Mathematical Analysis on a Conforming Finite Element Scheme for Advection-Dispersion-Decay Equations on Connected Graphs

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Theoretical stability and error analysis on a Conforming Petrov-Galerkin Finite Element (CPGFE) scheme with the fitting technique for solving the Advection-Dispersion-Decay Equations (ADDEs) on connected graphs is performed. This paper is the first research paper that applies the concept of the discrete Green’s function (DGF) to error analysis on a numerical scheme for the ADDEs on connected graphs. Firstly, the stability analysis shows that the scheme is unconditionally stable in space for steady problems and is stable in both space and time for unsteady problems if the temporal term is appropriately discretized with a lumping technique. Secondly, basic properties of the DGF on connected graphs, which provide key mathematical tools in the error analysis, are presented. The error analysis with the DGF reveals a direct relationship between the regularity conditions on the known functions and accuracy of the scheme, explicitly indicating that the accuracy of the scheme is strongly influenced by the accuracy of the discretized known functions. The error analysis also shows that the scheme is uniformly-convergent in the $L^r$-error norm with respect to the diffusivity, which cannot be achieved in the conventional numerical schemes. This unique and remarkable property is a significant advantage of the present CPGFE scheme over the conventional ones.

Key Words : advection-dispersion-decay equations, conforming Petrov-Galerkin finite element schemes, connected graphs, discrete Green’s functions, uniform convergence

1. INTRODUCTION

Advection-Dispersion-Decay Equations (ADDEs) are elliptic or parabolic partial differential equations arising naturally in scientific and engineering problems. Longitudinal dispersion of solute in open channels are formulated as initial (or terminal) and boundary value problems of the ADDEs on connected graphs with appropriate internal boundary conditions (IBCs) at junctions. Other examples are longitudinal movements of aquatic organisms living in river systems. Yoshioka and Unami deduced the conservative ADDE governing the cross-sectionally averaged solute concentration based on a stochastic differential equation for Lagrangian particles movement. Yoshioka et al. proposed a system of non-conservative ADDEs for evaluating hydrodynamic uncertainties involved in the longitudinal dispersion of solute in open channels. Although the ADDEs are typically simplified counterparts of 3-D mathematical models, their usefulness in practical applications is still in high demand. Since analytical solutions to the ADDEs are available only for limited cases, numerical schemes are utilized in applications.

Accurate and stable numerical resolution of the ADDEs has been a challenging task. A major difficulty encountered in numerically solving the ADDEs is instability due to the existence of the advection terms. A large reaction term is also a source of numerical instability. Another difficulty, an issue particular to the ADDEs on connected graphs, is
consistent and efficient treatment of the IBCs. Some researches pointed out that an improper treatment of the IBCs yields inaccurate numerical solutions\(^{(3, 13)}\). Most of the existing numerical schemes separately discretize an ADDE at junctions and in reaches\(^{(4, 15, 16)}\), resulting in loss of computational efficiency.

The authors\(^{6}\) previous researches revealed that the finite volume (FV) schemes and the conforming Petrov-Galerkin finite element (CPGFE) schemes are effective in solving the ADDEs on connected graphs, the former being fully lumped counterparts of the latter\(^{(3, 6)}\). Their advantages over the conventional ones are high stability and the use of compact computational stencils with the fitting technique\(^{6}\), achieving a unified spatial discretization that consistently and efficiently handles the IBCs. On the other hand, their disadvantage is lower computational accuracy than the high-resolution schemes, such as the slope-limited schemes. However, the numerical schemes with the fitting technique have potential robustness for advection-dominant and reaction-dominant problems as shown in this paper.

The main purpose of this paper is to show a uniformly-convergent property of a CPGFE scheme with appropriately chosen basis functions from the spaces of exponential splines, utilizing the discrete Green\(^{\theta}\) function (DGF)\(^{(10)}\) as a mathematical tool. Stability analysis on the CPGFE schemes is also performed in this paper. The analysis in this paper mainly focuses on steady problems. The presented CPGFE scheme is a modified version of the previous one\(^{(3)}\), in which only the basis discretization for the trial space is changed. It is shown that the impact of this minor change is significant; numerical solutions to the presented scheme uniformly converge to exact solutions under the \(L^\infty\) error norm with respect to the diffusivity, which are not the cases for the conventional numerical schemes. This unique and remarkable property, which has not been reported in the conventional researches, indicates potential robustness and higher consistency of the present scheme for ADDEs with significantly large advection and reaction terms as encountered in practical problems.

The remainder of this paper is organized as follows. Some preliminaries and a concise introduction for the ADDE are given in section 2. The new CPGFE scheme is presented in section 3. Stability analysis on the CPGFE schemes is performed in section 4. Error analysis on the schemes is performed in section 5. Section 6 provides conclusions of this paper. Appendices contain supplement information not presented in the main text of this paper.

2. MATHEMATICAL MODEL

(1) Preliminaries

\[ \text{Eq. (1)} \]

Let \( \Omega \) be a connected graph that consists of finite numbers of 0-D vertices and 1-D reaches\(^{(18)}\). There exists at least one path connecting arbitrary distinct two vertices in the domain \( \Omega \). A vertex that connects more than one reaches is referred to as a junction where an appropriate IBC is specified. A vertex connected with exactly one reach is referred to as a boundary vertex. Arbitrary position in the domain \( \Omega \) is uniquely specified with the \( x \) abscissa taken along the reaches. Fig. 1, presents a sketch of a connected graph. The norm of a function \( v = v(x) \) in the domain \( \Omega \) with a functional space \( X \) is expressed as \( \|v\|_{X} \). The Sobolev\(^{\phi}\) spaces for the functions are expressed following Berkolaiko and Kuchment\(^{(18)}\). For example, the \( L^2 \) -norm for the function \( v \in L^2(\Omega) \) is defined as

\[ \|v\|_{L^2} = \left( \int_{\Omega} v^2dx \right)^{1/2} \]

and the \( L^\infty \) -norm for the function \( v \in L^\infty(\Omega) \) as

\[ \|v\|_{L^\infty} = \sup_{x \in \Omega} |v| \]

where the integral in the right hand side of Eq.(1) means the integration in the reaches of the domain \( \Omega \). By the Sobolev\(^{\phi}\) embedding theorem\(^{(10)}\), the function \( v \in H^1(\Omega) \) is spatially continuous in the domain \( \Omega \). The inner product for the two functions \( v_1, v_2 \in L^2(\Omega) \) is defined as

\[ \langle v_1, v_2 \rangle = \int_{\Omega} v_1 v_2 dx. \]

In this paper, a numerical scheme for steady problems is said to be uniformly-convergent if

\[ \lim_{h \to 0} \lim_{\varepsilon \to 0} \|u - u_h\|_{L^\infty} = \lim_{\varepsilon \to 0} \lim_{h \to 0} \|u - u_\varepsilon\|_{L^\infty} = 0 \]

where \( u \) is the exact solution, \( u_h \) is the numerical solution, \( h \) is the element size and \( \varepsilon \) is the diffusivity defined in the next sub-section. The first term in Eq.(4) is not satisfied in the conventional schemes as shown in Appendix A. Eq.(4) holds if
\[ \| u - u_0 \|_C \leq C h^{C_2} \]  \hspace{1cm} (5)

is satisfied with positive constants \( C_1 \) and \( C_2 \) independent of the diffusivity.

(2) Advection-dispersion-decay equation

This paper focuses on the ADDE with the time-independent known functions expressed as \(^{2,19}\)

\[
\frac{\partial u}{\partial t} - a \frac{\partial u}{\partial x} - \epsilon \frac{\partial^2 u}{\partial x^2} + cu = f
\]  \hspace{1cm} (6)

where \( u \in H^1(\Omega) \) is the unknown, \( a \in L^r(\Omega) \) is the drift coefficient, \( c \in L^s(\Omega) \) is the decay coefficient, \( f \in H^1_0(\Omega) \) is the source, and \( \epsilon > 0 \) is the diffusivity given by a constant over the domain \( \Omega \). Eq.(6) is assumed to be non-dimensionalized. The drift coefficient \( a \) represents the velocity of passive solute particles or aquatic species in most of the applications for the problems in hydraulics and related applied research fields. In order to deal with the ADDEs on connected graphs in a consistent manner, Eq.(6) is understood as the weak form \(^{19}\)

\[
\int_{\Omega} w \left( \frac{\partial u}{\partial t} - a \frac{\partial u}{\partial x} + cu \right) dx + \epsilon \int_{\Omega} \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} dx = \int_{\Omega} w f dx + B \]  \hspace{1cm} (7)

where \( w \in H^1(\Omega) \) is the test function. The boundary term \( B \) in the right-hand side of Eq.(7) is given by

\[
B = \left[ \epsilon \frac{\partial w}{\partial x} \right]_{\Gamma}. \]  \hspace{1cm} (8)

Eq.(7) associates the Kirchhoff-type IBC

\[
\sum_{j} \left[ \epsilon \frac{\partial u}{\partial x} \right]_{j} = 0 \]  \hspace{1cm} (9)

where the subscript \( \Gamma_0 \) represents each junction in the domain \( \Omega \). It is assumed that the coefficients in Eq.(7) are bounded with the upper and lower bounds \((a_{\min}, a_{\max}, c_{\min}, c_{\max})\) as

\[ 0 < a_{\min} \leq a \leq a_{\max} \quad \text{and} \quad 0 < c_{\min} \leq c \leq c_{\max}. \]  \hspace{1cm} (10)

The coefficients \( a\) and \( c \) are further assumed to be piecewise \( C^1 \)-class in the domain \( \Omega \) and their singular points are regarded as junctions connecting two reaches, at which the IBC Eq.(9) is specified. The boundary \( \Gamma \) of the domain \( \Omega \) is assumed to consist of the two parts, which are the set of upstream-ends \( \Gamma_u \) and the set of downstream-ends \( \Gamma_d \). Since \( a > 0 \), they are uniquely determined by the unit outward normal \( \sigma (= \pm 1) \) on \( \Gamma \) as

\[ \Gamma_u = \{ x \in \Gamma; \sigma = -1 \} (\neq \emptyset) \]  \hspace{1cm} (11)

and

\[ \Gamma_d = \{ x \in \Gamma; \sigma = +1 \} (\neq \emptyset). \]  \hspace{1cm} (12)

The total number of the junctions in the domain \( \Omega \) is denoted by \( N_J \). The total numbers of the upstream and downstream boundary vertices are denoted by \( N_u \) and \( N_d \), respectively. In this paper, the following two cases of the boundary conditions are considered; the one is the Dirichlet-Dirichlet case and the other is the Neumann-Dirichlet case. In the former case, the homogenous Neumann condition

\[
\epsilon \frac{\partial u}{\partial n} = 0 \]  \hspace{1cm} (14)

is specified at both the upstream-ends and the downstream-ends. On the other hand, in the latter case, the homogeneous Neumann condition

\[ \epsilon \frac{\partial u}{\partial n} = 0 \]  \hspace{1cm} (13)

is satisfied with positive constants \( \delta \) and arbitrary \( v \in H^1(\Omega) \), which is assumed throughout this paper. The Lax-Milgram Lemma \(^{20}\) shows that the steady counterpart of Eq.(15) given by

\[
A(w,v) = \langle w, f \rangle \]  \hspace{1cm} (18)

admits a unique solution \( u \).

3. Finite Element Scheme

(1) Spatial discretization

The CPGFE scheme presented in this paper uses the compact computational stencils that efficiently incorporate Eq.(9) in spatial discretization \(^{19}\). The coefficients \( a \) and \( c \) are attributed to the elements and the source \( f \) to the nodes.

The domain \( \Omega \) is discretized into a computational mesh consisting of elements bounded by two nodes, so that a junction exactly falls on one of the nodes. The elements and the nodes are indexed with the...
natural numbers. The kth element is denoted by \( \Omega_k \). The length of the element \( \Omega_k \) is denoted by \( h_k \). The i th node is denoted by \( \mathbf{P}_i \). The number of elements shearing the node \( \mathbf{P}_i \) is denoted by \( \nu(i) \). The jth element connected to the node \( \mathbf{P}_i \) is referred to as the \( \kappa(i,j) \) th element \( \Omega_{k(i,j)} \). Two nodes bound the element \( \Omega_{k(i,j)} \); one of them is \( \mathbf{P}_i \) and the other is referred to as \( \mathbf{P}_{\mu(i,j)} \). Direction of the x abscissa in the element \( \Omega_{k(i,j)} \) is identified with the sign parameter \( \sigma_{i,j} \), which equals to -1 when \( x \) is directed to \( \mathbf{P}_i \) and equals to 1 otherwise. The local abscissa

\[
0 \leq z_{ij} = \frac{x-x_i}{\sigma_{i,j}h_{k(i,j)}} \leq 1
\]

is defined in \( \Omega_{k(i,j)} \) for the sake of brevity. Fig. 2 shows sketches of the computational mesh for the CPGFE schemes where \( \nu(i) = 4 \) in Fig. 2(a).

The present CPGFE scheme is based on the fitting technique\(^6\), in which the trial and test functions are determined from the analytical solutions to local two-point boundary value problems. In the present scheme, the trial function \( \phi \) and the test function \( w \) are the analytical solutions to

\[
\varepsilon \frac{d^2 \phi}{dx^2} + a_e \frac{d \phi}{dx} - c_e \phi = 0
\]

and

\[
\varepsilon \frac{d^2 w}{dx^2} - a_e \frac{d w}{dx} - c_e w = 0
\]

in each element, respectively where the subscript \( \varepsilon \) denotes each element. With the fitting technique, the unknown \( u \) is interpolated in the element \( \Omega_{k(i,j)} \) as

\[
u = u_{\phi_{i,j,0}} + u_{\phi_{\mu(i,j),j}} \phi_{i,j,1}
\]

with the trial functions \( \phi_{i,j,0} \) and \( \phi_{i,j,1} \) given by

\[
\phi_{i,j,0} = \frac{e^{\lambda_{i,j,0}z_{ij}} - e^{-\lambda_{i,j,0}z_{ij}}}{e^{\lambda_{i,j,0}} - e^{-\lambda_{i,j,0}}}
\]

and

\[
\phi_{i,j,1} = \frac{e^{\lambda_{i,j,1}z_{ij}} - e^{-\lambda_{i,j,1}z_{ij}}}{e^{\lambda_{i,j,1}} - e^{-\lambda_{i,j,1}}},
\]

respectively where \( \lambda_{i,j} \) are the non-dimensional numbers defined by

\[
\lambda_{i,j} (x_{ij}) = \frac{1}{2} \left( - \text{Pe}_{k(i,j)} \pm \sqrt{ \text{Pe}_{k(i,j)}^2 + 4 \text{Da}_{k(i,j)}} \right)
\]

with the cell Peclot number

\[
\text{Pe}_{k(i,j)} = \frac{\sigma_{i,j}h_{k(i,j)}}{\varepsilon}
\]

and the cell Damköhler number

\[
\text{Da}_{k(i,j)} = \frac{\sigma_{i,j}}{\varepsilon}
\]

The exact expressions for the non-zero \( (i,m) \)th entry of the mass matrix \( M \) is given by

![Fig. 2 Sketches of the computational mesh: (a) around the node \( \mathbf{P}_i \) and (b) in the element \( \Omega_{k(i,j)} \).](image)

\[
\text{Da}_{k(i,j)} = \frac{c_{k(i,j)} \beta_{k(i,j)}}{\varepsilon} \geq 0.
\]

The trial functions \( \phi_{i,j,0} \) and \( \phi_{i,j,1} \) satisfy

\[
\phi_{i,j,0} (0) = \phi_{i,j,1} (1) = 1
\]

and

\[
\phi_{i,j,0} (1) = \phi_{i,j,1} (0) = 0.
\]

The test function associated with the node \( \mathbf{P}_i \), denoted by \( w_j \), is expressed in the element \( \Omega_{k(i,j)} \) as

\[
w_j = \frac{e^{\lambda_{i,j}(1-z_{ij})} - e^{\lambda_{i,j}(1-z_{ij})}}{e^{\lambda_{i,j}(1-z_{ij})} - e^{\lambda_{i,j}(1-z_{ij})}}
\]

The test function \( w_j \) satisfies

\[
w_j (0) = 1 - w_j (1) = 1.
\]

The trial and test functions presented in Eqs. (23), (24), and (30) with the degeneration of the coefficients \( a_{k(i,j)} \) and \( c_{k(i,j)} \) are accordingly derived with the Hospital rule\(^6\).

(2) Temporal discretization

Application of the present CPGFE scheme to Eq. (7) leads to the linear system of ordinary differential equations (ODEs)

\[
M \left( \frac{du}{dt} - f = -Lu = - \sum_i u_{ix} (w_j, \phi) \right)
\]

with the nodal solution vector \( \mathbf{u} \), the mass matrix \( M \), the coefficient matrix \( L \), the vector of the source \( f \), and the discrete bilinear form

\[
A_{ix} (w_j, \phi) = \int_\Omega \left( -a_{k(i,j)} w_j + e \frac{\partial w_j}{\partial x} \frac{\partial \phi}{\partial x} + c_{k(i,j)} w_j \phi \right) dx.
\]

The exact expressions for the non-zero \( (i,m) \)th entry of the mass matrix \( M \) is given by
Fig. 3 A plot of the numerical diffusivity ratio.

\[ M_{ij} = \frac{\sum_{\alpha} h_{\alpha(i,j)} \left( e^{\lambda^{\alpha}_{\text{h}+\text{d}}(j)} - e^{\lambda^{\alpha}_{\text{h}+\text{d}}(i)} - 2(e^{\lambda^{\alpha}_{\text{h}+\text{d}}(j)} - e^{\lambda^{\alpha}_{\text{h}+\text{d}}(i)}) \right)}{\Delta t (\lambda^{\alpha}_{\text{h}+\text{d}}(j) - \lambda^{\alpha}_{\text{h}+\text{d}}(i)) (e^{\lambda^{\alpha}_{\text{h}+\text{d}}(j)} - e^{\lambda^{\alpha}_{\text{h}+\text{d}}(i)})^2} \]  

with \( m = i \) and

\[ M_{\mu(i,j),m} = \frac{h_{\mu(i,j)} \left( \lambda^{\mu(i,j)}_{(j)} - \lambda^{\mu(i,j)}_{(i)} \right) \left( e^{\lambda^{\mu(i,j)}_{(j)}} - e^{\lambda^{\mu(i,j)}_{(i)}} \right) - 2(e^{\lambda^{\mu(i,j)}_{(j)}} - e^{\lambda^{\mu(i,j)}_{(i)}})}{\Delta t (\lambda^{\mu(i,j)}_{(j)} - \lambda^{\mu(i,j)}_{(i)}) (e^{\lambda^{\mu(i,j)}_{(j)}} - e^{\lambda^{\mu(i,j)}_{(i)}})^2} \]

with \( m = \mu(i,j) \). Similarly, the exact expressions for the non-zero \((i,m)\)th entry of the coefficient matrix \( L \) is given by

\[ L_{i,j} = \frac{\sum_{\alpha} h_{\alpha(i,j)} \left( e^{-\lambda^{\alpha}_{\text{h}+\text{d}}(j)} - e^{-\lambda^{\alpha}_{\text{h}+\text{d}}(i)} \right)}{h_{\alpha(i,j)} (e^{\lambda^{\alpha}_{h}(j)} - e^{\lambda^{\alpha}_{h}(i)}) (e^{\lambda^{\alpha}_{h}(j)} - e^{\lambda^{\alpha}_{h}(i)})^2} \]  

with \( m = i \) and

\[ L_{i,m} = \frac{e^{-\lambda^{\mu(i,j)}_{(j)}} - \lambda^{\mu(i,j)}_{(j)}}{e^{\lambda^{\mu(i,j)}_{(j)}} - \lambda^{\mu(i,j)}_{(j)}} \]  

with \( m = \mu(i,j) \). The Hospital rule is applied to Eqs.(34) through (37) for the degenerated cases.

(3) Numerical diffusivity and numerical decay coefficient

For steady problems with a 1-D interval, analytical expressions of the numerical diffusivity \( \varepsilon_k \) and the numerical decay coefficients \( c_k \) involved in the CPGFE scheme are derived through the application of a modified equation analysis in each element \(^{21}\).

These numerical coefficients imply that the net diffusivity and the net decay coefficient in the spatial discretization are evaluated as \( \varepsilon + \varepsilon_k \) and \( c + c_k \) in each element, respectively. Throughout this subsection, the subscript specifying the element \( \kappa(i,j) \) is omitted from the variables for the sake of brevity.

Based on the modified equation analysis, the numerical diffusivity \( \varepsilon_k \) attributed to the element \( \Omega_{\kappa(i,j)} \) is expressed as

\[ \frac{\varepsilon_k}{\varepsilon} = \frac{\text{Pe} e^{\text{Pe} + 1}}{2 e^{\text{Pe} - 1} - 1} \]  

The numerical diffusivity in Eq.(38) is identical to those in the literatures\(^{22,23}\). By Eq.(38), the asymptotic behavior of the ratio \( \frac{\varepsilon_k}{\varepsilon} \) for the limit \( \varepsilon \to +0 \) is

\[ \frac{\varepsilon_k}{\varepsilon} = \frac{\text{Pe}}{2 e^{\frac{\text{Pe}}{2}}} \]  

showing that the numerical diffusivity \( \varepsilon_k \) asymptotically approaches that of the conventional fully upwind scheme for the limit \( \varepsilon \to +0 \). In fact, Eq.(39) leads to the order estimate

\[ \varepsilon_k = O(h) \]  

The numerical decay coefficient \( c_k \) attributed to the element \( \Omega_{\kappa(i,j)} \) is expressed as

\[ c_k = 2 \left( 1 + \frac{c_k}{\text{Pe}} \right) \left( e^{\varepsilon - 1}(1 - e^{\varepsilon}) \right) - 1, \]

which is an increasing function of the variables \( \text{Pe} \) and \( \text{Da} \). By Eq.(41), the limit behaviour

\[ c_k \to +0 \]  

holds for \( c \to +0 \). Assuming the absence of the drift \((a \to +0)\) in particular, Eq.(41) degenerates to

\[ c_k = 2 \left( \frac{1}{12} \text{Da} + O(\text{Da}^2) \right) = O(h^2) \]  

which leads to the order estimate

\[ c_k = c \left( \frac{1}{12} \text{Da} + O(\text{Da}^2) \right) = O(h^2) \]  

For a generic drift \( a \), by Eqs.(25), (39), and (41), the asymptotic behavior of the ratio \( \frac{c_k}{c} \) for \( \varepsilon \to +0 \) is

\[ \frac{c_k}{c} = 2 \left( 1 + \frac{|\text{Pe}|}{2} \right) (e^{\varepsilon} - 1) = O(h) \]  

Figs. 3 and 4 provide the plots for the numerical coefficients \( \varepsilon_k \) and \( c_k \). Eqs.(40) and (44) show that
these numerical coefficients are of the order of $O(h)$ and $O(h^2)$ for a generic drift $a$, and $O(h)$ and $O(h^2)$ for the limit $a \to +0$ respectively. The coefficients satisfy $a \gg c$ for typical transport phenomena in surface water bodies, indicating larger contribution of the numerical diffusivity $\varepsilon_k$ on computational accuracy for the problems where the advection is dominant in particular. However, influences of the numerical decay coefficient are considered not to be negligible for the reaction-dominant problems appearing in the other research fields.  

(4) Differences between the two schemes

The present and previous CPGFE schemes share an analogous spatial discretization procedure that implicitly incorporates the IBC Eq.(9) into spatial discretization. These schemes are conforming in the sense that their trial and test functions are of $H^2$-class and do not explicitly add the stabilization terms that the conventional schemes employ. 

Only difference between the present and the previous CPGFE schemes is that the former uses the exponential splines for the trial functions but the latter uses the piecewise linear splines. For a steady problem without source terms, spatially discretized systems with the two schemes are identical; they give an identical nodal solution. However, their numerical solutions differ in each element because of the use of the different trial functions. The piecewise exponential splines exactly resolve sharp transitions of the solutions even with small diffusivity $\varepsilon$, which on the other hand is not true for the piecewise linear splines as shown in APPENDIX A.  

4. STABILITY ANALYSIS

Stability analysis on the previous and present CPGFE schemes is performed for steady and unsteady cases. The main purposes of the stability analysis are the followings. The first purpose is to show unconditional stability of the schemes for steady problems. The second purpose is to show a qualitative difference between the schemes for unsteady problems. The last purpose, related to the second one, is to indicate necessity of the use of the selective lumping algorithm in temporal discretization of the schemes so that the parabolic discrete maximum principle is rigorously satisfied.

(1) Steady case

The steady counterpart of Eq.(32) is

$$ Lu = f. \quad (46) $$

As described in the last sub-section, the previous scheme also leads to the same spatial discretization for the steady case. It is shown that the two schemes are unconditionally stable in space because the matrix $L$ in Eq.(46) is an $M$-matrix, which is a diagonally-dominant square matrix whose inverse is positive definite. Diagonal and non-diagonal entries of an $M$-matrix are positive and negative, respectively. A proof for the unconditional stability has been presented in Yoshioka et al., which is briefly described below for the sake of self-containedness of this paper. Firstly, Eqs.(36) and (37) are rewritten as

$$ L_{ij} = \sum_{j=1}^{n} \sigma_{i,j} \left( a_{i,j} - \varepsilon_{i,j} \frac{\partial W_i}{\partial x} \right) \quad (47) $$

and

$$ L_{ij} = \sigma_{i,j} \varepsilon_{i,j} \frac{\partial W_i}{\partial x} \quad (48) $$

respectively. By Eq.(48) and the elliptic discrete maximum principle to the test function $w_i$, the entry $L_{ij}$ is negative. By Eqs. (30) and (33), Eq.(47) is rewritten in the integral form as

$$ L_{ij} = \sum_{j=1}^{n} \sigma_{i,j} \left( a_{i,j} + \varepsilon_{i,j} \frac{\partial W_i}{\partial x} \right) \quad (49) $$

showing that the entry $L_{ij}$ is positive. The diagonal dominance of the matrix $L$ follows from

$$ \left| L_{ij} \right| - \sum_{j=1}^{N_x} \left| L_{i,j} \right| = L_{ij} + \sum_{j=1}^{n} L_{ij} = 0. \quad (50) $$

The stability presented ensures the elliptic discrete maximum principles of the two schemes. Namely, they can preserve non-negativity of numerical solutions for arbitrary values of the known functions, which cannot be achieved in some of the FE schemes with stabilization techniques. The proposed CPGFE scheme is superior to the other FE schemes in this sense.

(2) Unsteady case

Stability of the CPGFE schemes for the unsteady case in general depends not only on the element size $h$ but also on the time increment $\tau$. The stability here is defined in the Lax-Richtmyer sense, which provides a stronger stability criterion than the Neumann sense. The Lax-Richtmyer stability is
equivalent to the absolute stability, with which numerical solutions computed with the CPGFE schemes comply with the parabolic discrete maximum principle\(^\text{[30]}\). The stability analysis is performed assuming the \(\theta\)-method in time \((0 \leq \theta \leq 1)\). Eq.(32) without the source is then discretized in time as

\[
\Xi u^{n+1} = \Psi u^n
\]

with the square matrices

\[
\Xi = M + \theta rL \quad \text{and} \quad \Psi = M - (1 - \theta) rL.
\]

The scheme is stable if the matrices \(\Xi\) and \(\Psi\) are positive definite matrix and an \(M\)-matrix, respectively. Because \(M\) is a positive definite matrix and \(\Psi\) is an \(M\)-matrix, these conditions are equivalent to

\[
M_{i,j} + \theta r L_{i,j} < 0 \quad \text{and} \quad M_{i,j} - (1 - \theta) r L_{i,j} > 0,
\]

showing that the increment \(\tau\) has a lower bound if \(0 < \theta \leq 1\) and has an upper bound if \(0 \leq \theta < 1\). By Eq.(53), the present CPGFE scheme is unconditionally stable if the condition

\[
1 - \theta^2 > -L^-1_{i,j} M_{i,j} L^-1_{i,j}
\]

with \(0 < \theta < 1\) is satisfied. The condition Eq.(54) is restrictive in applications. For example, consider the case where the domain \(\Omega = (0,1)\), the known functions \(a = \pm 1\), \(c = 0\), and \(f = 0\) are given and \(\Omega\) is uniformly discretized into \(m\) elements. The minimum value of the parameter \(\theta\) that Eq.(54) holds is plotted in Fig. 5 for both of the CPGFE schemes where Pe in the figure is the cell Peclet number defined by

\[
Pe = \frac{a}{|\theta| m e}.
\]

Fig. 5 shows that the parameter \(\theta\) has to be almost equal to 1 in the present scheme, meaning that the fully implicit method (\(\theta = 1\)) seems to be the most reasonable choice, which although may suffer from the lower bound of the time increment by Eq.(54). The minimum value of the parameter \(\theta\) with the previous scheme is smaller than that of the present scheme. The difference between them is more clearly seen for larger absolute value of the cell Peclet number Pe. Application of the selective lumping algorithm of Yoshioka et al.\(^\text{[30]}\) or the full mass lumping is sufficient to overcome this situation, which involves a linear numerical diffusion smaller than that of the fully-lumped counterpart. The CPGFE scheme perturbs the mass matrix \(M\) so that the conditions in Eq.(54) is satisfied. The order of the perturbation introduced is \(O(h)\).

5. ERROR ANALYSIS

(1) Discrete Green’s function

The DGF is a discrete counterpart of the Green\(\hat{\Phi}\) function for continuous problems\(^{[17]}\). The DGF has been introduced in the error analysis of elliptic problems in 1-D intervals\(^{[31, 32]}\). However, no or at least only a few applications have been made for the problems on connected graphs except for the authors’ contribution\(^{[33]}\). This paper therefore provides basic properties of the DGF on connected graphs. The DGF associated with the \(j\)th node in the domain \(\Omega\), denoted by \(G_j\), solves the elliptic problem

\[
A(G_j, \Phi) = \Phi_{x x}
\]

for arbitrary \(\Phi \in H^1_0(\Omega)\). The DGF \(G_j\) is represented as a linear combination of the basis functions in the test space. Unique existence and non-negativity of the DGF \(G_j\) are guaranteed owing to the positive-definiteness of the coefficient matrix \(L^{-1}\). Furthermore, the DGF \(G_j\) has the classical \(C^2\)-regularity in each element because of the element-wise regularity of the basis functions in the test space. Hereafter, the subscript for the coefficients specifying element is omitted for the sake of brevity. The abbreviations

\[
G_j' = \frac{dG_j}{dx} \quad \text{and} \quad G_j'' = \frac{d^2G_j}{dx^2}
\]

are utilized throughout this sub-section. In addition, hereafter \(C\) represents generic positive constants independent of the diffusivity \(\varepsilon\).

A representation formula for the DGF \(G_j\), which is used in this paper in order to prove its bound under the \(L^\infty\)-norm, is derived as follows. According to Eq.(56), the DGF \(G_j\) satisfies

\[
\varepsilon G_j'' - a G_j - c G_j = 0
\]

in each element in the classical sense subject to

\[
\sum_{j=1}^N (\varepsilon G_j' - a G_j) = -\delta_{j,j}.
\]

By Eq.(59), integrating Eq.(58) over the domain \(\Omega\) yields

![Fig. 5 Plots of the minimum values of the parameter \(\theta\).](image-url)
\[ 0 = \int_{\Omega} \left( \varepsilon G_j^* - a G_j' - c G_j \right) dx \]
\[ = \int_{\Omega} \left( \varepsilon G_j^* - a G_j' \right) dx - \int_{\Omega} c G_j dx \]
\[ = -\sum_{k=1}^{N_U} \left( \varepsilon G_j' - a G_j \right)_{U_k} + \sum_{k=1}^{N_D} \left( \varepsilon G_j' - a G_j \right)_{D_k} - \sum_{j=1}^{N_U} \left( \varepsilon G_j' - a G_j \right)_{j} - \int_{\Omega} c G_j dx \]
\[ = -\sum_{k=1}^{N_U} \left( \varepsilon G_j' - a G_j \right)_{U_k} + \sum_{k=1}^{N_D} \left( \varepsilon G_j' - a G_j \right)_{D_k} - \int_{\Omega} c G_j dx + 1 \]  
where the subscripts \( U_k \) and \( D_k \) represent the \( k \)th upstream and downstream vertices that belong to the sets \( \Gamma_U \) and \( \Gamma_D \), respectively. The formula Eq.(60) accordingly reduces to the conventional counterpart for the problems in one reach \(^7\).

In the following, the bound of the DGF \( G_j \) in the \( L^\infty \)-norm is derived for both the Dirichlet-Dirichlet and the Neumann-Dirichlet cases. In the former case, the homogenous Dirichlet condition
\[ G_j = 0 \]  
(61)
is specified at both the upstream-ends and the downstream-ends. On the other hand, in the latter case, the homogeneous Neumann condition
\[ \varepsilon G_j' = 0 \]  
(62)
is specified at the upstream-ends and the homogeneous Dirichlet condition Eq.(61) at the downstream-ends. Eqs.(61) and (62) are a consequence of the fact that the DGF \( G_j \) is a linear combination of the test functions subject to the specified boundary conditions for the unknown \( u \). By Eqs.(60), (61), and (62), the representation formulae for the DGF \( G_j \) in the Dirichlet-Dirichlet case and the Neumann-Dirichlet case are derived as
\[ \sum_{k=1}^{N_U} \left( \varepsilon G_j' \right)_{D_k} = \sum_{k=1}^{N_U} \left( \varepsilon G_j' \right)_{U_k} + \int_{\Omega} c G_j dx - 1 \]  
(63)
and
\[ \sum_{k=1}^{N_D} \left( \varepsilon G_j' \right)_{D_k} = -\sum_{k=1}^{N_U} \left( a G_j \right)_{U_k} + \int_{\Omega} c G_j dx - 1, \]  
(64)
respectively. In both of the cases, the norm \( \| G_j \|_{L^\infty} \) of the DGF \( G_j \) in the domain \( \Omega \) is bounded regardless of the value of the diffusivity \( \varepsilon \). This statement is proven in the following.

a) Dirichlet-Dirichlet case

The bound for the norm \( \| G_j \|_{L^\infty} \) is derived with the proof of contradiction. Firstly, the conditions
\[ G_j' \big|_{U_k} \geq 0 \]  
and
\[ G_j' \big|_{D_k} \leq 0 \]  
are satisfied because of the non-negativity of the DGF \( G_j \) in the domain \( \Omega \) and the elliptic maximum principle. Assuming the condition
\[ \| G_j \|_{L^\infty} > \frac{1}{c_{\min} \left( \Omega \right)} \]  
(66)
with Eqs.(63) and (65) yields
\[ \sum_{k=1}^{N_U} \left( \varepsilon G_j' \right)_{D_k} = \sum_{k=1}^{N_U} \left( \varepsilon G_j' \right)_{U_k} + \int_{\Omega} c G_j dx - 1 \]
\[ \geq \int_{\Omega} c G_j dx - 1, \]  
(67)
\[ > c_{\min} \left( \Omega \right) \frac{1}{c_{\max} \left( \Omega \right)} - 1 = 0 \]
which is a contradiction, yielding the bound
\[ \| G_j \|_{L^\infty} \leq \frac{1}{c_{\min} \left( \Omega \right)}. \]  
(68)

b) Neumann-Dirichlet case

By Eq.(65), Eq.(64) reduces to
\[ 1 + \sum_{k=1}^{N_U} \left( a G_j \right)_{U_k} \geq \int_{\Omega} c G_j dx. \]  
(69)
By the non-negativity of the term \( a G_j \), Eq.(69) is satisfied if
\[ 1 \geq \int_{\Omega} c G_j dx. \]  
(70)
holds. Assuming the condition
\[ \| G_j \|_{L^\infty} \leq \frac{1}{c_{\max} \left( \Omega \right)} \]  
(71)
leads to Eq.(70). Eq.(71) thus serves as a bound for the Neumann-Dirichlet case. Eqs.(68) and (71) show that the DGF \( G_j \) is bounded as
\[ \| G_j \|_{L^\infty} \leq C. \]  
(72)

(2) Pointwise error estimate

An estimate for the pointwise error with the present CPGFE scheme, which does not depend on the value of the diffusivity \( \varepsilon \), is derived in this sub-section. The discretized coefficients and sources are represented with the subscript \( h \). Firstly, by Eq.(56), the equality
\[ (u - u_h)(x_i) = (G_j, u - (G_j, u_h) \]  
\[ = A_h (G_j, u) - A_h (G_j, u_h) + (A(G_j, u) - (G_j, f)) \]  
(73)
\[ = (G_j - f_h) - A_h (G_j, u_h) + (A(G_j, u) - (G_j, f)) \]
\[ = (A - A_h)(G_j, u) - (G_j, f - f_h) \]
holds. Eq.(73) leads to
\[(u - u_h)(x)\]  
\[\leq \left| (A - A_h)(G, u) - (G, f - f_h) \right| \]
\[\leq \left\| A - A_h \right\|_{\infty} \left( \left\| G \right\|_{L^\infty} + \left\| f - f_h \right\|_{L^\infty} \right), \quad (74)\]
\[\leq \left( \left\| a - a_h \right\|_{L^\infty} + \left\| e - c_h \right\|_{L^\infty} \left\| u \right\|_{L^\infty} \right) C\left\| G \right\|_{L^\infty} + \left\| f - f_h \right\|_{L^\infty} C\]

which explicitly relates the regularity conditions of the known functions and their discretization errors with the accuracy of the present CPGFE scheme. Eq.(74) also shows that numerical solutions with the scheme are nodally-exact if the discretized known functions do not contain the errors. This condition is satisfied for the problems with piecewise constant sources. The 1-D counterpart of Eq.(74) has been presented in Miller et al.\(^{(7)}\).

In the following, the error estimates on the coefficient and sources
\[\left\| a - a_h \right\|_{L^\infty}, \left\| e - c_h \right\|_{L^\infty}, \left\| f - f_h \right\|_{L^\infty} \leq Ch \quad (75)\]
are assumed, which are not considered to be so restrictive in applications. The first term in the left hand side of Eq.(75) is more restrictive than the others since it is given as an \(L^\infty\) -error norm, which is stronger than the \(L^1\) -norm. Assuming the error estimates in Eq.(75), Eq.(74) reduces to
\[\left| (u - u_h)(x) \right| \leq C \left( \left\| u \right\|_{L^1} + Ch \right) \quad (76)\]
\[\leq \max \left\{ \left\| u \right\|_{L^1}, C \right\} Ch\]
Because the two norms \(\left\| u \right\|_{L^1}\) and \(\left\| u \right\|_{L^\infty}\) have the bounds that do not depend on the diffusivity \(e\) as shown in APPENDIX B, the pointwise estimate
\[\left| (u - u_h)(x) \right| \leq Ch \quad (77)\]
is derived. Eq.(77) means that the numerical solutions with the present CPGFE scheme satisfies
\[\lim_{h \to 0} \left| (u - u_h)(x) \right| = \lim_{x \to 0} \left| (u - u_h)(x) \right| = 0 \quad (78)\]
showing that it is nodally uniformly-convergent. The previous CPGFE scheme is also nodally uniformly-convergent because the estimate Eq.(77) is also satisfied with the piecewise linear test functions.

(3) Global error estimate
For steady problems, an \(L^\infty\) -error norm between the exact and numerical solutions with the present CPGFE scheme is derived. A key to derive the error estimate is the use of the weak maximum principle\(^{(34)}\) that the exact solution \(u\) complies with. Under the assumed regularity conditions on the coefficients and the source, it is shown that the solution \(u\) has \(C^2\) -regularity in each reach. The maximum principle of Stynes and O’Riordan\(^{(34)}\) is then satisfied. For the steady case, applying the maximum principle to Eq.(18) yields
\[\left| (u - u_h)(x) \right| \leq r \leq Ch \quad (79)\]
in the domain \(\Omega\) with a barrier function \(r = r(x)\), a positive function with piecewise \(C^2\) -regularity in each element. Since the position \(x\) in Eq.(79) is arbitrary in the domain \(\Omega\), Eq.(79) consequently leads to the \(L^\infty\) -error estimate present CPGFE scheme as
\[\left\| u - u_h \right\|_{L^\infty} \leq Ch \quad (80)\]
showing that Eq.(4) is satisfied and the scheme is uniformly convergent with the \(L^\infty\) -error norm. The error estimates presented in this sub-section have not been reported in the conventional researches. The results therefore guarantee potential robustness of the present CPGFE scheme and indicate usefulness of the DGF for the error analysis.

(4) Some remarks
For unsteady problems with the fully lumped mass matrix, just following Stynes and O’Riordan\(^{(34)}\), it is shown that both the present and previous CPGFE schemes satisfy the pointwise estimate
\[\left| (u - u_h)(x) \right| \leq C \left( h + \tau \right) \quad (81)\]
at each time step. In addition, the present scheme satisfies the estimate
\[\left\| u - u_h \right\|_{L^\infty} \leq C \left( h + \tau \right) \quad (82)\]
at each time step if it is fully lumped or lumped with the selective lumping algorithm\(^{(19)}\). Note that Eq.(82) means the uniform convergence of the scheme at each time step, but not the uniform convergence in both space and time.

Remarks on the problems with the variable diffusivity \(e = e(x)\) with a constant \(e_h\) and some positive function \(\eta\) is also mentioned. Such a situation is common in practical problems of solute transport phenomena in steady water flows. For a variable \(\eta \in H^1(\Omega)\), Eq.(74) is replaced by
\[\left| (u - u_h)(x) \right| \leq \left( \left\| 1 - \tilde{a}_h \right\|_{L^\infty} + \left\| \eta - c_h \right\|_{L^\infty} \right) \left\| u \right\|_{L^1} + \left\| f - f_h \right\|_{L^\infty} \quad (83)\]
with the auxiliary variable \(\tilde{a}\) defined by
\[\tilde{a} = a + \frac{\tilde{c}}{\tau} \quad (84)\]
The error estimates Eqs(77), and (80) are then satisfied for arbitrary \(e_h\).

6. CONCLUSIONS
A fitting technique-based CPGFE scheme with the
exponential splines for the ADDEs was proposed and its theoretical stability and error analysis was performed. The obtained results in this paper are summarized as follows.

Firstly, the present and previous CPGFE schemes are unconditionally stable in space and are pointwise uniformly-convergent. Both of them thus handle steady ADDEs with a small diffusivity. Secondly, the concept of DGF serves as a useful mathematical tool relating accuracy of the numerical schemes for the ADDEs on connected graphs with the regularity conditions on the known functions. Thirdly, the present scheme is uniformly-convergent under the $L^\infty$-error norm while the previous one is not, indicating the critical importance of the choice of the trial functions for advection-dominant and reaction-dominant ADDEs; however, the exponential splines for the trial functions results in the narrower stability condition for unsteady problems. It was also mentioned that this drawback could be overcome by the use of a lumping algorithm.

The high stability and the uniformly-convergent property of the present CPGFE scheme guarantee its potential robustness for problems with significantly large advection and/or decay terms as frequently encountered in scientific and engineering practical applications. This is mainly because that numerical errors in spatial and temporal discretization procedures in the scheme are well-behaved in the sense of Eqs.(80) and (82). The results obtained in this paper are therefore expected to contribute to developing a robust numerical tool for a variety of practical problems in civil and environmental engineering that ultimately reduce to solving the ADDEs on connected graphs, such as analytical assessment of nutrient dynamics in irrigation and drainage systems. This research topic is currently under investigation where the flow field is computed with the shallow water equations and the nutrient dynamics with the ADDEs. Applicability of the CPGFE schemes to nonlinear problems will also be investigated in future researches, focusing on the problems related to hydrodynamics in particular. Finally, the present numerical technique that consistently incorporates the IBCs into spatial discretization is also applicable to the other FE schemes. Comparison of numerical performances among the conventional and the CPGFE schemes utilizing the numerical technique is another important research topic and will also be carried out in future researches in order to clarify their qualitative and quantitative differences and to develop better numerical schemes.

ACKNOWLEDGMENT: This research is supported by the JSPS under grant No. 25・2731.

APPENDIX A Counter examples

This appendix presents two examples that the previous scheme fails to satisfy Eq.(80). The domain $\Omega$ is set as the interval (0,1). Assume that the domain $\Omega$ is uniformly discretized into $m$ elements with the length of $h=m^{-1}$ where the $x$-abscissa of the $i$th node is given by $x_i=ih$ ($0 \leq i \leq m$).

The first example is the Dirichlet problem 33)
\[
\frac{1}{p} \frac{d^2 u}{dx^2} - \frac{du}{dx} = 0 \quad (85)
\]
with the coefficient $p=e^{-1}$ subject to the boundary conditions $u(0) = 0$ and $u(1) = 1$. The exact solution to Eq.(85) is given by
\[
u = \frac{e^{p} - 1}{e^{p} - 1} \quad (86)
\]
Assuming that the previous CPGFE scheme is applied to this problem, the $L^\infty$-error norm between the exact solution $u$ and the numerical solution $u_h$ is
\[
\|u-u_h\|_{L^\infty} = \left| \frac{e^p}{e^{p} - 1}\left[1 + \frac{1}{ph} \ln \left(\frac{1 - e^{-ph}}{ph}\right)\right] \right|, \quad (87)
\]
which leads to the estimate
\[
\lim_{h \to 0} \|u-u_h\|_{L^\infty} = 0 \quad (90)
\]
for fixed $p$ but to
\[
\lim_{e \to a^{+}} \|u-u_h\|_{L^\infty} = \lim_{e \to a^{-}} \|u-u_h\|_{L^\infty} = 1 \quad (91)
\]
for fixed $h$.

The second example is the Dirichlet problem
\[
\frac{1}{p^2} \frac{d^2 u}{dx^2} - u = 0 \quad (92)
\]
with the coefficient $p=e^{-1}$ subject to the boundary conditions $u(0) = 0$ and $u(1) = 1$. The exact solution to Eq.(92) is given by
\[
\nu = \frac{e^{p} - e^{-p}}{e^{p} - e^{-p}} \quad (93)
\]
The $L^\infty$-error norm between the exact solution $u$ and the numerical solution $u_h$ is
\[
\|u-u_h\|_{L^\infty} = \left| \frac{e^p}{e^{p} - 1}\left[1 + \frac{1}{ph} \ln \left(\frac{1 - e^{-ph}}{ph}\right)\right] \right|, \quad (88)
\]
where
\[
\beta = \frac{1}{p} \left[ \frac{1}{2} (\gamma + \sqrt{\gamma^2 - 4}) \right] \quad (95)
\]
with

\[ \gamma = \frac{1}{\rho h} \left[ e^{\nu (1-h)} (e^{ph} - 1) + e^{-\nu (1-h)} (1 - e^{-ph}) \right] \geq 2 . \]  

(96)

Substituting Eq.(96) into Eq.(94) yields

\[ \| u - u_0 \|_r = \left( \beta - x_{n+1} \right) p_{y} \left( e^{ph} - e^{-ph} \right) + e^{p_{n+1} - e^{-p_{n+1}}} \]

(97)

again leading to Eqs.(90) and (91).

Miller et al.17 reported that the conventional fully upward scheme does not comply with Eq.(80) as well. The cause of these failures is the use of the linear trial functions with which numerical schemes do not resolve the sharp transitions of the solutions. The above examples show that the finite element scheme with the polynomial test functions40 also fails in this type of solutions since it uses the linear splines for the trial functions. The results indicate that the addition of an excessive numerical diffusivity does not always lead to the uniform-convergence. The situation is reported to be worse in multi-dimensional cases41. Linear and non-linear multi-dimensional problems that the fully upward scheme fails are presented in Brandt and Yavneh12. The disadvantages of the numerical schemes presented in this appendix can be improved if the fitting mesh technique13 is employed. However, this technique necessitates a priori knowledge on the locations of sharp transitions in the solutions, which cannot be available in real applications. Adaptive remeshing techniques appropriately considering regularity conditions on the solutions will be a possible way to overcome this problem.

**APPENDIX B  Bounds on the solution \( u \)**

This appendix proves that the norms \( \| u \|_{L^1} \) and \( \| u \|_{L^r} \) have bounds independent of the diffusivity \( \varepsilon \) for steady problems. The procedure presented in this appendix is an extended version of Roos et al.44.

Firstly, the estimate

\[ \| u \|_{L^r} \leq C \]  

(98)

follows from the weak maximum principle in each reach, owing to the boundedness and coercivity of the bilinear form \( A_\varepsilon \). Secondly, denote a reach in the domain \( \Omega \) by \( R \). The length of the reach is denoted by \( L \). The two vertices that bound the reach \( R \) are denoted by \( P_0 \) and \( P_1 \). Without the loss of generality, the values of the \( x \) abscissa at these nodes are specified as 0 and \( L \) so that the reach is identified with the 1-D interval \( (0, L) \). The values of the solution \( u \) at the vertices \( x = 0 \) and \( x = L \) are denoted by \( u_0 \) and \( u_L \), respectively. Following the 1-D analogue44, the solution \( u \) in the reach \( R \) satisfies

\[ u = u_L - \int_0^L \psi(t)\,dt - \frac{d}{dx} \int_{x=R-0}^R \eta(R,t)\,dt \]  

with

\[ \psi(t) = \int_0^L \frac{1}{\varepsilon} (f - cu) \eta(t,x)\,dt , \]  

(100)

\[ \eta(t,x) = \exp \left(-\frac{1}{\varepsilon} (a(t) - \hat{a}(x)) \right) > 0 , \]  

(101)

and

\[ \hat{a}(x) = \int_0^L a(t)\,dt . \]  

(102)

By Eqs.(10) and (101), the inequality

\[ \exp \left(-\frac{a_{\text{max}}}{\varepsilon} (t - x) \right) \leq \eta(t,x) \leq \exp \left(-\frac{a_{\text{min}}}{\varepsilon} (t - x) \right) \]  

(103)

follows, which leads to the estimates

\[ \| u \|_{L^r} \leq \frac{1}{a_{\text{max}}} \| f - cu \|_{L^r} \leq C \]  

(104)

and

\[ \left\| \frac{d}{dx} \right\|_{L^1} \leq C \]  

(105)

By Eq.(99), the equality

\[ \frac{d}{dx} \psi(x) + \frac{d}{dx} \eta(R,x) = \eta(R,x) \]  

(106)

holds, leading to the estimate

\[ \int_0^L \frac{d}{dx} \psi(x)\,dx + \int_0^L \frac{d}{dx} \eta(R,x)\,dx \leq C + C \int_0^L \exp \left(-\frac{a_{\text{max}}}{\varepsilon} (t - x) \right)\,dx = C \]  

(107)

Eq.(107) follows the estimate

\[ \| u \|_{L^1} \leq C . \]  

(108)

Eqs. (98) and (108) show that the norms \( \| u \|_{L^1} \) and \( \| u \|_{L^r} \) are bounded independent of the diffusivity \( \vareference


(Received June 20, 2014)