GENERAL FORMS OF GALERKIN VECTOR
IN CYLINDRICAL COORDINATES

By Hiroyoshi Hirai* and Masao Satake**

1. INTRODUCTION

The various forms of solution of Navier’s equation expressed in terms of potentials have been proposed for the analysis of three-dimensional elastic problems. In the case of cartesian coordinates, many solutions are proposed by Neuber1), Papkovich1), Boussinesq2), Galerkin3) and so on. In the case of cylindrical coordinates, there is the solution by Love or Michell1) for axisymmetric problems and its generalization was studied by Hasegawa4). Further, Muki5) proposed a solution for asymmetric problems and the completeness of Muki’s solution was discussed by Hata6).

The above solutions in cylindrical coordinates are considered to be closely related with the solution expressed in terms of Galerkin vector. From this viewpoint, the general forms of Galerkin vector in cylindrical coordinates are proposed and their application to an asymmetric problem is shown in this paper.

2. GALERKIN VECTOR IN CYLINDRICAL COORDINATES

If the body force is absent, Navier’s equation is expressed as

\[ V^2u + \frac{1}{1-2\nu} \nu (\nabla \cdot u) = 0, \quad (1) \]

where \( u \) is the displacement vector and \( \nu \) is Poisson’s ratio. Galerkin introduced a solution of Eq. (1) with the form

\[ \nabla^2 \mathbf{G} = \mathbf{g}, \quad (2) \]

\[ V^2 \mathbf{g} = 0, \quad (3) \]

where \( \mathbf{G} \) is called the Galerkin vector and \( \lambda \) denotes Lamé’s modulus. In the following, general expressions of Galerkin vector in cylindrical coordinates are investigated. As Eq. (3) is considered to be a biharmonic equation, \( \mathbf{G} \) is to be a biharmonic vector. Putting

\[ \nabla^2 \mathbf{G} = \mathbf{g}, \quad (4) \]

\[ V^2 \mathbf{g} = 0, \quad (5) \]

\[ \mathbf{G} = (G_r, G_\theta, G_z), \quad (6) \]

\[ \mathbf{g} = (g_r, g_\theta, g_z), \quad (7) \]

Eq. (5) is expressed in the form

\[ \left( \nabla^2 - \frac{1}{r^2} \right) g_r - \frac{2}{r^2} \frac{\partial}{\partial r} g_z = 0, \]

\[ \frac{2}{r^2} \frac{\partial}{\partial r} g_r + \left( \nabla^2 - \frac{1}{r^2} \right) g_z = 0, \]

\[ V^2 g_z = 0, \quad (8) \]

where

\[ V^2 = \frac{\partial}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right), \quad (9) \]

Referring the formulas for derivatives and recurrence relations in the Bessel functions, it is easy to show that the solutions of Eq. (8) can be given from combination of \( g_r, g_\theta \) and \( g_z \) listed in Table 1. On the other hand, Eq. (4) is expressed in the form

\[ \left( \nabla^2 - \frac{1}{r^2} \right) G_r - \frac{2}{r^2} \frac{\partial}{\partial \theta} G_z = g_r, \]

\[ \frac{2}{r^2} \frac{\partial}{\partial \theta} G_r + \left( \nabla^2 - \frac{1}{r^2} \right) G_z = g_\theta, \]

\[ V^2 G_z = g_z. \quad (10) \]

Substituting the solutions of Eq. (8) into the right-hand side of Eq. (10) and using the theorem of Almansi7) in cylindrical coordinates, the solutions of Eq. (10) can be given from combination of \( G_r, G_\theta \) and \( G_z \) listed in Table 1. In Table 1, \( J_\alpha(ax) \) and \( Y_\alpha(ax) \) are the Bessel functions of the first kind and the second kind of order \( \alpha \) respectively, and \( I_\beta(\beta r) \) and \( K_\beta(\beta r) \) are the modified Bessel functions of the first and the second kind of order \( \beta \) respectively, where \( \alpha \) and \( \beta \) are constants.
3. EXAMPLE

Okumura treated with the problem of an elastic short circular cylinder loaded semicircularly on its upper and lower sides by using the generalized Neuber's solution. We shall deal with the problem of an elastic short circular cylinder subjected to asymmetric loads (Fig. 1) as an application of the general forms of Galerkin vector explained in the previous chapter.

The boundary conditions are to be specified as follows:

\[ \tau_r = a \]
\[ \tau_\theta = 0 \]  
\[ \sigma_r = 0 \]
\[ \text{on } r = a \]
\[ \tau_\theta = 0 \]  
\[ \sigma_r = 0 \]
\[ \text{on } z = \pm h \]

\[ \tau_r = 0 \]  
\[ \tau_\theta = 0 \]  
\[ \sigma_r = 0 \]  
\[ \text{on } z = \pm h \]

Table 1: General forms of biharmonic vector in cylindrical coordinates.

<table>
<thead>
<tr>
<th>( g_{r,1} )</th>
<th>( r^{k+3} )</th>
<th>( \frac{1}{z} )</th>
<th>( \cos(\theta) )</th>
<th>( \sin(\theta) )</th>
<th>( \tau_{r,1} )</th>
<th>( r^{k+1} )</th>
<th>( \frac{1}{z} )</th>
<th>( \sin(\theta) )</th>
<th>( -\cos(\theta) )</th>
<th>( \tau_{r,2} )</th>
<th>( r^{k+1} )</th>
<th>( \frac{1}{z} )</th>
<th>( \sin(\theta) )</th>
<th>( -\cos(\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_{r,2} )</td>
<td>( I_k(a \sigma) )</td>
<td>( \cosh(\alpha z) )</td>
<td>( \cos(\theta) )</td>
<td>( \sin(\theta) )</td>
<td>( \tau_{r,3} )</td>
<td>( r f_{k+2}(a \sigma) )</td>
<td>( \cosh(\alpha z) )</td>
<td>( \cos(\theta) )</td>
<td>( \sin(\theta) )</td>
<td>( \tau_{r,4} )</td>
<td>( Y_k(a \sigma) )</td>
<td>( \sin(\alpha z) )</td>
<td>( \sin(\theta) )</td>
<td>( -\cos(\theta) )</td>
</tr>
<tr>
<td>( g_{r,3} )</td>
<td>( J_k(a \sigma) )</td>
<td>( \sinh(\alpha z) )</td>
<td>( \cos(\theta) )</td>
<td>( -\cos(\theta) )</td>
<td>( \tau_{r,5} )</td>
<td>( r f_{k+2}(a \sigma) )</td>
<td>( \cosh(\alpha z) )</td>
<td>( \sin(\theta) )</td>
<td>( -\cos(\theta) )</td>
<td></td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>( g_{r,4} )</td>
<td>( I_{k+1}(\beta r) )</td>
<td>( \cos(\beta z) )</td>
<td>( \cos(\theta) )</td>
<td>( \sin(\theta) )</td>
<td>( \tau_{r,6} )</td>
<td>( r K_{k+1}(\beta r) )</td>
<td>( \sin(\beta z) )</td>
<td>( \cos(\theta) )</td>
<td>( \sin(\theta) )</td>
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<td>( J_{k+1}(\beta r) )</td>
<td>( \sin(\beta z) )</td>
<td>( \cos(\theta) )</td>
<td>( -\cos(\theta) )</td>
<td>( \tau_{r,7} )</td>
<td>( r K_{k+1}(\beta r) )</td>
<td>( \sin(\beta z) )</td>
<td>( \cos(\theta) )</td>
<td>( -\cos(\theta) )</td>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>( g_{r,6} )</td>
<td>( 1 )</td>
<td>( \log r )</td>
<td>( \frac{1}{z} )</td>
<td>( \theta )</td>
<td>( \tau_{r,8} )</td>
<td>( 1 )</td>
<td>( \frac{1}{z} )</td>
<td>( \theta )</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>( g_{r,7} )</td>
<td>( 1 )</td>
<td>( \frac{1}{z} )</td>
<td>( \cos(\theta) )</td>
<td>( \sin(\theta) )</td>
<td>( \tau_{r,9} )</td>
<td>( 1 )</td>
<td>( \frac{1}{z} )</td>
<td>( \cos(\theta) )</td>
<td>( \sin(\theta) )</td>
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<tr>
<td>( g_{r,8} )</td>
<td>( \frac{1}{z} )</td>
<td>( \sin(\theta) )</td>
<td>( \frac{1}{z} )</td>
<td>( \cos(\theta) )</td>
<td>( \tau_{r,10} )</td>
<td>( 1 )</td>
<td>( \frac{1}{z} )</td>
<td>( \theta )</td>
<td></td>
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* The meaning of the form is, for example,
\[ g_{r,1} = (a_1 r^{k+1} + a_2 r^{k-1} + a_3 r^{k+1} + a_4 r^{k-1}) (b_1 \cos(\theta) + b_2 \sin(\theta)) \]
where \( a_1, ..., a_4, b_1, b_2 \) are constants.
As the distributed loads are symmetrical for \( z = 0 \) and proportional to \( \cos \theta \), we choose the forms of Galerkin vector in the following manner:

\[
\sigma_z = \begin{cases} 
\frac{p}{d} r \cos \theta, & (0 \leq r \leq d) \\
0, & (d < r \leq a)
\end{cases}
\]  
\[
(16)
\]

Substituting Eq. (17) into Eq. (2), expressions of displacements are obtained. Using the relations between displacements and stresses, expressions of stresses are obtained. From the conditions of Eqs. (11) and (14), we obtain

\[
C_m = (v + \alpha_m h)/(2 \tanh (\alpha_m h)) \{ D_m \sinh (\alpha_m h)/\alpha_m + 12aA \} = 0,
\]
\[
(20)
\]

\[
E_n = L_1 B_n + L_2 F_n,
\]
\[
(21)
\]

where

\[
L_1 = [(v - 3/2)\beta_n I_v(\beta_n a) + (3/2-2)/a \beta_n I_1(\beta_n a)]/\{I_1(\beta_n a)/a\} - I_1(\beta_n a)/a,
\]
\[
(22)
\]

\[
L_2 = [(1/2 - v)\beta_n I_0(\beta_n a) + (v/a) + \alpha_m h]/\{I_1(\beta_n a)/a\} - I_1(\beta_n a)/a.
\]
\[
(23)
\]

From the conditions of Eqs. (12) and (13), we have

\[
\sum_{m=1}^{\infty} M_n D_m + M_3 B_n + M_5 F_n = 0,
\]
\[
(24)
\]

\[
v/a^2 h \sum_{m=1}^{\infty} D_m f_1(\alpha_m) \sinh (\alpha_m h)/\alpha_m + 12aA = 0,
\]
\[
(25)
\]

where

\[
M_1 = 2(-1)^n \alpha_m h f_1(\alpha_m) \sinh (\alpha_m h)
\]

\[
\cdot [v^2 - 1/(2\alpha_m^2 + 2a \alpha_m h)] /\{a^2 [\{\alpha_m h]^2 + (n \pi /h)^2]\}
\]

\[
(26)
\]

\[
M_2 = [(3v - 7)/2 + \alpha_m h]/\{a^2 [\{\alpha_m h]^2 + (n \pi /h)^2]\}
\]

\[
+ (4 - 3v) \beta_n I_1(\beta_n a) I_1(\beta_n a)
\]

\[
+ (2(v - 1)/a^2 + \alpha_m h)/\{a^2 I_1(\beta_n a)\}
\]

\[
(27)
\]

\[
M_3 = [(3v - 7)/2 + a \alpha_m h]/\{a^2 [\{\alpha_m h]^2 + (n \pi /h)^2]\}
\]

\[
+ \beta_n \beta_n (1 + \alpha_m h)/\{a^2 I_1(\beta_n a)\}
\]

\[
+ [v^2 - 1/a^2 + \alpha_m h]/\{a^2 I_1(\beta_n a)\}
\]

\[
(28)
\]

\[
M_4 = [(v - 3)/2 + a \alpha_m h]/\{a^2 [\{\alpha_m h]^2 + (n \pi /h)^2]\}
\]

\[
+ \beta_n \beta_n (1 + \alpha_m h)/\{a^2 I_1(\beta_n a)\}
\]

\[
+ [v^2 - 1/a^2 + \alpha_m h]/\{a^2 I_1(\beta_n a)\}
\]

\[
(29)
\]

From the condition of Eq. (16), we get

\[
\sum_{i=1}^{\infty} N_i D_i + \sum_{n=1}^{\infty} \{S_i B_n + S_2 F_n\} = U,
\]
\[
(27)
\]
where

\[
N_l = \begin{cases} 
\left[ \alpha^m / 2 \left( (1 + \alpha_m \tan h(\alpha_m h)) \right) \\
\tan h(\alpha_m h) - \alpha_m h \\
-8\alpha^l / \{ (\alpha_m h^2 - (\alpha_m h)^2 - 1) \} \sinh(\alpha_m h), \end{cases} \\
l = m \\
\end{cases}
\]

\[
S_l = [\beta_n T_l / 2 + (\beta_n I_0(\beta_n a)) / 2 - (2 - \nu) / a \\
+ \beta_n^2 a / 2] I_1(\beta_n a) T_l / (\beta_n I_0(\beta_n a) \\
- I_1(\beta_n a) / a)] (-1)^n \beta_n^2 I_1(\beta_n a),
\]

\[
T_1 = 2\alpha_m [a^2 \beta_n - a(\beta_n^2 - \alpha_m^2) I_0(\beta_n a) / I_1(\beta_n a) \\
- \beta_n (1 + \alpha_m f_0(\alpha_m) f_1(\alpha_m)) / [(\alpha_m^2 + \beta_n^2)]] \\
/[ (\alpha_m^2 - 1) f_1(\alpha_m) (\alpha_m^2 + \beta_n^2)] \\
U = 2p [\alpha_m f_0(\alpha_m) - 2 f_1(\alpha_m)] \\
/[ (\alpha_m^2 - 1) f_1(\alpha_m) f_1(\alpha_m)],
\]

The condition of Eq. (15) is satisfied identically by Eqs. (19) and (20). From Eqs. (23), (24), (25) and (27), we obtain the simultaneous equations for \( A, B_n, D_m \) and \( F_n \). The calculations are carried out numerically, in which the first \( m_0 \) and \( n_0 \) roots of \( \alpha_m \) and \( \beta_n \) are used. Assuming the following values

\( d/a = 0.5, \ a/h = 1.0, \ \nu = 0.25, \)

we get the results illustrated in Figs. 2 and 3, which have 3 digits of efficient numbers, taking \( m_0 = n_0 = 15 \).

4. CONCLUDING REMARKS

In this paper, the solution of Navier's equation represented by Galerkin vector was investigated in cylindrical coordinates. General forms of Galerkin vector in cylindrical coordinates were listed in Table 1 and the proposed forms of Galerkin vector were applied efficiently to an asymmetric problem in cylindrical coordinates.

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