Hydrostatically pressurized buckling of complete spherical shells filled with an elastic medium

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The critical buckling characteristics of hydrostatically pressurized complete spherical shells filled with an elastic medium are presented. A model based on small deflection thin shell theory, the equations of which are solved using exact methods in conjunction with variational principles, is presented. In the current formulation, axisymmetric and inextensional assumptions are not used initially and the elastic medium is modelled as a Winkler foundation, *i.e.* using uncoupled radial springs with a constant foundation modulus that is independent of wave numbers of shell buckling modes. Critical buckling pressures and characteristic modal shapes are demonstrated for a wide range of material and geometric parameters. A phase diagram is established to obtain the requisite thickness to radius, and stiffness ratios for a desired mode profile. The present formulation can be readily extended to apply to more general cases of non-axisymmetric buckling problems.

Key Words : *hydrostatically pressurized buckling, complete spherical shell, Winkler foundation*

1. Introduction

The analytical study of the structural behaviour of spherical shells is of great importance in the fields of not only civil, mechanical and aeronautical engineering but also nanoscience and biomechanics. Notable examples include pressure vessels, spherical honeycombs¹⁾, carbon onions²⁾ and spherical viruses³⁾ and so on. In the more conventional engineering research fields, some pioneering theoretical works on the elastic instability issues of empty complete spherical shells were conducted in 1960s by Thompson⁴⁾, Hutchinson⁵⁾ and Koiter⁶⁾.

In recent years, analytical works which considered an interaction between a spherical shell and an internal elastic medium have also been conducted. The interaction leads to novel buckling patterns which depend on the stiffness and thickness to radius ra-

tios. Recently for example, Yin *et al.*⁷⁾ successfully demonstrated the stress-driven buckling patterns in spheroidal core/shell structures, where the core implies an elastic medium, by using the finite element approach. Interestingly, the authors insisted in that work⁷⁾ that shapes of many natural fruits and vegetables can be reproduced by anisotropic stress-driven buckles on the spheroidal core/shell system (see Fig. 1).

Purely from a structural mechanics perspective, Timoshenko⁸⁾, and Flügge⁹⁾ introduced the formulation for buckling of a complete spherical shell in their respective books that are now regarded as classics in the field. In these, the exact approaches to solve the buckling problem of hydrostatically pressurized complete spherical shells without an internal elastic medium were introduced. However, both of these formulations were based on the axisymmetric

Fig. 1 Comparison between several fruits (or vegetables) and examples of corresponding simulated buckle shapes on model spheroids (reprinted with permission from Yin et al. (2008) ⁷⁾).

assumption and formulations for more general cases including non-axisymmetric deformations were not included. More recently, Fok and Allwright ¹⁰⁾ analysed the elastic axisymmetric buckling behaviour of a complete spherical shell embedded in an elastic material and loaded by a far-field hydrostatic pressure. In that study, the energy method in conjunction with a Rayleigh–Ritz trial function was used for simplicity but the validation of the obtained results was not discussed in detail with buckling deformation modes being omitted. Hence, to our knowledge, no general non-axisymmetric formulation, in conjunction with exact methods of solution, has been developed for the buckling behaviour of complete spherical shells with an internal elastic medium thus far. Moreover, much attention has been given to the structural morphology of the core/shell structure. Some authors have recently demonstrated the cross-sectional morphology of carbon nanotubes embedded in an elastic medium ^{11),12)}. These results clearly show that interactions between shells and cores lead to some novel wavy-shaped buckling deformation patterns.

From the background described above, the buckling properties of hydrostatically pressurized complete spherical shells filled with an elastic medium are demonstrated in the current study. Presently, the elastic medium is modelled as a Winkler foundation, *i.e.* with uncoupled radial springs and a constant foundation modulus. An exact approach for the solution of the developed equations based on the formulation without using the axisymmetric and inextensional assumptions is presented. This approach therefore avoids any discussion about the validity of the solution and allows to model to be extended to cover more generic non-axisymmetric cases with relative ease. The analytical results are presented with a phase diagram and illustrative buckling modes that are compared with earlier results. Conclusions are then drawn.

Fig. 2 Hydrostatically pressurized spherical shell filled with an elastic medium.

2. Analytical model

Figure 2 shows the analytical model considered in the current study. The critical pitchfork bifurcation phenomenon of the hydrostatically pressurized complete spherical shell is investigated. The spherical shell is constructed from an homogeneous and isotropic linear elastic material with Young's modulus E and Poisson's ratio ν , which is filled with an elastic material that is modelled as a Winkler foundation, *i.e.* with uncoupled springs in the radial direction with a constant foundation modulus k_f . For a complete spherical shell with radius a and thickness h , spherical angular coordinates in the latitude and meridian directions θ and ϕ are used respectively. Displacement functions: u , v and w are in the θ , ϕ and the (outward) radial directions, respectively.

3. Formulation

3.1 Energy formulation

The following analysis is based on classical small deformation theory of thin shells ^{13),14)}. The total potential energy V is expressed by the sum of the strain energy and work done by external load as:

$$V = U_M + U_B + U_F + \Omega, \quad (1)$$

in which U_M is the membrane (in-plane) strain energy term, where:

$$\begin{aligned} U_M &= U_{M\phi} + U_{M\theta} + U_{M\phi\theta}, \\ U_{M\phi} &= \frac{1}{2} \int_{\phi} \int_{\theta} N_{\phi} \epsilon_{\phi} a^2 \sin \phi \, d\phi \, d\theta, \\ U_{M\theta} &= \frac{1}{2} \int_{\phi} \int_{\theta} N_{\theta} \epsilon_{\theta} a^2 \sin \phi \, d\phi \, d\theta, \\ U_{M\phi\theta} &= \int_{\phi} \int_{\theta} N_{\phi\theta} \epsilon_{\phi\theta} a^2 \sin \phi \, d\phi \, d\theta, \end{aligned} \quad (2)$$

U_B is the bending (out-of-plane) component, where:

$$\begin{aligned} U_B &= U_{B\phi} + U_{B\theta} + U_{B\phi\theta}, \\ U_{B\phi} &= \frac{1}{2} \int_{\phi} \int_{\theta} M_{\phi} \chi_{\phi} a^2 \sin \phi \, d\phi \, d\theta, \\ U_{B\theta} &= \frac{1}{2} \int_{\phi} \int_{\theta} M_{\theta} \chi_{\theta} a^2 \sin \phi \, d\phi \, d\theta, \\ U_{B\phi\theta} &= \int_{\phi} \int_{\theta} M_{\phi\theta} \chi_{\phi\theta} a^2 \sin \phi \, d\phi \, d\theta, \end{aligned} \quad (3)$$

U_F is the strain energy term due to a Winkler foundation, thus:

$$U_F = \frac{1}{2} \int_{\phi} \int_{\theta} a^2 k_f w^2 \sin \phi \, d\phi \, d\theta, \quad (4)$$

and Ω is the potential energy of the applied hydrostatic pressure p :

$$\Omega = \int_{\phi} \int_{\theta} a^2 p w \sin \phi \, d\phi \, d\theta. \quad (5)$$

Based on the assumptions of thin shell theory¹³⁾, the strain-displacement relations can be expressed as:

$$\begin{aligned} \epsilon_{\phi} &= \frac{1}{a}(v_{,\phi} + w), \\ \epsilon_{\theta} &= \frac{1}{a}(\cot \phi v + \frac{u_{,\theta}}{\sin \phi} + w), \\ \epsilon_{\phi\theta} &= \frac{1}{a}(\frac{v_{,\theta}}{\sin \phi} - \cot \phi u + u_{,\phi}), \\ \beta_{\phi} &= \frac{1}{a}(-w_{,\phi} + v), \\ \beta_{\theta} &= \frac{1}{a}(-\frac{w_{,\theta}}{\sin \phi} + u), \\ \chi_{\phi} &= \frac{1}{a^2}(-w_{,\phi\phi} + v_{,\phi}), \\ \chi_{\theta} &= \frac{1}{a^2}[-\frac{w_{,\theta\theta}}{\sin^2 \phi} + u_{,\theta} + \cot \phi(v - w_{,\phi})], \\ \chi_{\phi\theta} &= \frac{\sin \phi u_{,\phi} + v_{,\theta} - 2w_{,\phi\theta} + 2 \cot \phi w_{,\theta} - \cos \phi u}{2a^2 \sin \phi}, \end{aligned} \quad (6)$$

where “ x ” denotes differentiation with respect to x . The constitutive relationships are as follows:

$$\begin{aligned} N_{\phi} &= C(\epsilon_{\phi} + \nu \epsilon_{\theta}), \quad N_{\theta} = C(\epsilon_{\theta} + \nu \epsilon_{\phi}), \\ N_{\phi\theta} &= C \frac{1-\nu}{2} \epsilon_{\phi\theta}, \quad M_{\phi} = D(\chi_{\phi} + \nu \chi_{\theta}), \\ M_{\theta} &= D(\chi_{\theta} + \nu \chi_{\phi}), \quad M_{\phi\theta} = D \frac{1-\nu}{2} \chi_{\phi\theta}, \end{aligned} \quad (7)$$

where $C = Eh/(1 - \nu^2)$ is the membrane stiffness and $D = Eh^3/12(1 - \nu^2)$ is the flexural rigidity of the spherical shell, respectively. Substituting the expressions in Eq.(6) and (7) into Eqs.(2)–(4) gives the energy expressions in terms of the displacements u , v and w .

3.2 Fundamental state

When uniform hydrostatic pressure p is acting on the spherical shell, only the inward radial static displacement w_0 can occur in the pre-buckling state. Hence, the stress-strain relationship of the spherical shell is assumed to be linear up to the point of instability. In the fundamental pre-buckling state, the potential energies $V^{(0)}$ are given by:

$$V^{(0)} = U_M^{(0)} + U_B^{(0)} + U_F^{(0)} + \Omega^{(0)} = \int_{\phi} \int_{\theta} F^{(0)} \, d\phi \, d\theta, \quad (8)$$

where:

$$\begin{aligned} U_M^{(0)} &= C(1 + \nu) \int_{\phi} \int_{\theta} w_0^2 \, d\phi \, d\theta, \quad U_B^{(0)} = 0, \\ U_F^{(0)} &= \frac{1}{2} \int_{\phi} \int_{\theta} a^2 k_f w_0^2 \sin \phi \, d\phi \, d\theta, \\ \Omega^{(0)} &= \int_{\phi} \int_{\theta} a^2 p w_0 \sin \phi \, d\phi \, d\theta. \end{aligned} \quad (9)$$

For an equilibrium state, the first variation of the total potential energy V must equal zero. This condition gives the following static displacement under uniform hydrostatic pressure:

$$w_0 = -\frac{a^2(1 - \nu)}{2Eh + a^2k_f(1 - \nu)}p. \quad (10)$$

3.3 Critical buckling analysis

(1) Minimum potential energy criterion

To obtain an expression for the second variation of the total potential energy, the following infinitesimally small increments are defined:

$$u = u_1, \quad v = v_1, \quad w = w_0 + w_1, \quad (11)$$

which correspond to buckling displacement modes. The expression for the potential energy due to the external force is a linear functional of the displacement components and makes no contribution to the second variation expression, *i.e.* $\delta^2 \Omega = 0$ ¹³⁾. Consequently, the second variation of the total potential energy becomes:

$$\delta^2 U = \delta^2 U_M + \delta^2 U_B + \delta^2 U_F = \int_{\phi} \int_{\theta} F \, d\phi \, d\theta, \quad (12)$$

where:

$$\begin{aligned} \delta^2 U_M &= \delta^2 U_{stretch} + \delta^2 U_{shear}, \\ \delta^2 U_{stretch} &= \frac{Cw_0(1 + \nu)}{a} \int_{\phi} \int_{\theta} [(v_1 + w_{1,\phi})^2 \\ &\quad + (u_1 - w_{1,\theta} \csc \phi)^2] \sin \phi \, d\phi \, d\theta \\ &\quad + C \int_{\phi} \int_{\theta} [(v_1 + w_{1,\phi})^2 \\ &\quad + (u_1 - w_{1,\theta} \csc \phi)^2] \, d\phi \, d\theta, \\ \delta^2 U_{shear} &= \frac{C(1 - \nu)}{2} \int_{\phi} \int_{\theta} (\cot \phi u_1 + \csc \phi v_{1,\theta} \\ &\quad + u_{1,\phi})^2 \sin \phi \, d\phi \, d\theta, \end{aligned} \quad (13)$$

$$\begin{aligned}
\delta^2 U_B &= \delta^2 U_{bend} + \delta^2 U_{twist}, \\
\delta^2 U_{bend} &= \frac{D}{a^2} \int_{\phi} \int_{\theta} \{ (\cot \phi u_1 + \csc \phi v_{1,\theta} \\
&\quad - \csc^2 \phi w_{1,\theta\theta} - \cot \phi w_{1,\phi})^2 \\
&\quad (v_{1,\phi} + w_{1,\phi\phi}) [v_{1,\phi} + w_{1,\phi\phi} + \\
&\quad 2\nu (\cot \phi v_1 + \csc \phi u_{1,\theta} - \csc^2 \phi w_{1,\theta\theta} \\
&\quad - \cot \phi w_{1,\phi})] \} \sin \phi \, d\phi \, d\theta, \\
\delta^2 U_{twist} &= \frac{D(1-\nu)}{2a^2} \int_{\phi} \int_{\theta} [2(1+\nu)w_1 \\
&\quad (v_1 \cot \phi + w_1 + u_{1,\theta} + u_{1,\phi}) \\
&\quad + u_{1,\phi} (2\nu v_1 \cot \phi + 2\nu u_{1,\theta} + v_{1,\phi}) \\
&\quad + (v_1 \cot \phi + u_{1,\theta} \csc \phi)^2] \sin \phi \, d\phi \, d\theta,
\end{aligned} \tag{14}$$

$$\delta^2 U_F = \int_{\phi} \int_{\theta} a^2 k_f w_1^2 \sin \phi \, d\phi \, d\theta. \tag{15}$$

According to the Trefftz criterion, the buckling equations can be obtained by introducing F in Eq.(12) into the Euler–Lagrange equations with the calculus of variations. The Euler–Lagrange equations in this case are as follows ¹³⁾:

$$\begin{aligned}
\frac{\partial F}{\partial u_1} - \frac{\partial}{\partial \phi} \left(\frac{\partial F}{\partial u_{1,\phi}} \right) - \frac{\partial}{\partial \theta} \left(\frac{\partial F}{\partial u_{1,\theta}} \right) &= 0, \\
\frac{\partial F}{\partial v_1} - \frac{\partial}{\partial \phi} \left(\frac{\partial F}{\partial v_{1,\phi}} \right) - \frac{\partial}{\partial \theta} \left(\frac{\partial F}{\partial v_{1,\theta}} \right) &= 0, \\
\frac{\partial F}{\partial w_1} - \frac{\partial}{\partial \phi} \left(\frac{\partial F}{\partial w_{1,\phi}} \right) - \frac{\partial}{\partial \theta} \left(\frac{\partial F}{\partial w_{1,\theta}} \right) \\
&\quad + \frac{\partial^2}{\partial \phi^2} \left(\frac{\partial^2 F}{\partial w_{1,\phi\phi}} \right) + \frac{\partial^2}{\partial \phi \partial \theta} \left(\frac{\partial^2 F}{\partial w_{1,\phi\theta}} \right) \\
&\quad + \frac{\partial^2}{\partial \theta^2} \left(\frac{\partial^2 F}{\partial w_{1,\theta\theta}} \right) = 0.
\end{aligned} \tag{16}$$

(2) Stability equation

Substitution of a solution of the form:

$$\begin{aligned}
u_1 &= \sum \bar{u}(\phi) \sin n\theta, \\
v_1 &= \sum \bar{v}(\phi) \cos n\theta, \\
w_1 &= \sum \bar{w}(\phi) \cos n\theta,
\end{aligned} \tag{17}$$

where n is a positive integer, into Eqs.(12) – (15) gives the following ordinary differential equations for arbitrary n after some manipulation:

$$\begin{aligned}
a_u^{(i)} \frac{\partial^i \bar{u}}{\partial \phi^i} + b_u^{(i)} \frac{\partial^i \bar{v}}{\partial \phi^i} + c_u^{(i)} \frac{\partial^i \bar{w}}{\partial \phi^i} \\
- \frac{w_0}{a} (1+\nu) (\bar{u} \sin \phi + n \bar{w}) = 0,
\end{aligned} \tag{18}$$

$$\begin{aligned}
a_v^{(i)} \frac{\partial^i \bar{u}}{\partial \phi^i} + b_v^{(i)} \frac{\partial^i \bar{v}}{\partial \phi^i} + c_v^{(i)} \frac{\partial^i \bar{w}}{\partial \phi^i} \\
- \frac{w_0}{a} (1+\nu) \sin \phi (\bar{v} - \frac{\partial \bar{w}}{\partial \phi}) = 0,
\end{aligned} \tag{19}$$

$$\begin{aligned}
a_w^{(i)} \frac{\partial^i \bar{u}}{\partial \phi^i} + b_w^{(i)} \frac{\partial^i \bar{v}}{\partial \phi^i} + c_w^{(i)} \frac{\partial^i \bar{w}}{\partial \phi^i} + \frac{a^2 k_f}{C} \bar{w} \sin \phi \\
+ \frac{w_0}{a} (1+\nu) \sin \phi \left(\frac{\partial \bar{v}}{\partial \phi} + \bar{v} \cot \phi + \frac{n \bar{u}}{\sin \phi} \right. \\
\left. - \frac{\partial^2 \bar{w}}{\partial \phi^2} - \frac{\partial \bar{w}}{\partial \phi} \cot \phi + \frac{n^2}{\sin^2 \phi} \bar{w} \right) = 0,
\end{aligned} \tag{20}$$

in which the implied summation over i covers the range of non-zero coefficients, which are described, including the non-dimensional quantity $k = h^2/12a^2$, as follows:

$$\begin{aligned}
a_u^{(0)} &= (1+k) \left[\frac{1-\nu}{2} (1 - \cot^2 \phi) \sin \phi - \frac{n^2}{\sin \phi} \right], \\
a_u^{(1)} &= (1+k) \frac{1-\nu}{2} \cos \phi, \quad a_u^{(2)} = (1+k) \frac{1-\nu}{2} \sin \phi, \\
a_u^{(3)} &= a_u^{(4)} = 0, \\
b_u^{(0)} &= (1+k) \frac{\nu-3}{2} n \cot \phi, \quad b_u^{(1)} = -(1+k) \frac{1+\nu}{2} n, \\
b_u^{(2)} &= b_u^{(3)} = b_u^{(4)} = 0, \\
c_u^{(0)} &= kn \left(2 - \frac{n^2}{\sin^2 \phi} \right), \\
c_u^{(1)} &= kn \cot \phi, \quad c_u^{(2)} = kn, \\
c_u^{(3)} &= c_u^{(4)} = 0, \\
a_v^{(0)} &= (1+k) \frac{\nu-3}{2} n \cot \phi, \quad a_v^{(1)} = (1+k) \frac{1+\nu}{2} n, \\
a_v^{(2)} &= a_v^{(3)} = a_v^{(4)} = 0, \\
b_v^{(0)} &= -\frac{1+k}{\sin \phi} \left(\cos^2 \phi + \nu \sin^2 \phi + \frac{1-\nu}{2} n^2 \right), \\
b_v^{(1)} &= (1+k) \cos \phi, \quad b_v^{(2)} = (1+k) \sin \phi, \\
b_v^{(3)} &= b_v^{(4)} = 0, \\
c_v^{(0)} &= -2kn^2 \frac{\cot \phi}{\sin \phi}, \quad c_v^{(1)} = -k \left(2 - \frac{1+n^2}{\sin^2 \phi} \right) \sin \phi, \\
c_v^{(2)} &= -k \cos \phi, \quad c_v^{(3)} = -k \sin \phi, \quad c_v^{(4)} = 0, \\
a_w^{(0)} &= (1+k)(1+\nu)n - kn \left(2 + \frac{1-n^2}{\sin^2 \phi} \right), \\
a_w^{(1)} &= kn \cot \phi, \quad a_w^{(2)} = -kn, \quad a_w^{(3)} = a_w^{(4)} = 0, \\
b_w^{(0)} &= (1+k)(1+\nu) \cos \phi - k \cos \phi \left(2 + \frac{1-n^2}{\sin^2 \phi} \right), \\
b_w^{(1)} &= (1+k)(1+\nu) \sin \phi + \frac{k(n^2 + \cos^2 \phi)}{\sin \phi}, \\
b_w^{(2)} &= -2k \cos \phi, \quad b_w^{(3)} = -k \sin \phi, \quad b_w^{(4)} = 0, \\
c_w^{(0)} &= -k \left[2(1+\nu) \sin \phi + n^2 \frac{3-\nu+4 \cot^2 \phi}{\sin \phi} \right. \\
&\quad \left. - \frac{n^4}{\sin^3 \phi} \right], \quad c_w^{(1)} = \left(1-\nu + \frac{1+2n^2}{\sin^2 \phi} \right) \cos \phi, \\
c_w^{(2)} &= -\frac{1+2n^2+\nu \sin^2 \phi}{\sin \phi}, \\
c_w^{(3)} &= 2k \cos \phi, \quad c_w^{(4)} = k \sin \phi.
\end{aligned} \tag{21}$$

These governing differential equations (18)–(20) are difficult to solve exactly by only assuming periodic

functions for $\bar{u}$, $\bar{v}$ and $\bar{w}$ directly. However, the implementation of the following special linear differential operator enables the solution of these equations with an exact approach:

$$H(\dots) = (\dots)'' + (\dots)' \cot \phi + (\dots) \left(2 - \frac{n^2}{\sin^2 \phi} \right), \quad (22)$$

where primes denote differentiation with respect to ϕ . From the definition of H in Eq.(22), the following relations are readily obtained:

$$\begin{aligned} HH(\dots) &= (\dots)'''' + 2(\dots)''' \cot \phi \\ &+ (\dots)'' \left(3 - \frac{1+2n^2}{\sin^2 \phi} \right) \\ &+ (\dots)' \cot \phi \left(4 + \frac{1+2n^2}{\sin^2 \phi} \right) \\ &+ (\dots) \left(4 - \frac{2n^2}{\sin^2 \phi} - \frac{n^2(4-n^2)}{\sin^4 \phi} \right), \quad (23) \end{aligned}$$

$$\begin{aligned} H'(\dots) &= (\dots)''' + (\dots)'' \cot \phi \\ &+ (\dots)' \cot \phi \left(2 - \frac{1+n^2}{\sin^2 \phi} \right) \\ &+ 2n^2(\dots) \frac{\cos \phi}{\sin^3 \phi}. \quad (24) \end{aligned}$$

In addition, a new variable $\hat{u}$ is defined in terms of $\bar{u}$ by:

$$\bar{u} = \hat{u}'. \quad (25)$$

By making use of the relations of Eqs.(22)–(24) and the new definition of Eq.(25), Eqs.(18) and (19) can be rewritten respectively as:

$$\begin{aligned} (1+k) \sin \phi \frac{\partial}{\partial \phi} \left(\frac{1-\nu}{2} H(\hat{u}) - \frac{1+\nu}{2} n \frac{\bar{v} \sin \phi + n \hat{u}}{\sin^2 \phi} \right) \\ - 2(1+k)n \frac{\bar{v} \sin \phi + n \hat{u}}{\sin^2 \phi} \cos \phi \\ - (1+k)(1+\nu)nw + knH(w) \\ - \frac{w_0(1+\nu)}{a} (\hat{u}' + n\bar{w}) = 0, \quad (26) \end{aligned}$$

$$\begin{aligned} (1+k) \left[\frac{1+\nu}{2} (\bar{v} \sin \phi + n \hat{u})'' + \frac{1-\nu}{2} H(\bar{v} \sin \phi) \right. \\ \left. - \frac{3-\nu}{2} \cot \phi (\bar{v} \sin \phi + n \hat{u})' + (1+\nu)w' \sin \phi \right] \\ - k \sin \phi H'(w) - \frac{w_0(1+\nu)}{a} \sin \phi (\bar{v} - \bar{w}') = 0. \quad (27) \end{aligned}$$

Differentiation of Eq.(27) with respect to ϕ and then multiplying it by $\sin \phi/n$ leads to the following:

$$\begin{aligned} (1+k) \frac{\sin \phi}{n} \left\{ \frac{1+\nu}{2} (\bar{v} \sin \phi + n \hat{u})''' \right. \\ \left. + \frac{1-\nu}{2} H'(\bar{v} \sin \phi) + \frac{3-\nu}{2} \left[\frac{1}{\sin^2 \phi} (\bar{v} \sin \phi + n \hat{u})' \right. \right. \end{aligned}$$

$$\begin{aligned} \left. \left. - \cot \phi (\bar{v} \sin \phi + n \hat{u})'' \right] \right\} \\ + (1+\nu)(w'' \sin \phi + w' \cos \phi) \Big\} \\ - k \frac{\sin \phi}{n} [\cos \phi H'(w) + \sin \phi H''(w)] = 0. \quad (28) \end{aligned}$$

By adding Eq.(28) to Eq.(26), some algebraic manipulation gives:

$$\begin{aligned} (1+k)[H(\Psi) - (1+\nu)(\Psi - H(w) + 2w)] \\ - k(HH(w) - 2H(w)) - \frac{w_0}{a}(\Psi - H(w) + 2w) = 0, \quad (29) \end{aligned}$$

in which:

$$\Psi = \frac{(\bar{v} \sin \phi + n \hat{u})'}{\sin \phi}. \quad (30)$$

Similarly, Eq.(20) can be written using the operator H and the new variable Ψ , defined in Eq.(30), as follows:

$$\begin{aligned} (1+k)(1+\nu)(\Psi + 2w) \\ + k[HH(w) - (3+\nu)H(w) - H(\Psi)] \\ + \frac{a^2 k_F}{C} w + (1+\nu) \frac{w_0}{a} (\Psi - H(w) + 2w) = 0. \quad (31) \end{aligned}$$

Equations (29) and (31) are the governing equations to be solved. Now, the solutions of Ψ and w of Eq.(29) and (31) are assumed to be in the form:

$$\Psi = \sum_{m=0}^{\infty} A_m P_m(\cos \phi), \quad w = \sum_{m=0}^{\infty} B_m P_m(\cos \phi), \quad (32)$$

where A_m and B_m are the constant deformation amplitudes, and the spherical harmonic $P_m(\cos \phi)$ is a series of Legendre functions of degree n which satisfies the following Legendre differential equation:

$$P_m'' + P_m' \cot \phi + m(m+1)P_m = 0. \quad (33)$$

By using the operator H , defined in Eq.(22), the following relation can be obtained:

$$H(P_m) = -\lambda_m P_m, \quad (34)$$

where:

$$\lambda_m = m(m+1) - 2, \quad (35)$$

and

$$HH(P_m) = -\lambda_m^2 P_m. \quad (36)$$

Substituting the series given in Eq.(32) into Eqs.(29) and (31) and using Eqs.(34) and (36), we obtain:

$$\begin{aligned} \sum_{m=0}^{\infty} (c_m^{(11)} A_m + c_m^{(12)} B_m) P_m(\cos \phi) = 0, \\ \sum_{m=0}^{\infty} (c_m^{(21)} A_m + c_m^{(22)} B_m) P_m(\cos \phi) = 0, \quad (37) \end{aligned}$$

Fig. 3 Variation of the nondimensional critical buckling pressure p_{cr}/p_0 with ak_f/E .

Fig. 5 Variation of the nondimensional critical buckling mode number m_{cr} .

Fig. 4 Nondimensional critical pressure curves with ak_f/E and a/h .

in which:

$$\begin{aligned} c_m^{(11)} &= (1+k)(\lambda_m + 1 + \nu) + \frac{w_0}{a}(1 + \nu), \\ c_m^{(12)} &= (1+k)(1 + \nu) + k(\lambda_m^2 + 2k\lambda_m) \\ &\quad + \frac{w_0}{a}(1 + \nu)(1 + \lambda_m), \\ c_m^{(21)} &= (1+k)(1 + \nu) + k\lambda_m + \frac{w_0}{a}(1 + \nu), \\ c_m^{(22)} &= 2(1+k)(1 + \nu) + k\lambda_m(\lambda_m + 3 + \nu) \\ &\quad + \frac{a^2 k_F}{C} + \frac{w_0}{a}(1 + \nu)(\lambda_m + 2). \end{aligned} \quad (38)$$

Hence, homogeneous linear equations for A_m and B_m are obtained for all m as follows:

$$\begin{bmatrix} c_m^{(11)} & c_m^{(12)} \\ c_m^{(21)} & c_m^{(22)} \end{bmatrix} \begin{pmatrix} A_m \\ B_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (39)$$

This is a standard eigenvalue problem for which non-trivial values are obtained when the coefficient matrix in Eq.(39) becomes singular. The corresponding

pressure p is thus calculated from the determinantal equations:

$$\det \begin{bmatrix} c_m^{(11)} & c_m^{(12)} \\ c_m^{(21)} & c_m^{(22)} \end{bmatrix} = 0. \quad (40)$$

By solving Eq.(40), the pressure p with the corresponding mode m for several combinations of geometric and material constants can be obtained as an eigenvalue. For the particular values of the constants, the minimum value of the eigenvalues is the critical buckling pressure p_{cr} . It should be noted that the above equation is independent of the mode number n . This shows that only axisymmetric modes can occur in this problem in spite of the inclusion of the Winkler foundation term.

4. Analytical results and discussion

4.1 Buckling pressure

Figure 3 shows the variation of the nondimensionalized critical buckling pressure p_{cr}/p_0 , where p_0 is the critical pressure for the empty complete spherical shell that is given by:

$$p_0 = \frac{2E}{\sqrt{3(1-\nu^2)}} \left(\frac{h}{a} \right)^2. \quad (41)$$

As can be easily seen in this figure, the value of p_{cr}/p_0 increases with the increasing stiffness ratio ak_f/E . This fact can also be found in Fig. 4 which illustrates the nondimensional critical pressure curves with ak_f/E and a/h . The variation of the critical buckling mode number m_{cr} is plotted in Fig. 5. Larger values of ak_f/E cause higher buckling modes to be triggered, as in the case of beams on an elastic Winkler foundation¹⁵.

Figure 6 shows a phase diagram of the buckling mode number m . This figure gives significant information on the various axisymmetric buckling modes depending on the values of ak_f/E and a/h . It is

Fig. 6 Phase diagram of the buckling mode number m . Various axisymmetric buckling modes can be found depending on the values of ak_f/E and a/h .

observed from this figure that larger ak_f/E and a/h values favour buckling modes with higher mode numbers. We also find that when the spherical shells are relatively thin with larger a/h values, the effect of the foundation modulus becomes more significant when compared with the case for the relatively thick shells with smaller a/h values.

4.2 Eigenmodes

Figure 7 illustrates the buckling eigenmodes for the empty complete spherical shell of $a/h = 100$. The characteristic wavy-shaped axisymmetric buckling deformation with the mode number $m = 18$ is found in this case. The obtained mode number for this shell is the same as that determined by Koiter⁶⁾.

On the other hand, Fig. 8 shows the eigenmodes for the “filled” spherical shell with $a/h = 100$ and $ak_f/E = 0.1$; the axisymmetric buckling deformation of the mode number $m = 33$ is seen in that figure. As can be understood in the developed formulation, non-axisymmetric modes cannot occur under the condition of a constant foundation modulus. However, the cross-sectional shape for this case is moderately different from that shown in Fig. 7.

The results shown in Fig. 7 and 8 suggest that the eigenmodes change corresponding to the mode number m . When m is even, the corresponding deformation of the spherical shell is symmetric about the equator, as shown in Fig. 7. On the other hand, when m is odd, the deformation is anti-symmetric about the equator, that is, an inward deformation at the north pole is accompanied by an outward deformation at

Fig. 7 Buckling modes of an hydrostatically pressurized “empty” complete spherical shell with $a/h = 100$ and $ak_f/E = 0$. (Left) A 3-dimensional view through a great circle. (Right) Cross-sectional view. The buckling mode with mode number $m = 18$ is found.

Fig. 8 Buckling modes of an hydrostatically pressurized “filled” complete spherical shell with $a/h = 100$ and $ak_f/E = 0.1$. (Left) A 3-dimensional view through a great circle. (Right) Cross-sectional view. The buckling mode with mode number $m = 33$ is found.

the south pole. This is owing to the property of the geometry of the sphere and the Legendre functions, not whether the shell is empty or otherwise.

5. Conclusions

The characteristic buckling modes in hydrostatically pressurized complete spherical shells filled with an elastic medium have been demonstrated. The governing equations of equilibrium have been derived from a theoretical formulation based on small-displacement thin shell theory; these have been solved using an exact methodology. From the formulation developed currently it has been shown that only axisymmetric eigenmodes can occur in spite of adding the Winkler foundation term; critical modes that are symmetric or anti-symmetric about the equator may be determined depending on the combination of the foundation stiffness and the radius to wall-thickness

ratio. A phase diagram has been established to obtain the requisite values of a/h and ak_f/E for a corresponding buckling mode.

This work is a fundamental investigation that has been developed with future studies in mind. These will address the post-buckling behaviour^{16),17)} and non-axisymmetric modes which may occur when a foundation modulus is not a constant value or a function of the wave number of the shell deformation.

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