Measurements of Vertical Pressure Profile beneath a Planar Valley of Loose Sand and Its Estimation Based on Self-Similar Solution of Elliptic Equation System

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In this paper, a series of physical modeling with different sample preparation methods and types of basal support was conducted to investigate vertical pressure profiles beneath a planar valley of loose sand. A linear elastic solution of stress field for semi-infinite valley loaded by self weight was derived and compared with the experimental results. Despite of different geometry, the solution obtained from an elliptic equation system with an assumption of self-similarity was found equivalent to a classical Fillunger's solution of a planar semi-infinite wedge with body force. The Fillunger-based solution is found to underestimate the measured vertical pressure. Moreover, the analytic stress field was considered as an inadmissible state because linear elasticity failed to satisfy the limit stress condition along cascading down slopes.

Key Words: granular materials, stress distribution, analytic solution, physical modeling

1. Introduction

The studies on the load transmission in static granular assemblies have recently drawn much attention in physics and mechanics. The mechanisms of load transmission in granular material have been theoretically and experimentally elucidated by many researchers. Recently, load transmission focusing to a slope inclined at angle of repose such as sand heap has attracted much of interest in physics of soft-condensed matter and granular mechanics¹⁻⁸⁾. Continuum mechanics, finite element method and discrete element method are the main theoretical approaches for solving this fundamental problem of free-standing bulk of sand loaded by self weight. This study concentrates on a continuum mechanics approach because its idealization provides the simplest assumptions to explain how the body force is transferred to the base of granular body. Another reason is because the authors have already gained a certain achievement in an analytical study of self weight distribution in loose sand mound whose side boundaries of the wedge symmetrically

incline at the angle of repose⁸⁻¹¹⁾. Therefore, more investigations are aimed at different geometry of the problem.

The present work provides a preliminary study of a sand valley which is considered as a conjugate problem of sand heap. A sand valley is comparable to the V-shaped notch problem in solid mechanics¹²⁻¹⁴⁾ but additionally includes the stability of slopes. A typical geometry of sand valley lied at the angle of repose is illustrated by combination of two infinite slopes intersecting at the sharp edge. Despite being a conjugate problem of sand heap, theoretical and experimental on weight transmission in a sand valley is rare. The knowledge on this problem can be applied to explain the mechanics of sand dunes in desert and undersea as well as load transmission and stress concentration around the intersecting ridge of two slopes in cut slopes and open-pit mining sites.

In this paper, a series of physical modeling is conducted to supply the vertical pressure profile beneath a planar valley of loose sand for comparison and validation with the developed analytic solutions. The reason of choosing a loose and dry

condition of sand in this study is because an angle of repose can be taken equivalent to an angle of internal friction^{15,16)}, hence reducing arduous effort in a process of theoretical and experimental examinations of this fundamental study. Moreover, experimental realization of sand valley in a planar shape can be simply achieved; therefore, this study is limited to two-dimensional plane condition.

According to past studies, there are two kinds of governing partial differential equations (PDE) of stress configuration which are an elliptic equation system and a hyperbolic equation system. By the nature of mathematics, a rigid-plastic model constitutes a hyperbolic system while a linear elastic model forms an elliptic system. Both systems have their own mathematical characteristics with different requirements on boundary conditions. Therefore, classification of governing equations gives a hint on how to select suitable boundary conditions in order to obtain the stable and unique solution.

Both configuration systems will be considered in a whole plan of study; however, an elliptic system is selected for this study and will compare with results obtained from hyperbolic system which is carried out independently of this article. The experimental data which is originally released to this article will validate results of both configuration systems for further discussions on their performance of prediction.

This fundamental study might be useful for those who are interested in load transmission in granular media as well as the related analytical and experimental techniques, and would present a basic knowledge about stress distribution in excavation and mining problems with the intersecting ridge.

2. Theoretical Background

For two-dimensional idealization, configuration of stress with the gravity force acting along the z-axis in polar coordinate system (r, θ) is shown in Fig.1, where r is a radius and θ is an angle rotating anticlockwise from the vertical plane. According to geometry, the relationships with rectangular coordinate system (x, z) are obtained.

$$x = r \sin(\theta), z = r \cos(\theta), r^2 = x^2 + z^2 \quad (1)$$

Fig. 1 Stress components under static equilibrium condition in polar coordinate system for a given differential element

Three components of stress must satisfy the static equilibrium conditions in the polar coordinate system under a gravity field in a downward vertical direction as shown in Eq.(2) and Eq.(3), where σ_r is a normal stress in the radial direction, σ_θ is a normal stress in the circumferential direction, $\tau_{r\theta}$ is a tangential shear stress and V is a body-force scalar potential. These equations are formulated for an angular θ measured from the z-axis.

Fig. 2 Notations of stress components in rectangular and polar coordinate systems

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = \frac{1}{r} \frac{\partial V}{\partial r} \quad (2)$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = -\frac{\partial V}{\partial r} \quad (3)$$

$$V = V(r, \theta) = \gamma r \sin \theta \quad (4)$$

As outlined in Fig. 2, the stress components $(\sigma_r, \sigma_\theta, \tau_{r\theta})$ in a polar coordinate system which rotates anticlockwise at an angle θ can be transformed into the stress components $(\sigma_x, \sigma_z, \tau_{xz})$ in a rectangular coordinate system.

$$\sigma_x = \frac{\sigma_r + \sigma_\theta}{2} - \frac{\sigma_r - \sigma_\theta}{2} \cos 2\theta + \tau_{r\theta} \sin 2\theta \quad (5)$$

$$\sigma_z = \frac{\sigma_r + \sigma_\theta}{2} + \frac{\sigma_r - \sigma_\theta}{2} \cos 2\theta - \tau_{r\theta} \sin 2\theta \quad (6)$$

$$\tau_{xz} = \frac{\sigma_r - \sigma_\theta}{2} \sin 2\theta + \tau_{r\theta} \cos 2\theta \quad (7)$$

In two-dimensional problems, it is convenient to represent the stress components in a potential form using Airy stress function. Airy¹⁷⁾ found that a state of stress which satisfies the equilibrium equations can be expressed independently of deformations as second-order partial derivatives of a single scalar function with respect to the coordinates. The basic feature of Airy stress function of satisfying the equilibrium equations enables broad applications to both elastic and plastic states, with and without body forces, either in plane stress or plane strain problem. By introducing the Airy stress function $F(r, \theta)$ in polar coordinates, the stress solutions can be expressed in the forms:

$$\sigma_r = \sigma_r(r, \theta) = \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} + \gamma r \cos \theta \quad (8)$$

$$\sigma_\theta = \sigma_\theta(r, \theta) = \frac{\partial^2 F}{\partial r^2} + \gamma r \cos \theta \quad (9)$$

$$\tau_{r\theta} = \tau_{r\theta}(r, \theta) = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial F}{\partial \theta} \right) \quad (10)$$

For linear stress-strain relations, the Beltrami-Mitchell equation provides the necessary condition of compatibility where Young's modulus E is cancelled out in the process of derivation. In plane strain condition, F must satisfy the following Beltrami-Mitchell equation where ν is Poisson's ratio.

$$\nabla^4 F = \nabla^2 \nabla^2 F = -\frac{1-2\nu}{1-\nu} \nabla^2 V \quad (11)$$

where Laplacian operator is defined below:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \quad (12)$$

Due to homogeneous body force defined in Eq.(4), $\nabla^2 V=0$; thus the right-hand side of Eq.(11) becomes zero and the effect of ν is nullified. Consequently, Eq.(11) is reduced to a biharmonic equation whose feature is obviously an elliptic system of PDE. According to Michell¹⁸, the solution of $\nabla^4 F=0$ in polar coordinate system can be generally expressed by:

$$\begin{aligned} F = & A_{01}r^2 + A_{02}r^2\theta + A_{03}\ln r + A_{04}\theta \\ & + (A_{11}r^3 + A_{12}r\ln r + A_{13}\theta r + A_{14}r^{-1})\cos\theta \\ & + (B_{11}r^3 + B_{12}r\ln r + B_{13}\theta r + B_{14}r^{-1})\sin\theta \\ & + \sum_{n=2}^N (A_{n1}r^{n+2} + A_{n2}r^{2-n} + A_{n3}r^n + A_{n4}r^{-n})\cos n\theta \\ & + \sum_{n=2}^N (B_{n1}r^{n+2} + B_{n2}r^{2-n} + B_{n3}r^n + B_{n4}r^{-n})\sin n\theta \end{aligned} \quad (13)$$

Unknown sets of coefficients A and B , as well as the choice of suitable forms of Eq.(13) are determined by identifying the possibility of the highest order polynomial term to match the imposing boundary conditions of the problem.

2.1 Self-similar Solution

A semi-infinite body of valley problem is a main idealization of geometry adopted in this study. The characteristics of this typical problem are self-similar and ideally have no inherent length scale like that of heap problem. Therefore, stress distributions appear in the similar geometric shape at all distances from the intersecting edge of valley. Moreover, the self-similarity can be achieved with its immediate consequence that the stress solutions can be expressed in the separated-variable form between its radius and angular coordinate. Thus, Airy stress function generally expressed by Eq.(13) can be reduced to Eq.(14) as separable functions of r^n and $f_n(\theta)$ in the particular case of an infinite self-similar body.

$$F(r, \theta) = r^n f_n(\theta) \quad (14)$$

The power of $n=3$ agrees with a physical quantity of zero traction along the sliding plane where A_{11} , B_{11} , A_{33} , and B_{33} are constant coefficients based on Eq.(13).

$$F(r, \theta) = \begin{pmatrix} A_{11}\cos\theta + B_{11}\sin\theta \\ + A_{33}\cos 3\theta + B_{33}\sin 3\theta \end{pmatrix} r^3 \quad (15)$$

Stress function F expressed in Eq.(15) consists of four unknowns which are solvable by four independent equations. Substitution of Eq.(15) into Eqs.(8)–(10) obtains the stress field:

$$\sigma_r = r \begin{pmatrix} \gamma \cos\theta + 2A_{11}\cos\theta + 2B_{11}\sin\theta \\ -6A_{33}\cos 3\theta - 6B_{33}\sin 3\theta \end{pmatrix} \quad (16)$$

$$\sigma_\theta = r \begin{pmatrix} \gamma \cos\theta + 6A_{11}\cos\theta + 6B_{11}\sin\theta \\ +6A_{33}\cos 3\theta + 6B_{33}\sin 3\theta \end{pmatrix} \quad (17)$$

$$\tau_{r\theta} = r \begin{pmatrix} 2A_{11}\sin\theta - 2B_{11}\cos\theta \\ +6A_{33}\sin 3\theta - 6B_{33}\cos 3\theta \end{pmatrix} \quad (18)$$

Due to zero traction along the slope, we require:

$$\sigma_\theta(r, \pi/2 + \phi) = 0, \quad \tau_{r\theta}(r, \pi/2 + \phi) = 0 \quad (19), (20)$$

Taking into account the static equilibrium for body with constant r in the directions of x and z where ϕ is the repose angle of slope at loose condition as shown in Fig. 3, we obtain:

$$\int_{-(\pi/2+\phi)}^{\pi/2+\phi} (\sigma_r \sin\theta + \tau_{r\theta} \cos\theta) r d\theta = 0 \quad (21)$$

$$\int_{-(\pi/2+\phi)}^{\pi/2+\phi} (\sigma_r \cos\theta - \tau_{r\theta} \sin\theta) r d\theta = \gamma r^2 (\pi/2 + \phi) \quad (22)$$

The constant coefficients are obtained by solving Eqs.(19)–(22).

$$\begin{Bmatrix} A_{11} \\ B_{11} \\ A_{33} \\ B_{33} \end{Bmatrix} = \frac{\gamma}{48\sin^2\phi} \begin{Bmatrix} 1 \\ 3\cos 3\phi/\cos\phi \\ 0 \\ 0 \end{Bmatrix} \quad (23)$$

Since the coefficients were obtained from Eq.(23), the radial stress field in sand valley is expressed by:

$$\sigma_r = \gamma r \cos\theta (\sin^2\theta + \sin^2\phi) / 2\sin^2\phi \quad (24)$$

$$\sigma_\theta = \gamma r \cos\theta (\cos^2\theta - \sin^2\phi) / 2\sin^2\phi \quad (25)$$

$$\tau_{r\theta} = \gamma r \sin\theta (\cos^2\theta - \sin^2\phi) / 2\sin^2\phi \quad (26)$$

Substitutions of Eqs.(24)–(26) back into Eqs.(5)–(7) obtain:

$$\sigma_x = \frac{\gamma z}{2} \cot^2\phi, \quad \sigma_z = \frac{\gamma z}{2}, \quad \tau_{xz} = \frac{\gamma x}{2} \quad (27), (28), (29)$$

Fig. 3 Semi-infinite valley loaded by self weight

Despite of different geometry, the obtained solutions are consistent with Fillunger¹⁹⁾'s classical solution for a symmetrical wedge-shaped mound. These solutions ignore boundary effect by referring to Saint-Venant's principle because a basal and wall effects produced by a self-equilibrating system of tractions would decay with an increasing distance from the boundaries.

Fig. 4 Stress components of semi-infinite valley problem in polar coordinate system adopted from Fillunger-based self-similar solution using $\phi = \pi/6$.

Fig. 5 Stress components of semi-infinite valley problem in rectangular coordinate system adopted from Fillunger-based self-similar solution using $z = d = 1$ and $\phi = \pi/6$.

As illustrated in Figs.4 and 5, the angle repose $\phi = \pi/6$ and non-dimension depth $z = d = 1$ are chosen for the sake of simplicity for visualizing the solutions of Eqs.(24)–(29). The scaling stresses in positive and negative hatched areas indicate the positive and negative sign of stresses. The appearance of tensile stress shown in Fig.4(a) and 4(b) is acceptable for solid mechanics but unacceptable for soil mechanics.

Fillunger-based self-similar solution of an elliptic system derived in this study is theoretically correct but unrealistic because development of plastic zone along slopes because limiting stress condition is ignored. Actually, conclusion drawn from this theoretical outcome can sufficiently reject the analytic solution obtained from an elastic model and its consequence of elliptic system at this stage.

Let us continue proving this drawback experimentally in the next section because error might be occurred only for regions neighboring to slopes. For a broader view, additional comparison is made using a self-similar solution of a hyperbolic system based on the closure of polarized principle axes (PPA)⁸⁾ in order to understand the influence of governing equations on stress field solutions.

3. Experimental Program

A summary of the testing conditions undertaken in this study is given in Table 1. A small scale physical model of load transmission of sand valley inclined at angle of repose in a transparent plastic rectangular bin was carried out on 15 samples of dry Silica sand. The reliable stress measurement in granular materials is extremely difficult due to many factors. In order to figure out various effects on the pressure distribution, sample preparation method, initial condition, stiffness of basal support, and type of basal support were considered in the experiments. Three patterns of locations for attaching pressure gauges on the floor of bin are shown in Fig.6.

Fig. 6 Outline of the pressure gauges on the floor of bin along plane (L) and plane (W)

In the paper, the initial condition referred to a condition of tapping or no tapping a sample after the filling procedure completed. The tapping technique is specifically defined as a method in which a small vibration was applied to a sample by slightly tapping the wall of the physical model bin using a wooden hammer in order to have a more homogeneous granular packing and to reduce the fluctuation of the measured vertical pressure. Apart from the initial condition, the stiffness of the basal support was varied by using a flexible plate and a rigid plate to examine the effect of deflection of the base on stress distribution. The flexible basal support referred to an apparently bending of raised floor due to the insufficient rigidity of a plate supporting on four sides of the wall. The basal support which was reinforced by filling all space of a raised floor to prevent flexural deformation of a plate was assumed to be rigid.

3.1. Physical Modeling

The geometry of a physical model for sand valley used in the present study is illustrated in Fig. 7. Its dimensions are length $L = 43.5$ cm, width $W = 28.5$ cm, and height $H = 29.3$ cm. A scale was stuck on the edge of the bin to aid visual inspection of excavation depth. The vertical pressures of the granular material exerting to the floor were measured by a set of miniature pressure cells having a maximum capacity of 10 kN/m^2 (6 mm in diameter) and of 20 kN/m^2 (10 mm in diameter) for capturing low level of pressure due to self weight of sand. A different number of pressure gauges were installed on a floor of bin along a plane of length (L) and a plane of width (W), and were outlined in different patterns of locations (symmetric and asymmetric) as presented in Table 1 and Fig. 6 to catch a wide range of the load transmission beneath the sand valley. In the present study, the effect of basal deformation of the bin was examined. According to Lee & Herington²⁰, Michalowski & Park⁶, and Trollope²¹,

the arching effect occurred and the stress distribution beneath a mound of granular material was in passive condition as its base underwent the concave deflection.

Fig. 7 Physical model of load transmission of sand valley with a smooth basal support

(a) Dimension of rectangular bin

(b) Miniature pressure cells embedded in the Teflon plate covered by sand paper

Fig. 8 Physical model of load transmission of sand valley with a rough basal support

Table 1 Testing program of small scale physical model of sand valley

Sample no.	Sample preparation method	No. of pressure gauges	Pattern of location	Initial condition	Stiffness of basal support	Type of basal support
S01	Air pluviation by sieve type 1 (AP1)	8	Symmetric	Tapping	Flexible	Smooth
S02		8	Symmetric	Tapping	Flexible	Smooth
S03		8	Symmetric	No tapping	Flexible	Smooth
S04		8	Symmetric	No tapping	Flexible	Smooth
S05		8	Symmetric	No tapping	Flexible	Smooth
S06		8	Symmetric	No tapping	Flexible	Smooth
S07		8	Asymmetric	No tapping	Rigid	Smooth
S08		11	Asymmetric	No tapping	Rigid	Smooth
S09		8	Asymmetric	No tapping	Rigid	Rough
S10		8	Asymmetric	No tapping	Rigid	Rough
S11	Air pluviation by sieve type 2 (AP2)	11	Asymmetric	No tapping	Rigid	Rough
S12		11	Asymmetric	No tapping	Rigid	Rough
S13		11	Asymmetric	No tapping	Rigid	Rough
S14	Dry deposition (DP)	8	Asymmetric	No tapping	Rigid	Rough
S15		8	Asymmetric	No tapping	Rigid	Rough

Note: the right parenthesis “}” refers to a group of samples repeated under the same testing conditions

Fig. 9 Particle size distribution curve of Silica sand No. 8

Additionally, two types of basal supports were prepared to observe the discrepancy in load transmission as shown in Figs. 7 and 8. First, the pressure cells were attached on the smooth basal support as shown in Fig. 7 without placing the pressure gauges face flush with the basal support. Second, in addition to the smooth basal support, a set of sensors was also embedded in the Teflon plate as shown in Fig. 8. This plate has a thickness of 3 mm and is covered by sand papers to ensure the rough basal support. As a result, the net height of the physical model bin reduces to 29 cm as shown in Fig. 8(a). Pressure gauges were mounted flush with the basal support. The level of the base was confirmed by a vernier caliper before carrying out the physical model experiment as illustrated in Fig. 8(b).

Three series of tests were performed by air pluviation technique using the sieve type 1, air pluviation technique using the sieve type 2, and by dry deposition technique, respectively. In the first series of tests, a set of samples S01 and S02 was tested under the same testing condition of tapping technique while other sets of S03–S10 were tested without tapping technique. The latter can be categorized into three subsets of different conditions of the basal support, each of which was the flexible and smooth condition for S03–S06, the rigid and smooth condition for S07 and S08, and the rigid and rough condition for S09 and S10. Moreover, the second and the last series of tests for samples S11–S13, and S14 and S15 were carried out on no tapping/rigid and rough condition.

3.2. Sample Preparation Method

Dry Silica sand No. 8 was used in the experimental program. The particle size distribution curve of this sand is plotted as a percent finer by weight against particle size as shown in Fig. 9. The tested sand properties consist of specific gravity $G_s = 2.65$, median particle size $d_{50} = 0.09$, uniformity coefficient $C_u = 2.93$, maximum void ratio $e_{max} = 1.333$ (or minimum unit weight $\gamma_{min} = 11.14 \text{ kN/m}^3$), and minimum void ratio $e_{min} = 0.703$ (or maximum unit weight $\gamma_{max} = 15.26 \text{ kN/m}^3$).

As the sample preparation is one of the most important aspects for soil testing, the uniformity of the sample has been a major concern. Various techniques from theoretical and experimental viewpoints have been proposed to reconstitute

homogeneous samples²²⁻²⁵. The sand samples were, therefore, prepared by three different types of techniques to observe the effect on load transmission as depicted in Fig. 11.

As shown in Fig. 11(a) and (b), the first and second techniques are air pluviation²⁵ using sieve types 1 and 2, which are hereafter abbreviated to AP1 and AP2, respectively. The third technique is called dry deposition²⁴ and is abbreviated to DP as presented in Fig. 11(c). To obtain loose samples, sands were filled in a bin through a funnel with sieve opening of 5 mm for AP 1 and that of 1 mm for AP2. Sand was then deposited layer by layer, each of which was formed by spreading sand at a constant 2 cm drop height in consecutive lines along the long and short direction of the bin as shown in Figs. 11(a) and 11(b). Approximate mass rate of pouring for AP1 and AP2 were 2 kg/minute and 0.67 kg/minute. In regard to DP in Fig. 11(c), sand was first filled into a funnel having a 2-cm-diameter nozzle which initially rested on the center of the bin's floor. The funnel was removed vertically with nearly zero drop height and with a mass rate of pouring 5 kg/minute. The subsequent funnel of sand was filled accordingly until the sand exceeded the boundary.

The excessive sand on the top was trimmed off as shown in Fig. 12(a). The average bulk unit weight of loose sand, based on weight/volume at storage condition, was approximated $\gamma = 12.46 \text{ kN/m}^3$ or relative density $D_r = 33\%$. After the sand filling process into the bin, the process of excavation along the center of sand valley is followed by using a nozzle of vacuum in order to remove sand with minimal disturbance. The vacuum with a sharp nozzle shown in Fig. 12(b) was used to suck sand along the centerline; therefore, loose sand deposited on both sides of slope cascades downslope once the slopes lose their supports during excavation process. Because subsequent layers of sand are formed immediately after sliding of the previous layer with the maximum angle that slopes can resist. This angle represents an angle of repose of sand. As seen in Fig. 10, the angle of repose $\phi = 35^\circ$ from the sand valley experiment is almost equal to internal friction angle $\phi_{DS} = 35.3^\circ$ from direct shear test under the maximum applied pressure at 310 kPa. The sands were excavated until depth $h = 20 \text{ cm}$, which is equal to the height $z = 9 \text{ cm}$ measured from the basal support as shown in Fig. 12 (c).

Fig. 10 Direct shear test on dry Silica sand No. 8

3.3. Calibration of Pressure Gauges

As shown in Fig. 13, the miniature pressure cells were calibrated in water to ensure the reliability of their measurement before starting to investigate the stress distribution induced by self weight of sand valley in the physical model. The hydrostatic pressures at desired depths can be measured after amount of water was filled into the bin as illustrated in Fig. 13(a). Owing to the pressure cells having the low pressure capacity and being sensitive to capture the pressure, the gentle flow of water circulates slowly into the bin to reduce the disturbance on the sensors. As shown in Fig. 6, one pressure gauge was installed at centerline, three on the right and four on the left asymmetrically in order to effectively catch a pressure profile by a limited number of sensors. For the simple representation of measured data, a half plane of the physical model can be considered due to its symmetric geometry.

In addition, as shown in Fig. 13(b), the vertical pressure σ_v measured on the floor of bin along the central plane of the length L of the physical model normalized by its initial hydrostatic pressure γH was plotted against χ representing the distance from the centerline along the floor normalized by half-length $L/2$. A circular symbol denotes a profile of pressures undergoing

loading after water was circulated into the bin till the height $H=15$ cm, and a rectangular symbol represents a series of pressures subjected to unloading after removing an amount of water. It can be seen from Fig. 13(b) that the experiment shows reliable results with the average value of normalized vertical pressure $\sigma_v/\gamma H = 1.01$.

4. Interpretation of Experimental Results

Figures 14–28 display profiles of the scaling vertical pressure on the basal support along the length L versus the normalized distance $2\chi/(L/2)$. The pattern of the vertical stress profiles was observed before and after excavation procedures. Rectangular and circular symbols represent the vertical pressures measured by pressure cells at excavation depth $h=0$ cm and 20 cm, respectively. The latter depth was chosen due to the pronounced pressure drop after a certain amount of sand was sucked to take shape of sand valley. Scattering data and stress irregularities are apparent and pressure profiles vary from test to test and from place to place even for the initial overburden pressure in a storage condition. The non-uniform overburden pressure tends to be more obvious for a deeper depth of deposition. It is likely

that the sand portion near the center settles more than the sand portion near the walls during deposition process, causing both non-uniform stress distribution and non-uniform bulk density. Problems and effects on the measurements are reported in this section. Difficulties in stress measurement, problems in stress and density inhomogeneity as well as dependence of bin stresses on filling method are also discussed by Savage²³⁾.

(a) Water filled into the bin

(b) Vertical pressure at depth $H = 15$ cm

Fig. 13 Calibration of miniature pressure gauges in water

4.1. Initial Condition

Figures 16–19 show the vertical stress distribution on the base along plane (L), which was obtained from a set of four samples S03–06 prepared by AP1. The sensors were installed symmetrically as outlined in Fig. 6. It can be seen that the observed trend of stress distribution on the left and the right of the central plane are fairly symmetric and reproducible. However, the measured vertical stress at $h = 0$ cm shows higher value than the overburden pressure with the average value of normalized pressure $\sigma_v / \gamma H = 1.23$. This is contrary to the classical perspective such as the silo effect as described by Janssen²⁶⁾ because the measured pressure should be approximately equal or less than the overburden pressure. Janssen pointed out that the pressure at a depth near the surface of the silo is lower than overburden pressure due to the shear stress along the wall supporting the weight of granular. The reason of high and scattered pressure measurement might be due to the non-uniformity of sample prepared by AP1. As pointed out by Ishihara²⁴⁾, Vaid & Negussey²⁵⁾, and Vanel & Clément²²⁾, the mechanical behavior of granular material depends on the soil fabrics, which can be packed by different sample preparation methods.

Local arching might be occurred in the vicinity of the pressure gauge; therefore, small vibration applied to the sample

would progressively erase a local asymmetry and promote an ensemble average value of weight distribution due to a gentle resettlement of the sand. Consequently, preliminary tests of samples S01 and S02 (see Figs. 14–15) was performed with the tapping technique²²⁾. After filling stage and before excavation, the wall of the physical model bin was tapped lightly by a wooden hammer to have a more homogeneous granular packing and to reduce the fluctuation of the measured vertical pressure. As the sample density increases due to the vibration by tapping, the vertical pressure decreases to some degrees. Though gauges response showed minor significance on the reduction in vertical pressure at $h = 0$ cm, the measured vertical pressure in Figs. 14–15 shows detectable difference in behavior if compared with that in Figs. 16–19 at the excavation depth $h = 20$ cm. Nevertheless, degree and location of tapping is difficult to define as a reproducible testing procedure, S03–S15 were prepared with no tapping method.

4.2. Stiffness of Basal Support

The effect of deflection of the base on stress distribution was observed. For sample S01–S06, the rigidity of basal support is not sufficiently high because it is a raised floor of plate supporting on four sides of wall; thus, a slight bending deformation can be seen after completing the filling procedure. Subsidence of basal support causes arching effect across a base²⁷⁾; therefore, flexibility of basal support will cause a problem of non-uniformity. In order to avoid this problem, a basal support was reinforced by filling all space under a raised floor to increase the basal rigidity for sample S07–S15.

It is apparent from Figs. 14–19 that the observed trend of the measured vertical pressure obtained from flexible base gives similar pattern with that of rigid base. However, the result from sample S08 in Fig. 21 gives large scattering and non-reproducible data in comparison with sample S07 in Fig. 20. This is due to an error during testing. In general, more reliable results are obtained after replacing a flexible base to a rigid base. This result is in agreement with Lee & Herington²⁰⁾ carrying out the physical modeling to probe the pressures along the base of granular embankment using an articulated base from independently supported narrow strips. As the base deflects downward, the arching is promoted and there is a progressive transmission of vertical pressures from the central regions. At large deflection, the pressures obtained along the base are in the passive condition. As seen in Figs. 16–19, the vertical pressures obtained from the flexible base do not differ significantly from those of the rigid base. The justification is probable that the flexural deflection of the base in the present study does not exceed a certain critical value to trigger substantial changes on the state of the stress.

4.3. Types of Basal Support

The rough base and the smooth base, as described in the

above section, were two supporting conditions employed to find the variation of stress distribution in the sand mass stored in a rectangular bin. The experimental data of measured vertical pressures obtained from these two types of bases were plotted in Figs. 20–23. Under the same condition of AP1 and rigid base, the experimental result shows that whatever the type of the basal support is used, the normalized vertical pressures at $h = 0$ cm give the same value of $\sigma_v/\gamma H = 1.19$. As a consequence, it can be implied that the effect of rough base does not affect to the overburden pressures. However, the experimental results at excavation depth $h = 20$ cm showed that the profiles of the vertical pressure corresponding to rough base lie lower if compared to those of smooth base. The measured vertical pressures beneath the ridge of the valley apparently drop when the rough base was used.

Michalowski & Park⁶⁾ and Savage²³⁾ discussed the effect of rough base in a problem of sand heap by pointing out that the shearing resistance and its direction along the base has a substantial effect to the vertical pressure profile. Despite difference in boundary conditions between sand heap and sand valley, effect of basal roughness to sand valley can be confirmed experimentally.

So far, the review of improvement on the physical model of sand valley was explained and discussed in an attempt to obtain the reliable and reproducible results of stress distribution. The profile of overburden pressure after the filling process is examined to evaluate the uniformity of density. It seems that sample preparation with the tapping method and container with the rigid base does not significantly reduce the variation of density. Although some of the sensors give convincing result, the averages of vertical pressures still remain 23% higher than geostatic pressure. The reason for nonuniform overburden pressure is likely that the air pluviation method using sieve type 1 (AP1) is not a suitable method to provide the uniform density.

4.4. Sample Preparation Method

In this section, the interest is centered mainly on the effect of sample preparation, which can provide different behaviors of granular material due to particle arrangement. The typical evolution of the stress distribution corresponding to samples prepared by air pluviation technique using type 2 (AP2) and by dry deposition (DP) is shown in Figs. 24–28.

As seen in Figs. 24–26, at $h = 0$ cm the measured vertical pressures obtained from samples S11–13 prepared AP2 are in good consistency with geostatic pressure with the average ratio $\sigma_v/\gamma H = 1.11$ although showing slight scattering. It can be seen that as the mass rate of pouring decreases from 2 kg/min to 0.67 kg/min, the density of sample reduces to 2.4%. Then, the samples tend to be uniform and give reliable results because the density variation is acceptably small. This fact was also confirmed by some researchers. Vaid & Negussey²⁵⁾ investigated the factor affecting the uniformity of the sample.

They elucidated that a decrease in mass rate of pouring would reduce interference effects which may prevent sand particles reaching the most stable state and therefore result in uniformly loose sample. Besides, Vanel & Clément²²⁾ also stipulated that although there is a small change in density, its effect on the mechanical behavior is drastic.

Additionally, as illustrated in Figs. 27 and 28, samples reconstituted by DP at $h = 0$ cm provides acceptable data of vertical pressures with $\sigma_v/\gamma H = 1.06$. The result obtained from DP evidenced that the trend of recorded pressures appears to decrease slightly from the wall to the centerline. It is also noticed that the vertical pressure at the center seems lower than overburden pressure as the sand sample was prepared by dry deposition technique DP. This might cause by a particular pattern of force chain transmitting a central load to bin's wall which is not settled during the deposition process.

Assuming that the stress distribution depends solely of sample preparation method regardless of the effects of type of the base, after sands were excavated to depth $h = 20$ cm along centerline, the normalized vertical pressure for samples 14 and 15 in Figs. 27 and 28 is approximately equal to 0.46, which is less than 12% of that for samples S11–13 in Figs. 24–26 and 6% for samples S07, 09, and 10 in Figs. 20, 22, and 23.

5. Comparison and Discussion

The small scale physical model of load transmission of sand valley inclined at angle of repose in the transparent plastic rectangular bin was carried out on 15 samples of dry silica sand. In the previous section, we experimentally explained the vertical pressure profile along plane beneath a planar valley of loose sand. Though the experiments were carefully conducted by considering various effects to the models; such as roughness and stiffness on the bottom, the obtained results show a great variability even under the same conditions. In the experiment using loose sand, the variability of results is considered to be acceptable to some extent compared with other materials.

The key factor affecting the vertical stress is the influence of granular packing caused by different sample preparation techniques. The samples prepared by AP1 give widely scattered vertical pressures while samples prepared by DP appears to reveal highly reliable results with a local minimum pressure at the center of the physical model after the filling process.

A series of testing program and its interpretation is still insufficient to define the acceptable range of accuracy. The nonuniformity of density seems to be increased with depth of storage sand. This study can achieve an accuracy of the overburden pressure normalized by the geo-static pressure with the discrepancy 6% by using the DP method. Despite sand deposition using DP is recommended due to the reliable profile of overburden pressure, it calls for more experimental work with other materials, other boundary conditions and larger size of bin.

Fig. 14 Vertical stress distribution for S01

Fig. 15 Vertical stress distribution for S02

Fig. 16 Vertical stress distribution for S03

Fig. 17 Vertical stress distribution for S04

Fig. 18 Vertical stress distribution for S05

Fig. 19 Vertical stress distribution for S06

Fig. 20 Vertical stress distribution for S07

Fig. 21 Vertical stress distribution for S08

Fig. 22 Vertical stress distribution for S09

Fig. 23 Vertical stress distribution for S10

Fig. 24 Vertical stress distribution for S11

Fig. 25 Vertical stress distribution for S12

Fig. 26 Vertical stress distribution for S13

Fig. 27 Vertical stress distribution for S14

Fig. 28 Vertical stress distribution for S15

Fig. 29 Comparison of experimental results of stress distribution from the current physical modeling with exact solution based on the closure of polarized principal axes (PPA) of a valley problem²⁸⁾ and with the solution adopted from classical theory¹⁹⁾.

In addition, to observe the vertical pressure profile clearly, the measured pressure exerting on the base of the bin along the central plane (L) on long side and the central plane (W) on short side for all tested samples after reaching the excavation at the level $h=20$ cm or with the depth $z=9$ cm are plotted in Figs. 29 (a) and (b), respectively. These figures are featured by the normalized vertical pressure ($\sigma_v/\gamma H$) where $H=29$ cm and $\gamma=12.46$ kN/m³ versus the normalized distance $\chi/(L/2)$. The obtained results indicate a large fluctuations and variations of vertical stress along both plane L and plane W due to the effect of preparation techniques.

The dash and solid line denote the closed-form solution based on the closure of polarized principal axes (PPA) applying to a valley problem²⁸⁾ and the theoretical solution based on Fillunger's classical work¹⁹⁾, respectively. It is clear from Fig. 29 that the predicted solution based on Fillunger concept gives a constant value of vertical pressure $\sigma_v/\gamma H=0.15$ everywhere beneath sand valley. All experimental data of measured vertical pressures lie above the Fillunger-based solution. On the other hand, Fig. 29 indicates that the exact solution based on PPA results in concave curve passing with the normalized vertical stress $\sigma_v/\gamma H=0.73$ beneath the ridge of sand valley. This tractable solution stays mostly in upper part of measured vertical stress.

For convenience in discussion, let us define the apparent geostatic pressure exerting on the floor as a vertical pressure calculated solely from a geometrical outline of sand valley inclined at angle of repose ϕ . The expression shown in Eq.(30) is imposed to Fig. 29 and it represents outline of sand valley in normalized form. Due to the symmetry of sand valley, zero

shear stress can be assumed along the central plane because there is no relative displacement, in other words, $\tau_{xz}|_{x=0}=0$. According to the equilibrium along the gravity direction in plane condition shown in Eq.(31), Eq.(30) simply represents an unrealistic state of stress where the horizontal stress is zero.

$$\sigma_z = \gamma(z + x \tan \phi), \quad \partial \tau_{xz} / \partial x + \partial \sigma_z / \partial z = \gamma \quad (30), (31)$$

Thus, the differences in the results of each experimental case might be explained as difference in direction and magnitude of shear stress. The experimental data lying under the apparent geostatic pressure indicate that the horizontal shear stress turns towards the central plane (see Fig. 2) which is corresponded to the Fillunger-based solution; but the other is corresponded to the PPA-based solution with the opposite shear direction.

6. Conclusion

A linear elastic solution of stress field for semi-infinite valley loaded by self-weight was derived and compared with the experimental results. Despite of different geometry, the solution obtained from an elliptic system with an assumption of self-similarity was found equivalent to a classical Fillunger's solution of a planar semi-infinite wedge with body force. The Fillunger-based solution underestimates the measured vertical pressure while that obtained from a hyperbolic equation overestimate. Moreover, the Fillunger-based solution was considered as an inadmissible state because linear elasticity failed to satisfy the limit stress condition along cascading down slopes; thus giving unrealistic tensile stresses near the slopes.

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