{12} = \omega \Delta h \left[ \sum_{k=1}^{N} \bar{y}_{k} (v_{k} + v_{k+1}) \bar{y}_{k+1} \right]
+ 2 \Delta h \left\{ \sum_{k=1}^{N} \bar{y}_{k} (v_{k} + v_{k+1}) + \sum_{k=1}^{N} \bar{y}_{k} (v_{k} + v_{k+1}) \right\} / 2 A_{0} \]

\[
D_{21} = \sigma_{f} \omega \Delta h \left[ \sum_{k=1}^{N} \left\{ v_{k} (3 \bar{y}_{k} + \Delta h) (1 - \bar{y}_{k}) + v_{k+1} (3 \bar{y}_{k} + \Delta h) (1 - \bar{y}_{k+1}) \right\} 
+ 6 \Delta h \left\{ \sum_{k=1}^{N} \bar{y}_{k} (v_{k} + v_{k+1}) + \sum_{k=1}^{N} \bar{y}_{k} (v_{k} + v_{k+1}) \right\} / 6 A_{0} \right]
\]

\[
D_{22} = \sigma_{f} \omega \Delta h \left[ \sum_{k=1}^{N} \left\{ v_{k} (3 \bar{y}_{k} + \Delta h) \bar{y}_{k}
+ v_{k+1} (3 \bar{y}_{k} + \Delta h) \bar{y}_{k+1} \right\} 
+ 6 \Delta h \left\{ \sum_{k=1}^{N} \bar{y}_{k} (v_{k} + v_{k+1}) + \sum_{k=1}^{N} \bar{y}_{k} (v_{k} + v_{k+1}) \right\} / 6 A_{0} \right]
\]

\[
E_{11} = \omega \Delta h \left[ \sum_{k=1}^{N} \{ \bar{v}_{k} (3 \bar{y}_{k} + \Delta h) \bar{y}_{k}
+ \bar{v}_{k+1} (3 \bar{y}_{k} + \Delta h) \bar{y}_{k+1} \right\} 
+ 6 \Delta h \left\{ \sum_{k=1}^{N} \bar{y}_{k} (v_{k} + v_{k+1}) + \sum_{k=1}^{N} \bar{y}_{k} (v_{k} + v_{k+1}) \right\} / 6 A_{0} \right]
\]

where \( \omega \): width of web,

\( \gamma_{1}, \gamma_{2} \): depth of the centroid of lower and upper flanges, respectively.
h: depth of cross section,  
\( t \): thickness of flange,  
\( b \): width of flange,  
\( d b \): elemental length of \( b \),  
\( d h \): elemental length of \( h \),  
\( A_o \): total cross-sectional area,  
\( k^e \): number of element in flange, used as suffix,  
\( k \): number of element in web, used as suffix.

From the result obtained above, the moment-curvature relation for I-section\(^2\) of Fig. 8, named WF Section (\( \sigma_y = 2812 \text{kg/cm}^2, E_o = 2.1 \times 10^6 \text{kg/cm}^2 \)), will be obtained as in Fig. 9.

Good agreement is obtained between the result of this method and rigorous solution\(^2\).

(3) Complementary Energy Method

Consider a steel member AB subjected to axial force \( N \) and bending moment \( M \), the complementary energy \( dU^* \) stored in the infinitesimal element \( dS \) of the member AB will be given\(^8\) from Fig. 10, as

\[
dU^* = \left\{ \int \phi(1+\varepsilon^o) dMdS + \int \varepsilon^o dN \right\} - \int \theta \frac{dM}{dS} dS \quad \text{(9)}
\]

where \( \theta \): chord slope of the element,  
\( H \): horizontal force.

Then, the total complementary energy of the member AB is

\[
U^* = \sum \left\{ \int \phi(1+\varepsilon^o) dMdS + \int \varepsilon^o dN \right\} - \int \theta \frac{dM}{dS} dS \quad \text{(9)}
\]

For the small deformation theory, Eq. (9) yields by dropping the secondary effects of the deformation.

\[
U^* = \sum \left\{ \int \phi(1+\varepsilon^o) dMdS + \int \varepsilon^o dN \right\} - \int \theta \frac{dM}{dS} dS \quad \text{(10)}
\]

(4) Elasto-Plastic Slope-Deflection Equation

An elasto-plastic slope-deflection equation of steel member subjected to bending moment and axial force such as shown in Fig. 11 is derived herein, for the convenience of the indeterminate structural analysis.
\[ \nu_i = \sum_{j=1}^{i-1} (1 + \epsilon_j) \sin \theta_j, \]
\[ Q_{AB} = -(M_{AB} + M_{BA} + \tau_{AB} H_{AB})/\epsilon_{ii}, \]
\[ M_i^p : \text{moment of point } i \text{ due to applied loads}, \]
\[ M_{AB}, M_{BA} : \text{end moments at ends } A, B, \text{ respectively}. \]

From the complementary minimum principle, rotation angles \( \theta_A, \theta_B \) at ends \( A, B \) are given by using Eq. (11) as
\[
\theta_A = R = \frac{\partial U^*}{\partial M_{AB}} = \int_{A}^{B} \phi(1 + \epsilon^i) \frac{\partial M}{\partial M_{AB}} dS, \]
\[
\theta_B = R = \frac{\partial U^*}{\partial M_{BA}} = \int_{A}^{B} \phi(1 + \epsilon^i) \frac{\partial M}{\partial M_{BA}} dS, \]
where \( R \) : revolution angle of member AB.

Substituting Eq. (15) into Eq. (16) and replacing \( \phi \) by \( \phi_i + \tau_i M/E_0 I \), we get the following equation.
\[
(\epsilon_i + \epsilon^i) \phi_i + \phi_i = \phi(1 + \epsilon^i) \frac{\partial M}{\partial M_{AB}} dS, \]
where \( a_i = \int_{A}^{B} (\epsilon_i + \epsilon^i) dS/l, \)
\[ b_i = b_i = \int_{A}^{B} \phi_i(1 + \epsilon^i) dS/l, \]
\[ a_i = \int_{A}^{B} \phi_i(1 + \epsilon^i) dS/l, \quad \phi = \phi - M/E_0 I, \]
\[ K = l/1, \quad \kappa = \phi \phi_i, \]
\[ C_i = M_i \int_{A}^{B} f(M + H_{AB} + \tau_{AB} + \phi) (1 + \epsilon^i) dS/l, \]
Thus, the required equation can be obtained by solving Eq. (17)
\[
M_{AB} = E_0 K (\epsilon_{AB} + \epsilon_{BA} + \epsilon_{AB} + \epsilon_{BA}) + C_{AB}, \]
\[ M_{BA} = E_0 K (\epsilon_{BA} + \epsilon_{AB} + \epsilon_{BA} + \epsilon_{AB}) + C_{BA}, \]
where \( a_{AB} = -a_1, \quad a_{BA} = -a_1, \)
\[ \beta_{AB} = b_1, \quad \beta_{BA} = a_1, \quad \alpha_1 = a_1, \quad \alpha_{ii} = a_1, \]
\[ C_{AB} = (a_1 - b_1), \quad C_{BA} = (a_1 - b_1)/a_1. \]

Eq. (18) is transformed into non-dimensional form as
\[
M_{AB} = \frac{1}{\omega} (\epsilon_{AB} + \epsilon_{AB} + \epsilon_{AB} + \epsilon_{BA}) + C_{AB}, \]
\[ M_{BA} = \frac{1}{\omega} (\epsilon_{BA} + \epsilon_{AB} + \epsilon_{BA} + \epsilon_{AB}) + C_{BA}, \]
where \( C = C/M_2, \omega = M_2/E_0 K = \phi_i. \]

(5) Illustrative Examples

a) Example-1

To illustrate this method it is applied to a cantilever AB subjected to vertical and horizontal concentrated loads at free end B.

If the cantilever is divided into 10 elements as shown in Fig. 12, equilibrium conditions will be given as follows;
\[
Q_{i+1} = P_o \cos \theta_i + H_o \sin \theta_i, \]
\[ N_{i+1} = P_o \sin \theta_i - H_o \cos \theta_i, \]
\[ M_{i+1} = M_i + Q_i (1 + \epsilon_i) + \epsilon^i, \quad (i=1\sim10) \]
where \( \theta_i = -\lambda \left( \phi_i (1 + \epsilon_i) \right) + \left( \phi_i + \phi_{i+1} \right) (2 + \epsilon_i + \epsilon_i^i)/2, \]
\[ P_o : \text{vertical concentrated load}, \]
\[ H_o : \text{horizontal concentrated load}, \]
\[ Q : \text{shearing force}, \]
\[ \lambda = l/10, \quad l : \text{span length}, \]

From Eq. (20), unknown quantities \( M_i, N_i \) and \( Q_i \) will be evaluated by using trial and error method, where \( \phi_i, \theta_i \) and \( \epsilon_i \) can be determined by Eqs. (14) and (8).

With these quantities \( M_i, N_i \) and \( Q_i \) known, vertical displacement \( \phi_{B} \) at the end B can be calculated by using complementary minimum principle as follows;
\[
\phi_{B} = \frac{\partial U^*}{\partial P_o} \]
obtained for the beam with rectangular cross section by running computer program with \( h = h/10 \), \( h = 20.47 \text{ cm} \), \( b = 2.81 \text{ cm} \), \( f_{Dy} = 2 \times 10^6 \text{ kg/cm}^2 \), \( E_0 = 2.1 \times 10^6 \text{ kg/cm}^2 \), \( f_{E0} = 0.005 \), \( l = 10 \text{ h} \), \( f_{D} = 0.3 \) and \( H_0 ( = H_0/N_y) \) having the value 0 and 0.3, respectively, and are plotted in Fig. 13.

Comparison is made between the results of this solution and the solution given by the small deformation theory. And it is found that the bending rigidity of the beam under axial force decreases considerably by the secondary effect of the finite deformation (See Figs. 13, 14).

It is also recognized that the iteration procedure converges rapidly if initial values of the unknowns are chosen properly, and that the problem of obtaining suitably initial values of the unknowns is conveniently resolved by using the incremental load method.

Fig. 15 also shows curve for the I-section of Fig. 8.

While, the stress-distribution diagrams for the given load become as shown in Fig. 16.

b) Example-2

A structure of Fig. 17 is taken herein as the second example.

End moments \( M_{AB}, M_{BA}, M_{BC} \) can be expressed from the given structural and load conditions as follows:

\[
\begin{align*}
M_{AB} &= \frac{1}{\omega} (\beta_{AB} \theta_B + \gamma_{AB} R) + C_{AB} \\
M_{BA} &= \frac{1}{\omega} (\alpha_{BA} \theta_B + \gamma_{BA} R) + C_{BA} \\
M_{BC} &= \frac{1}{\omega} (\alpha_{BC} \theta_B + \gamma_{BC} R) + C_{BC}
\end{align*}
\tag{23}
\]

Moment equilibrium condition at joint B is

\[
M_{BA} + M_{BC} = 0 \quad \text{..................................................(24)}
\]

And, force equilibrium condition is

\[
\begin{align*}
\bar{H}_a &= 2 \bar{Q}_{BA} = 0, \\
\bar{P}_e + \bar{Q}_{BC} - \bar{H}_{BA} &= 0
\end{align*}
\tag{25}
\]

where \( \bar{H}_a = H_0/N_y, \bar{P}_e = P_e/N_y, \bar{Q} = Q/N_y, \bar{H} = H/N_y \).

From Eqs. (23), (24) and Eq. (25), \( \theta_B \) and \( R \) will be given as

\[
\begin{align*}
\theta_B &= \frac{\omega \left( T_{BA} \left( \frac{T_{H_a}}{2} + \frac{T_{H_B}}{2} - v_{36} H_{AB} + C_{AB} - C_{BC} \right) - T_{AB} (C_{BA} + C_{BC}) \right)}{T_{BA} (\alpha_{BC} + \beta_{BC}) \theta_B + \omega (C_{BA} + C_{BC})} \\
R &= \frac{1}{T_{BA}} \left( (\alpha_{BA} + \alpha_{BC} + \beta_{BC}) \theta_B + \omega (C_{BA} + C_{BC}) \right)
\end{align*}
\tag{26}
\]
Then, horizontal displacement $\Delta B$ of joint B is

$$\Delta B = n_1 \tan R \quad \text{(27)}$$

In the above obtained equations, chord slope $\theta_i$ ($i=2\sim8, 11\sim17$) is defined by Eq. (12).

While, $\theta_1, \theta_{13}$ are expressed by the following equations.

$$\begin{align*}
\frac{1}{8} \sum_{i=1}^{17} (1+\epsilon_i^b) \sin \theta_i &= 0, \\
\frac{1}{8} \sum_{i=1}^8 (1+\epsilon_i^b) \sin \theta_i &= \Delta B
\end{align*}$$

or

$$\begin{align*}
\theta_{13} &= \frac{1}{8} \left\{ \sum_{i=1}^{17} (18-i)J_i \\
&- \frac{1}{8} \sum_{i=1}^{17} (\epsilon_i^b \sin \theta_i + \sin \theta_i - \theta_i) \right\} \\
\theta_1 &= \frac{1}{8} \left\{ \sum_{i=1}^8 (9-i)J_i \\
&- \frac{1}{8} \sum_{i=1}^8 (\epsilon_i^b \sin \theta_i + \sin \theta_i - \theta_i) \right\} + \Delta B
\end{align*}$$

where $J_i = (2\phi_i (1+\epsilon_i^b) + (\phi_i + \phi_{i-1}) (2+\epsilon_i^b + \epsilon_{i-1}^b)/2 + (\phi_i + \phi_{i+1}) (2+\epsilon_i^b + \epsilon_{i+1}^b)/2)/6$.

Using Eqs. (23), (26), (27), (28) and Eq. (8), the displacement $\Delta B$ to the increasing loads $H_n, P_n$ can be calculated with the parameters of the ratio $P_n/H_n=0$ and 10.

Comparison is made in Fig. 18 between the $H_n-\Delta B$ curves due to the finite deformation theory and the small deformation theory.

**4. Problem of Plastic Stability**

**(1) General Theory**

A member AB subjected to bending moments $M_{AB}, M_{BA}$ and horizontal force $H_{AB}$ is shown in Fig. 19 with notation for the coordinate-axes, displacements and forces.

If the member AB is divided into n elements, moments at the every divided points of the beam are given by considering the effect of the deflection in the following matrix form.

$$[\bar{M}] = (\left[ 1 \right] + [-X]) \cdot \bar{M}_{AB} + \bar{M}_{BA} [-X]$$

**Fig. 18** Horizontal Displacement $\Delta B$ of the Frame of Fig. 17.

**Fig. 19** Steel Member AB Subjected to $M_{AB}, H_{AB}, Q_{AB}$ at End A and $M_{BA}, H_{BA}, Q_{BA}$ at End B.

**Fig. 20** Simple Beam AB Subjected to Virtual Concentrated Load $P_i=0$ at Point i.
In general, the above obtained equation can be transformed into the following matrix form,

\[ \begin{bmatrix} \mathbf{Y} \\ \mathbf{X} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{X} \\ \mathbf{X} \end{bmatrix} - \omega \begin{bmatrix} \mathbf{X} \\ \mathbf{X} \end{bmatrix} \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix} + \mathbf{C} \begin{bmatrix} \mathbf{X} \\ \mathbf{X} \end{bmatrix} \]

where \( [a] \) : square matrix of order \((n, n)\),
\( [\mathbf{B}] \) : column matrix of order \((n, 1)\).

On the other hand, the curvature at point \( i \) is generally rewritten from Eq. (8) as,
\[ \phi_i = \frac{\phi_i}{\theta_i} \]
where \( \phi_i \) is the curvature at point \( i \).

Substituting Eq. (29) into Eq. (34), we have
\[ \begin{bmatrix} \phi_i \end{bmatrix} = \begin{bmatrix} \phi_i \end{bmatrix} + \begin{bmatrix} \phi_i \end{bmatrix} \]
which is transformed into
\[ \begin{bmatrix} \phi_i \end{bmatrix} = \begin{bmatrix} \phi_i \end{bmatrix} + \begin{bmatrix} \phi_i \end{bmatrix} \]

where \( \phi_i = \frac{\phi_i}{\theta_i} \) is the curvature at point \( i \).

Substituting Eq. (33) into Eq. (36), we have
\[ \begin{bmatrix} \phi_i \end{bmatrix} = \begin{bmatrix} \phi_i \end{bmatrix} + \begin{bmatrix} \phi_i \end{bmatrix} \]
where \( \phi_i = \frac{\phi_i}{\theta_i} \) is the curvature at point \( i \).

Finally, we can get the required formula as
\[ \begin{bmatrix} \phi_i \end{bmatrix} = \begin{bmatrix} \phi_i \end{bmatrix} + \begin{bmatrix} \phi_i \end{bmatrix} \]
where \( [k] = [a]^{-1} [b] \), \( [a] = [a]^{-1} [c] \).

From the foregoing investigation, \( q = \frac{\phi_i}{\theta_i} \) curve can be obtained for the H-section including residual stress.
stress, shown in Fig. 22, by running computer program with \( H_0 = 0.7, \mu = 0.001, \delta h = 0.1, h = 37.56 \text{ cm}, \ t = 10 \text{ cm}, \ w = 4 \text{ cm}, \ l = 170 \text{ cm}, \ \sigma_y = 3300 \text{ kg/cm}^2, \ E_o = 2.1 \times 10^6 \text{ kg/cm}^2 \) and \( \varepsilon \), having the values 0, 0.1, 0.2, 0.3, respectively, and plotted in Fig. 23, where the effect of the stress in the web is disregarded.

The transition points from stable to unstable equilibrium in Fig. 23 are given in Table 2. In the case of \( \varepsilon_r = 0 \), the above obtained values (0.0270, 0.765 \times 10^{-2}) are good agreement with the ones\textsuperscript{10} (0.0286, 0.765 \times 10^{-2}) estimated by using the stress-strain curve of Fig. 24. Fig. 25 is cyclic load-deflection curve diagram of same cantilever beam with \( H_0 = 0.3, \mu = 0.5, \delta h = 0.1, h = 37.56 \text{ cm}, \ b = 20 \text{ cm}, \ l = 170 \text{ cm}, \ \sigma_y = 2400 \text{ kg/cm}^2, \ E_o = 2.1 \times 10^6 \text{ kg/cm}^2 \) and \( \varepsilon_r = 0 \).

5. Conclusion

In this paper, the authors have succeeded in deriving general formula for the curvature of steel member subjected to bending moment combined with axial force, based on the numerical integration technique and iteration method.

The derived formula is then used to analyze elasto-plastic finite deformations of steel structures and to determine the critical loads employing the extended complementary energy method, wherein the effect of residual stress can be taken into account.

The iteration procedure, as presented, converges rapidly for any variable load by utilizing the incremental load method.

By the way, if a steel member is subjected to axial force in addition to bending moment, it is reasonable to expect that the rigidity of the member is influenced by the residual stress and finite deformation.

This fact is clarified in Examples of Fig. 12, Fig. 17 and Fig. 21 in which the deflection increases...
remarkably by the effect of the residual stress or that of the finite deformation.

Although these numerical examples deal with such simple structures as cantilever or rectangular frame of Fig. 17, the presented method is, of course, applicable to more complicated structures.

Thus, it may be concluded that the advantages of this method are its simplicity and generality of formulation, and even its applicability to a great variety of problems with complicated loading and structural conditions.

REFERENCES