Cyclic steps formed by turbidity currents

Duleeka Dahampriya DIAS*, Norihiro IZUMI** and Miwa YOKOKAWA***

*Student Member, Graduate School of Engineering, Hokkaido Univ.
(North 13, West 8, Kitaku, Sapporo 060–8628)
**Member, PhD, Faculty of Engineering, Hokkaido Univ.
(North 13, West 8, Kitaku, Sapporo 060–8628)
***DSc, Faculty of Information Science and Technology, Osaka Institute of Technology
(1-79-1 Kitayama, Hirakata, Osaka 573–0196)

Levees of channels formed due to turbidity currents on submarine fans are often covered with step like bedforms. Circumstantial evidences emerged with numerical and experimental studies have suggested these sediment waves should probably be cyclic steps. The formation of cyclic steps in sub-aqueous environments through a mathematical perspective is presented in this research. A mathematical model preserving essential physics of the system is solved for one step wave length to obtain a preserved step shape of upstream migrating steps and the behavior of characteristic parameters governing this cyclic step formation phenomenon.

Key Words : cyclic steps, turbidity current, submarine bedform

1. Introduction

Uniform water flow over a steep erodible bed may give rise to periodic bedforms. These bed evolutions do not show random irregularities, but rather show an inherent features due to the coupling of water flow and the suspended sediment transport$^1$.

Supercritical flow in the Froude sense over a steep movable bed may generate ephemeral short wave bedforms like antidunes. In some cases, however, antidunes give way to a much more stable cousin bounded by hydraulic jumps in the flow above them, which stabilize the flow and morphodynamics$^2$. “Chute-and-pool” morphology documented by Simons et al.$^3$, and documentation of field observed step-like structures during sand dam construction in Netherlands, by Winterwerp et al.$^4$ disclosed the basic idea about this peculiar bed undulations. This long wave manifestation was christened as “cyclic steps” by Parker$^5$ with the light of experiments on the formation of alluvial fans.

Cyclic steps can be categorized into two main folds based on the formation mechanism, i.e. transportational cyclic steps and purely erosional cyclic steps. Corresponding to flow transition between two conterminous steps facilitates erosion on the steep lee side of the step and sediment deposition on the flat or gently sloping sill portion, which leads to a consequent step migration. Whipple et al.$^6$ proposed the name transportational cyclic steps for these bed evolutions and flume experiments carried out by Taki and Parker$^7$ constricted these bedforms in a laboratory experiment elucidating their formation and the evolution. In the absence of sediment deposition, origin of steps has been elaborated in terms of a purely erosional analogy by Parker and Izumi$^1$ with their theoretical study.

Most commonly observed cyclic steps are manifested once the noncohesive alluvium exposes to a sub-aerial flow. Sun and Parker’s theoretical study followed by Taki and Parker’s experimental elaboration of the cyclic step
formation demystified the evolution and propagation of sub-aerial cyclic steps when the suspended sediment redep-
position is entertained within the solution domain and the problem was successfully addressed through a numerical
approach by Fagherazzi and Sun. Wohl observed trains of purely incisional rhythmic steps in the beds of bedrock streams,
which was theoretically elucidated by Parker and Izumi and successfully regenerated in the laboratory by Koyama
and Ikeda. Moreover, experimental studies and model analysis for bedrock incision have proven that the same fundamental
instability of cyclic step formation can be formulated with the theory of bedrock incision as well.

Turbidity currents, sub-aqueous analog of a river flowing over the movable ocean floor, bear the potential of
carving well defined leveed channels formed due to overflow of turbidity currents from the main channel. These
levees are often covered with step-like bedforms identical with sub-aqueous cyclic steps in appearance. Numerical
study performed based on “three-equation model” by Kubo and Nakajima suggested that the turbidity current
sediment waves should not be interpreted as antidunes based only on their upstream migration. Similarly, numerical
model developed by Kostic and Parker has been applied to Monterey Submarine Channel off Monterey, California by Fildani, aided to conclude that the sediment waves on the outside levee of the Shepard bend appear to be net depositional cyclic steps. Outcomes of this numerical study further concreted the idea that the sediment waves observed on the levees of many other submarine channels should probably be cyclic steps, perfectly agreeing with previous research results. Numerical model proposed by Kostic and Parker based on “four-equation model” provided an insight into what can be expected once cyclic steps generated by turbidity currents in an experimental setting. Cyclic steps emerged in pelagic environments powered by turbidity currents not yet been widely explored and specifically mathematical interpretations of the subaqueous cyclic step evolutions are rather bereft in this research area.

In this paper, it is endeavored to develop an idealized mathematical model preserving the essential physics of the
system, carrying the potential of delineating the cyclic step formation phenomenon in a sub-aqueous environment and the behavior of characteristic features during the formation process.

2. Formulation

2.1 Governing Equations

The movement of turbidity currents is described by the layer-averaged momentum equations of the form

\[
\frac{\partial \bar{U}}{\partial t} + \bar{U} \frac{\partial \bar{U}}{\partial x} = -\frac{Rg}{h} \frac{\partial \bar{C}}{\partial x} - \frac{\partial \bar{h}}{\partial x} - RgC \frac{\partial \bar{h}}{\partial x} - RgC \frac{\partial \bar{h}}{\partial x} + RgC S - \frac{\bar{\tau}_b}{\rho \bar{h}}
\]

where \( \bar{U} \) is the layer-averaged velocity in the \( \bar{x} \) direction, \( h \) is the layer thickness, \( C \) is the layer-averaged suspended sediment concentration, \( \bar{h} \) is the bed elevation referenced to the slope of \( S \), \( R \) is the submerged specific gravity (= \( \rho_s/\rho - 1 = 1.65 \)), \( g \) is the gravity acceleration (= 9.8 m/s\(^2\)), \( \rho \) is the density of sea water (= 1,025 kg/m\(^3\)), \( \bar{\tau}_b \) is the bed shear stress, and \( \bar{C} \) denotes dimensional variables.

The layer-averaged continuity equation of turbidity currents takes the form

\[
\frac{\partial \bar{h}}{\partial t} + \bar{U} \frac{\partial \bar{h}}{\partial x} = \bar{e}_w \bar{U}
\]

where \( \bar{e}_w \) characterizes the rate of entrainment of ambient sea water into the turbidity current from above, assumed to be expressed as a function of Richardson number \( Ri \)

\[
\bar{e}_w = \frac{0.00153}{0.0204 + Ri}
\]

In the case of flow induced by turbidity differences \( Ri \) is defined by

\[
Ri = \frac{RgC \bar{h}}{\bar{U}^2}
\]

The dispersion equation of suspended sediment is given by

\[
\frac{\partial C \bar{h}}{\partial t} + \frac{\partial U C \bar{h}}{\partial x} = v_s \left( \bar{E} - \bar{D} \right)
\]

where \( \bar{E} \) is the entrainment rate of suspended sediment from the bottom, \( \bar{D} \) is the deposition rate of suspended sediment onto the bed, and \( v_s \) is the falling velocity of suspended sediment. The entrainment rate \( \bar{E} \) is assumed to be expressed by

\[
\bar{E} = \begin{cases} 
\alpha \left( \frac{\bar{\tau}_b}{\bar{\tau}_{th}} - 1 \right)^n & \text{when } \bar{\tau}_b \geq \bar{\tau}_{th} \\
0 & \text{when } \bar{\tau}_b < \bar{\tau}_{th}
\end{cases}
\]

where \( \alpha \) and \( n \) are empirical constants determined by experiments and \( \bar{\tau}_{th} \) is the critical bed shear stress below which no entrainment takes place. The deposition rate of suspended sediment \( \bar{D} \) onto the bed is governed by the relation

\[
\bar{D} = r_0 \bar{C}
\]
where \( r_0 \) is the ratio between the suspended sediment concentrations near the bed \( \tilde{c}_b \) and \( \tilde{C} (= \tilde{c}_b/\tilde{C}) \), which is known to be close to unity. We assume \( r_0 \) as unity for simplicity.

The Exner equation of sediment continuity is given by

\[
(1 - \lambda_p) \frac{\partial \tilde{c}}{\partial t} = v_s (\tilde{D} - \tilde{E})
\]  

(8)

where \( \lambda_p \) stands for the porosity of the ocean bed. The bed shear stress can be expressed by

\[
\tilde{\tau}_b = \rho c_D \tilde{U}^2
\]  

(9)

where \( c_D \) is the drag coefficient assumed to be a constant.

### 2.2 Normalization

The following normalization is introduced:

\[
\tilde{U} = U \tilde{U},
\]

\[
\tilde{C} = \tilde{C} \tilde{C},
\]

\[
\tilde{E} = \tilde{E} \tilde{E},
\]

\[
\tilde{D} = \tilde{D} \tilde{D},
\]

\[
(\tilde{h}, \tilde{\eta}) = (H/h, \eta/\tilde{h})
\]

\[
\tilde{\chi} = \frac{\tilde{H}}{c_D}
\]

\[
\tilde{t} = \left( \frac{(1 - \lambda_p) \tilde{H}}{v_s \tilde{D} c} \right) t
\]  

(10g)

In the above normalization, \( \tilde{U}, \tilde{C}, \tilde{E}, \tilde{D} \) are the velocity, flow depth, suspended sediment concentration, entrainment rate from the bottom, and deposition rate from suspension, at the Richardson critical point, respectively. Normalized governing equations can be written in the form

\[
\alpha \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = -\frac{1}{2} \frac{\partial C}{\partial x} - C \frac{\partial \eta}{\partial x} - C \frac{\partial h}{\partial x} + \sigma C - \frac{U^2}{h}
\]

(11)

\[
\frac{\partial h}{\partial t} + \frac{\partial U h}{\partial x} = e_u U
\]

(12)

\[
\alpha \frac{\partial C}{\partial t} + \beta \frac{\partial U C}{\partial x} = \gamma E - C
\]

(13)

\[
\frac{\partial \eta}{\partial t} = C - \gamma E
\]

(14)

with

\[
\alpha = \frac{v_s \tilde{D}_c}{(1 - \lambda_p) c_D \tilde{U} \tilde{C}}
\]

\[
\beta = \frac{c_D \tilde{U} \tilde{C}}{v_s \tilde{D}_c}
\]

(15a, b)

\[
\sigma = \frac{S}{c_D}
\]

\[
\gamma = \frac{\tilde{E}}{\tilde{D}_c}
\]

(15c, d)

\[
e_u = \frac{\tilde{E}}{c_D}, \quad \frac{R_g \tilde{C} \tilde{H}}{\tilde{U}^2} = 1
\]

(15e, f)

Defined normalization imposed on equation (6) that re-shapes the entrainment correlation of sediment as

\[
E = \left( \frac{U^2 - U_{th}^2}{1 - U_{th}^2} \right)^n
\]

when \( U \geq U_{th} \)

\[
0 \quad \text{when} \quad U < U_{th}
\]

(15g)

where \( U_{th} \) is the normalized threshold velocity below which entrainment does not occur. Being on previous research outcomes, \( \alpha \) terms are assumed to be small enough to be neglected. Quasi-steady assumptions facilitate for a solution of permanent form dropping terms with time derivatives, nonetheless unsteady terms in the Exner equation can not be omitted. Incorporating those logics normalized governing equations can be recasted into following form:

\[
U \frac{\partial U}{\partial x} = -\frac{1}{2} \frac{\partial C}{\partial x} - C \frac{\partial \eta}{\partial x} - C \frac{\partial h}{\partial x} + \sigma C - \frac{U^2}{h}
\]

(16)

\[
\frac{\partial U h}{\partial x} = e_u U
\]

(17)

\[
\beta \frac{\partial U C}{\partial x} = \gamma E - C
\]

(18)

\[
\frac{\partial \eta}{\partial t} = C - \gamma E
\]

(19)

### 2.3 Coordinate Transformation

Moving coordinate system is introduced with upstream migration speed of \( f \)

\[
\tilde{x} = x + ft, \quad \tilde{t} = t
\]

(20a, b)

Applying the above coordinate transformation to the governing equations, we have

\[
U \frac{\partial U}{\partial \tilde{x}} = -\frac{1}{2} \frac{\partial C}{\partial \tilde{x}} - C \frac{\partial \eta}{\partial \tilde{x}} - C \frac{\partial h}{\partial \tilde{x}} + \sigma C - \frac{U^2}{h}
\]

(21)

\[
\frac{\partial U h}{\partial \tilde{x}} = e_u U
\]

(22)

\[
\beta \frac{\partial U C}{\partial \tilde{x}} = \gamma E - C
\]

(23)

\[
\frac{\partial \eta}{\partial \tilde{t}} + f \frac{\partial \eta}{\partial \tilde{x}} = C - \gamma E
\]

(24)

In the analysis, we focus at self-preserving quasi-steady step shape migrating upstream without changing its shape. In order to preserve such a solution, time derivative is dropped in (24). Neglecting the \( \gamma \) for simplicity, novel version of governing equations is obtained replacing partial derivatives with total derivatives. That is

\[
U \frac{d U}{d x} = -\frac{1}{2} \frac{d C}{d x} - C \frac{d \eta}{d x} - C \frac{d h}{d x} + \sigma C - \frac{U^2}{h}
\]

(25)

\[
\frac{d U h}{d x} = e_u U
\]

(26)

\[
\beta \frac{d U C}{d x} = \gamma E - C
\]

(27)
Thus we can have
\[ \text{denominator of (29) is vanished} \]

where
\[ e_w = \frac{e_1}{e_2 + Ch/U^2} \]

with \( e_1 = 0.00153/c_D \) and \( e_2 = 0.0204 \).

Upstream migration speed of a step can be calculated by considering an additional condition. At the origin of the streamwise coordinate system where Richardson critical conditions are preserved denominator of (29) is vanished by the implication of corresponding critical flow parameters; i.e., \( U = 1, C = 1, \) and \( h = 1 \). To neutralize the effect of this singularity, that can not be expected in reality numerator must also vanish at the Richardson critical point. Thus we can have
\[ f = \frac{2\beta (\gamma - 1)}{\gamma \beta e_w - 2\beta \sigma + 2\beta - 1} \]

### 2.4 Boundary Conditions

When \( x = L \), a hydraulic jump takes place. Upstream and downstream of the hydraulic jump, the momentum is conserved, such that
\[ U^2(L^-)h(L^-) + \frac{1}{2}C(L)h^2(L^-) = U^2(L+)h(L+) + \frac{1}{2}C(L)h^2(L+) \quad \text{at} \quad x = L \]

where \( L^- \) and \( L^+ \) denote the upstream and downstream sides of the hydraulic jump, respectively. Note that \( C \) is continuous between the upstream and downstream sides of the hydraulic jump, so that \( C(L^-) = C(L^+) = C(L) \). With the use of the continuity of mass \( (U(L^-)h(L^-) = U(L^+)h(L^+) = q(L)) \), (34) can be reduced to
\[ \frac{U(L^-)}{U(L^+)} = \frac{1 + \left[ 1 + \frac{8U^3_{th}}{C(0)h(0+)} \right]^{1/2}}{4U^3_{th} / C(0)h(0+)} \]

where \( q \) is the discharge per unit width, and is continuous at the hydraulic jump. Downstream of a hydraulic jump, the bed is eroded until the velocity reduces to the threshold velocity for entrainment, such that
\[ U(0+) = U(L^+) = U_{th} \]

In addition, \( h(L^+) = q(L)/U_{th} \), therefore,
\[ \frac{U(L^-)}{U_{th}} = \frac{1 + \left[ 1 + \frac{8U^3_{th}}{q(L)C(L)} \right]^{1/2}}{4U^3_{th} / q(L)C(L)} \]

From (27) and (28), we obtain
\[ \frac{d}{dx}[(\beta UCh + f\eta)] = 0 \]

The above equation can be integrated to
\[ \beta UCh + f\eta = B \]

where \( B \) is an integral constant, which is determined by the condition at the upstream end of the step \( (x = 0) \)

\[ U = U_{th}, \quad C = C(0), \quad h = h(0+), \quad \eta = 0 \quad \text{at} \quad x = 0+ \]

By applying the above condition, (39) is rewritten as
\[ \beta UCh + f\eta = \beta U_{th}C(0)h(0+) \]

Assuming that \( \eta = 0 \) at the downstream end of the step \( (x = L) \), the concentration \( C \) is
\[ C(L) = \frac{U_{th}C(0)h(0+)}{U(L^-)h(L^-)} = \frac{U_{th}C(0)h(0+)}{q(L)} \]

Substituting (42) into (35), we obtain
\[ \frac{U(L^-)}{U_{th}} = \frac{1 + \left[ 1 + \frac{8U^3_{th}}{C(0)h(0+)} \right]^{1/2}}{4U^3_{th} / C(0)h(0+)} \]

where \( R(0+) \) is the Richardson number when \( x = 0+ \).

Moreover, at the Richardson critical point which is assumed to be the origin of the streamwise coordinate system flow depth, layer averaged velocity, and suspended sediment concentrations are assigned the value of unity.

\[ U = 1, \quad C = 1, \quad h = 1 \quad \text{at} \quad x = L_e \]

where \( L_e \) is the \( x \) coordinate of the Froude critical point. From (36) and (41), we find a very interesting fact that the Richardson number at the downstream side of every hydraulic jump is the same. This means that the velocity \( U \) just upstream of hydraulic jump at the downstream end of every step is constant. Note, however, that the flow depth \( h \) and the suspended sediment concentration \( C \) are not constant, resulting in differences in wavelength of each step.
3. Solution

A two-point boundary value problem composed of three differential equations (29)-(31) is solved with the use of the shooting method incorporated with the Newton-Raphson scheme, which generates a solution for specified parameters satisfying the boundary conditions.

Specifying the threshold velocity for sediment entrainment at the upstream end of the step, and specifying $c_D$, $\sigma$, $\beta$ and $\gamma$, calculation is initiated from the vicinity of the Richardson critical point to avoid singularities of the equations that can rise at the critical point. We obtain the variations of layer-averaged quantities of velocity $U$, layer thickness $h$, suspended sediment concentration $C$, and bed elevation $\eta$ over one step wavelength. Rigorous calculation loop is performed towards upstream and downstream directions until the specified boundary conditions are met.

4. Results and discussion

4.1 Typical values of parameters

Typical values of all parameters are estimated herein. The critical layer thickness $H_c$ is calculated by

$$H_c = \sqrt{\frac{Q^2}{RgCc}}$$  \hspace{1cm} (45)

where $Q$ is the total discharge per unit width. With the use of the above relation, the parameter $\beta$ is rewritten as

$$\beta = c_D\frac{R^{2/3}\bar{g}^{2/3}\bar{C}^{5/3}}{\bar{v}_c\bar{Q}^{1/3}}$$  \hspace{1cm} (46)

where the ranges of the parameters in the above equation is estimated to be

$c_D = 0.002 \sim 0.02$ \hspace{1cm} (47)

$\bar{C}_c = 0.01 \sim 0.3$ \hspace{1cm} (48)

$\bar{v}_c = 0.001 \sim 0.01$ m/s \hspace{1cm} (49)

$\bar{Q} = 0.1 \sim 10$ m$^2$/s \hspace{1cm} (50)

Then, the range of $\beta$ is

$$\beta = 0.00028 \sim 37$$  \hspace{1cm} (51)

The possible range of the parameter $\beta$ is rather wide.

The parameter $\gamma$ is the ratio between entrainment and deposition rates at the Richardson critical point. We have observed $\gamma$ should be smaller than unity for the model to behave in a reasonable manner, implying that the deposition has to be dominant over the entrainment. In the sub-aerial cyclic step formation, $\gamma$ takes values larger than unity, which suggest that the entrainment is dominant, achieving net bed degradation at the Froude critical point. The bed evolution process of the sub-aqueous cyclic steps contrast strongly with the sub-aerial equivalent in this sense. The value of $\gamma$ ranges between 0 and 1 in the case of sub-aqueous cyclic steps considered in this study.

Fig. 3 The wavelength as a function of $\beta$ and $\gamma$. $\sigma = 2$.

Fig. 4 The wave slope as a function of $\beta$ and $\gamma$. $\sigma = 2$.

As discussed above, a typical value of the drag coefficient $c_D$ ranges from 0.002 to 0.02. Considering $c_D$ is rather small when the bed material is small, a value of 0.005 is assumed as $c_D$ herein. The average slope of the ocean floor observed in Figure 5 is 0.01, when the value of $\sigma$ is estimated to be 2.

4.2 The wavelength and wave slope

For specified $c_D$, $\sigma$ and $\gamma$, step wavelength is obtained as a function of $\beta$. The variation of wavelength depending on $\beta$ for the cases that $c_D = 0.005$, $\sigma = 2$ and $\gamma = 0.3$, 0.5, 0.7 and 0.9 is shown in Figure 3. It is found that wavelength increases with increasing $\beta$, and with decreasing $\gamma$. The parameter $\beta$ represents sediment suspendability, so that step wavelength becomes longer as suspendability is higher. The parameter $\gamma$ is an index of non-equilibrium of the system of flow and sediment movement. While $\gamma$ is unity if the system is in perfect equilibrium, $\gamma$ is much smaller than unity far from equilibrium. It follows that, wavelength increases when the turbidity current is far from equilibrium. Note that wavelength is normalized by $H_c/c_D$, so that wavelength is sufficiently large compared with the layer thickness except in the range of small $\beta$ and $\gamma$ close to unity.

In Figure 4, wave slope, defined as wave height divided by wavelength, is shown for the same cases as in Figure 3.
Wave slope decreases with increasing $\beta$ and $\gamma$. The implication is that, as suspendability increases, and the turbidity current is farther from equilibrium, the ratio between vertical and horizontal dimensions of cyclic steps decreases.

### 4.3 Application

The Monterey East system including the area on the outside of the Shepard Meander is characterized by large scale sediment waves and the giant scour features. Ubiquitous scour features observed using high-resolution seismic-reflection techniques shown in Figure 5 in this marine environment possess similarity with cyclic steps generated by the model in this analysis. A series of steps observed at the same location is shown in Figures 5(a) and (b) in which the vertical scale is 10 and 2 times magnified respectively.

According to Figure 5, the height and length of steps observed in the field are approximately 90 m and 3300 m, respectively, so that the wave slope is estimated to be 0.027. In the range of parameters shown in Figure 4, the wave slope of 0.024 is achieved when $(\beta, \gamma) = (0.08, 0.03) \sim (0.12, 0.07)$. We take 0.1 as $\beta$ and 0.5 as $\gamma$ herein.

The bed elevation and the layer thickness of turbidity current are shown in Figure 6. Appearances of the observed and modeled steps are in good agreement. The result concretes the idea that cyclic steps are flow interactive bedforms generating an in-phase behavior of the layer thickness and bottom evolution. In the case of cyclic steps formed under subaqueous conditions, an abrupt bed elevation increase is often observed in the upper part of the step as shown in Figure 5, unlike the sub-aerial settings. This interesting feature can be observed in Figure 6.

Subaqueous cyclic steps are long waves, so that the wavelength of subaqueous cyclic steps must be some orders of magnitude larger than the layer thickness. As described before, the length scale in the direction normal to the ocean floor is normalized by the critical layer thickness, and that in the direction tangential to the ocean floor is normalized by the critical layer thickness divided by the drag coefficient $c_D$. Since $c_D$ is assumed to be 0.005, the wavelength seen in Figure 6 is more than fifty times larger than the critical layer thickness. The long wave assumption is confirmed to be satisfied.

### 5. Conclusion

With the light of this study, we could generate the shape of cyclic steps which migrates upstream preserv-
ing its shape analytically. Velocity, layer thickness, and suspended sediment concentration variations over one step wavelength are obtained as well. Sensitivity of the parameters \( \gamma \) and \( \beta \) delineates the essential flow conditions to be satisfied for the emerge of this step shape bed undulations driven by turbid underflow. Importantly, features of the generated steps (wavelength, appearance) and the adjustments in surrounding environment (average bed slope) to facilitate this phenomenon bear an acceptable agreement with the field observations.

REFERENCES

15) Fildani, A., Normark, W. R., Kostic, S. and Parker, G.: Channel formation by flow stripping: large-scale scour features along the Monterey East Channel and

(Received March 8, 2011)