Application of continuumnization and PDS-FEM for the analysis of wave propagation in brick structures

Sumet SUPPRASERT\(^1\), Lalith WIJERATHNE\(^2\), Jian CHEN\(^3\), Muneo HORI\(^4\) and Tsuyoshi ICHIMURA\(^5\)

\(^1\)PhD. Candidate, Department of Civil Engineering, University of Tokyo (Bunkyo, Tokyo 113-0032)
\(^2\)Associate Professor, Earthquake Research Institute, University of Tokyo (Bunkyo, Tokyo 113-0032)
\(^3\)Dr. of Eng. Research Associate, RIKEN AICS (7-1-26 Minatoshima-Minami, Kobe 650-0047)
\(^4\)Professor, Earthquake Research Institute, University of Tokyo (Bunkyo, Tokyo 113-0032)
\(^5\)Associate Professor, Earthquake Research Institute, University of Tokyo (Bunkyo, Tokyo 113-0032)

We developed an equivalent continuum form for brick structures based on continuumnization proposed by Hori et al.\(^1\). This allows one to analytically study the characteristic properties of masonry brick walls, and apply numerical techniques used in continuum mechanics to simulate brick structures. Further, the continuum form opens up the possibility of developing simplified models like shells or beams, which will be convenient in designing brick structures. Based on the continuum form, we study how the wave characteristics, like wave speeds, etc., change according to brick arrangement and material properties. Further, we develop a PDS-FEM model\(^2\) for simulating brick structures and verify the developed model comparing numerically obtained wave speeds with that of analytical predictions.

**Key Words:** brick structures, continuumnization, dynamic characteristics, rotational-waves, PDS-FEM

1. Introduction

Although there is a number of approaches to model brick structures, most of those are not useful in brick structural designs. For design proposes, it will be convenient if there are simplified models like beams or shells for brick structures. Further, in designing brick structures, which are subjected to vibrations and minor earthquake loadings, it is desired to know how the wave characteristics change according to brick size, arrangement, material properties, etc. According to authors knowledge, none of these designs related needs are addressed by existing methods for modeling brick structures.

Often brick structures are modeled as rigid-body-spring systems, assuming that each brick-unit is a rigid block and mortar is infinitesimal springs\(^3,4,5,6,7,8\). Such discrete models are convenient for simulating brick structures, including damage and nonlinear mortar behavior. Another widely used approach is using homogenization techniques to find average material properties\(^9,10\), like Young’s modulus, required for simulating brick structures using Finite Element Method (FEM). However, the accuracy of this method is low. Another alternative approach is to approximate the discrete vector field formed by the translations and rotations of bricks as a continuous vector field using Taylor series expansion\(^11,12\). Based on this approach, Stefanou et al.\(^11\) have obtained relations between wave frequency and wave number. Obviously, rigid-body-spring systems cannot be used to develop simplified models like beams or shells. Though dimension reduced models can be developed based on some homogenization approaches, these techniques have poor accuracy.

One can easily address the above mentioned design related requirements, like developing equivalent beam or shell models, and prediction of wave characteristics, if an equivalent continuum form for brick structures is developed. Motivated by this, the objective of this work is to develop an equivalent continuum form for brick structures based on continuumnization proposed by Hori et al.\(^1\). Continuumnization allows one to pose a set of partial differential equations based on the discrete governing equations of an interacting regularly packed particle systems. Based on the equivalent continuum form of governing equations, we explore the wave characteristics of certain brick arrangement. Further, we develop a Particle Discretization Scheme Finite Element Method (PDS-FEM)\(^2\) model for simulating brick structures.

The rest of the paper is organized as follows. The section 2 presents application of continuumnization for brick structures. Further, we explore wave characteristics for a simple brick arrangement, and estimate material properties for wave propagation simulations.
The section 3 presents the development of PDS-FEM model based on the continuumization, while section 4 presents some numerical results to verify the developed PDS-FEM model.

2. Continuumization of a brick wall

We idealize a single layer brick wall as a network of rigid rectangular blocks, with the size of $2a_1 \times 2a_2 \times 2a_3$, connected with linear elastic springs as shown in Fig.1. The springs represent elasticity of both the bricks and mortar layers, while the domain of each rigid blocks includes a portion of mortar layers so that space is perfectly tessellated.

Let $u^\mu$ and $\theta^\mu$ denote three dimensional translation and rotation of $\mu^{th}$ block. The two neighboring blocks in $\gamma^{th}$-direction are denoted by $\gamma \pm$. Let unit vectors, $n^\gamma$, $t^\gamma$, and $s^\gamma$, denote orthonormal coordinate system on contact surface with any neighboring block as shown in Fig.2. $r^\gamma +$ denotes the relative position of the centroid of the contact area with the neighbor $\gamma +$. The contact is of the size $2b_1^\gamma \times 2b_2^\gamma$. Let and $h$ denote normal and tangential spring constants, respectively. For given set of infinitesimal displacements and rotations, the relative displacement at the a point $(x_t,x_s)$ on the contact surface with the neighbor $\gamma +$ can be expressed as

$$L^\gamma + = (u^\gamma + - u^\mu) - (\theta^\gamma + + \theta^\mu) \times r^\gamma + + (\theta^\gamma + - \theta^\mu) \times (x_t t^\gamma + x_s s^\gamma).$$

Here, the origin of $(x_t,x_s)$ is located at the centroid of the contact area. The corresponding elastic energy stored in springs is

$$V^\mu \gamma + = \frac{1}{2} \int \int k(n^\gamma \cdot L^\mu \gamma +)^2$$

$$+ h \left((t^\gamma \cdot L^\mu \gamma +)^2 + (s^\gamma \cdot L^\mu \gamma +)^2\right) \, dx_t dx_s$$

Similarly, the elastic energy $V^\mu \gamma -$ can be obtained using $-n^\gamma$, $-t^\gamma$, $s^\gamma$, $-r^\gamma$. The Lagrangian for the whole discrete system is

$$\mathcal{L} = \sum_\mu \left( \frac{1}{2} m \ddot{u}^\mu - \mathbf{K}^\mu \gamma + (u^\gamma + - 2u^\mu + u^\gamma -) - \mathbf{K}^\mu \gamma - (\theta^\gamma + - \theta^\gamma -) \right)$$

$$+ \mathbf{I} \cdot \dot{\theta}^\mu = \sum_\gamma \left( \frac{1}{2} \dot{\mathbf{K}}^\mu \gamma + (u^\gamma + - u^\gamma -) - \mathbf{K}^\mu \gamma + (\theta^\gamma + + 2\theta^\mu + \theta^\gamma -)$$

$$+ \mathbf{K}^\mu \gamma - (\theta^\gamma - - 2\theta^\mu + \theta^\gamma -) \right),$$

where matrices $\mathbf{K}^\mu \gamma +$, $\dot{\mathbf{K}}^\mu \gamma +$, $\mathbf{K}^\mu \gamma -$, and $\dot{\mathbf{K}}^\mu \gamma -$ define different modes of interaction between neighboring blocks and are expressed by the following equations

$$\mathbf{K}^\mu \gamma + = 4b_1^\gamma b_2^\gamma \left( k(n^\gamma \otimes n^\gamma + h t^\gamma \otimes t^\gamma + h s^\gamma \otimes s^\gamma) \right)$$

$$\mathbf{K}^\mu \gamma - = 4b_1^\gamma b_2^\gamma \left( k(n^\gamma \otimes (r^\gamma \times n^\gamma) + h t^\gamma \otimes (r^\gamma \times t^\gamma)$$

$$+ h s^\gamma \otimes (r^\gamma \times s^\gamma)) \right)$$

$$\dot{\mathbf{K}}^\mu \gamma + = 4b_1^\gamma b_2^\gamma \left( k((r^\gamma \times n^\gamma) \otimes (r^\gamma \times n^\gamma)$$

$$+ h(r^\gamma \times t^\gamma) \otimes (r^\gamma \times t^\gamma) + h(r^\gamma \times s^\gamma) \otimes (r^\gamma \times s^\gamma)) \right)$$

$$\dot{\mathbf{K}}^\mu \gamma - = \frac{4}{3} \left( k(b_1^3 + b_2^3) n^\gamma \otimes n^\gamma + kb_1 b_2^2 t^\gamma \otimes t^\gamma$$

$$+ k(b_1 b_2^3 + b_2 b_1^3) s^\gamma \otimes s^\gamma$$

$$= + k b_1 b_2^3 s^\gamma \otimes s^\gamma) \right)$$

It is $\dot{\mathbf{K}}^\mu \gamma +$ which couples translations and rotations. The above obtained governing equations for the discrete system can be used to simulate a brick structure as a mass spring system, in which the $\mathbf{K}^\mu \gamma +$, $\dot{\mathbf{K}}^\mu \gamma +$, $\mathbf{K}^\mu \gamma -$ and $\dot{\mathbf{K}}^\mu \gamma -$ define spring constants for the interactions among different degrees of freedoms.
(1) Continuumnization

Often in engineering applications, it is important to know how the dynamic characteristics like wave speeds vary with brick size, brick arrangement and material properties. The discrete governing equation, Eq.(4), is inconvenient for such application. Surely, one can do a large number of simulations varying the required parameters and obtain some relations among the required parameters. Instead of such brute force approach, we can use the continuumnization proposed by Hori et al.\(^1\) to study system characteristics analytically. Continuumnization allows one to pose a set of partial differential equations based on the discrete governing equations. This continuum form has several advantages: can be used to analytically study the above mentioned wave characteristics of the system; make it possible to apply FEM or other numerical methods in continuum mechanics for simulating brick structure; make it possible to develop dimension reduced models, like beam and shells in structural mechanics, for brick structure analysis; etc.

Following continuumnization, we assume that there exists at least twice differentiable vector fields \(u(x,t)\) and \(\theta(x,t)\), which can be used approximate the discrete \(u^\gamma\) and \(\theta^\gamma\) for a sufficiently large wavelengths. Using Taylor series, we approximate the motions of neighbor \(u^{\gamma\pm}\) as \(u(x\pm r,t)\) and express the relative translation as \(u^{\gamma\pm} - u^\mu \approx \pm 2r^\gamma \cdot \nabla u\). This allows us to approximate relative motion as difference operators as

\[
\begin{align*}
    u^{\gamma+} - 2u^\mu + u^{\gamma-} & \approx 4r^\gamma \cdot \nabla (r^\gamma \cdot \nabla u) \\
    u^{\gamma+} - u^{\gamma-} & \approx 4r^\gamma \cdot \nabla u \\
    \theta^{\gamma+} + 2\theta^\mu + \theta^{\gamma-} & \approx 4\theta + 4r^\gamma \cdot \nabla (r^\gamma \cdot \nabla \theta).
\end{align*}
\]

Substituting these approximations to Eq.(4) we can obtain the following continuumnized equations of motion

\[
\begin{align*}
    \frac{m}{V_b} \ddot{u} & = \nabla \cdot (c \cdot \nabla u) - q \cdot \nabla \theta \\
    \frac{1}{V_b} \ddot{\theta} & = q^T \cdot \nabla u - d \cdot \dot{\theta} + \nabla \cdot (v \cdot \nabla \theta),
\end{align*}
\]

where \(V_b\) is the volume of a block. The following constants \(c, q, d,\) and \(v\) are 4th, 3rd, 2nd, and 4th-order tensors which include material and geometric (i.e. block geometry and packing) properties.

\[
c = \sum_\gamma \frac{16b^4_{\gamma}}{V_b} \left( k r^\gamma \otimes n^\gamma \otimes r^\gamma \otimes n^\gamma + h r^\gamma \otimes t^\gamma \otimes r^\gamma \otimes t^\gamma \\
    + h r^\gamma \otimes s^\gamma \otimes r^\gamma \otimes s^\gamma \right)
\]

\[
q = \sum_\gamma \frac{16b^4_{\gamma}}{V_b} \left\{ k n^\gamma \otimes r^\gamma \otimes (r^\gamma \times n^\gamma) + h t^\gamma \otimes r^\gamma \otimes (r^\gamma \times t^\gamma) \\
    + h s^\gamma \otimes r^\gamma \otimes (r^\gamma \times s^\gamma) \right\}
\]

\[
d = \sum_\gamma \frac{16b^4_{\gamma}}{V_b} \left\{ k (r^\gamma \times n^\gamma) \otimes (r^\gamma \times n^\gamma) \\
    + h (r^\gamma \times t^\gamma) \otimes (r^\gamma \times t^\gamma) + h (r^\gamma \times s^\gamma) \otimes (r^\gamma \times s^\gamma) \right\}
\]

\[
v = \sum_\gamma \frac{16}{V_b} \left\{ h \left( \frac{b^3_{\gamma} \gamma^3}{3} + \frac{b^3_{\gamma} \gamma^3}{3} \right) r^\gamma \otimes n^\gamma \otimes r^\gamma \otimes n^\gamma \\
    + \frac{k b^3_{\gamma} \gamma^3}{3} r^\gamma \otimes t^\gamma \otimes r^\gamma \otimes t^\gamma + \frac{k b^3_{\gamma} \gamma^3}{3} r^\gamma \otimes s^\gamma \otimes r^\gamma \otimes s^\gamma \right\}
\]

\[
= \frac{16b^4_{\gamma}}{V_b} \left\{ k r^\gamma \otimes (r^\gamma \times n^\gamma) \otimes r^\gamma \otimes (r^\gamma \times n^\gamma) \\
    + h r^\gamma \otimes (r^\gamma \times t^\gamma) \otimes r^\gamma \otimes (r^\gamma \times t^\gamma) \\
    + h r^\gamma \otimes (r^\gamma \times s^\gamma) \otimes r^\gamma \otimes (r^\gamma \times s^\gamma) \right\}.
\]

(2) The estimation of wave velocities

An advantage of Eq.(6) is that it makes it possible to analytically study the dynamic characteristics of the approximated discrete system. As an example, consider the two dimensional single layered brick wall shown in Fig.3. For this given packing, we can evaluate the four tensors, \(c, q, d,\) and \(v,\) and obtain the equivalent continuum form of governing equation (i.e. Eq.(6)). Taking the Fourier transform of the resulting equations, with the kernel of \(e^{i(\xi x - \omega t)},\) and solving the resulting characteristic equations, the relations between the wave frequencies and wave number can be obtained; \(\xi = \xi \{ \cos \theta, \sin \theta \} \) where \(\theta_\xi\) denotes the direction of the propagation of waves. Specifically, the relations between frequencies and wave numbers of pressure, shear and rotational waves due to in-plane deformation are obtained. With these relations, wave phase velocities, \(\omega/\xi,\) and corresponding modes can be obtained. Since the system is anisotropic, the wave velocities depend on the direction of the propagating wave. For example, the wave velocities of primary wave (p-wave) and secondary wave (s-wave) can be expressed in Table 1 and 2, where \(\zeta = a_2/a_1\) and \(n = h/k.\) Note that the finite value of rotational wave velocity cannot be obtained since \(\omega\) is non-zero when \(\xi\) closes to zero. Also, wave
Table 1: Wave velocities and corresponding modes \( \{ u_1, u_2, \theta_3 \} \) for \( \theta_\xi = 90^\circ \).

<table>
<thead>
<tr>
<th>wave</th>
<th>phase velocity</th>
<th>mode shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>( \sqrt{\frac{2k_{a_2}}{p}} )</td>
<td>{0,1,0}</td>
</tr>
<tr>
<td>s</td>
<td>( \sqrt{\frac{2k_{a_2}(1+4\eta \xi)}{p(1+4\eta \xi+4\eta \xi^2)}} ) ( \left{ 1,0,-\frac{4\eta \xi^2\xi+i}{1+4\eta \xi+4\eta \xi^2} \right} )</td>
<td></td>
</tr>
<tr>
<td>r</td>
<td>-</td>
<td>{0,0,1}</td>
</tr>
</tbody>
</table>

Table 2: Wave velocities and corresponding modes \( \{ u_1, u_2, \theta_3 \} \) for \( \theta_\xi = 0^\circ \).

<table>
<thead>
<tr>
<th>wave</th>
<th>phase velocity</th>
<th>mode shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>( \sqrt{\frac{2k_{a_2}}{2\rho \xi}} )</td>
<td>{1,0,0}</td>
</tr>
<tr>
<td>s</td>
<td>( \sqrt{\frac{2k_{a_2}(1+4\eta \xi)}{p(1+4\eta \xi+4\eta \xi^2)}} ) ( \left{ 0,1,-\frac{(1+4\eta \xi)\xi}{1+4\eta \xi+4\eta \xi^2} \right} )</td>
<td></td>
</tr>
<tr>
<td>r</td>
<td>-</td>
<td>{0,0,1}</td>
</tr>
</tbody>
</table>

modes corresponding to \( \{ u_1, u_2, \theta_3 \} \) including rotational wave (r-wave) are observed.

Fig. 3: A single layered 2 dimensional brick arrangement.

(3) The estimation of spring constants based on experiment

The above obtained wave speed predictions can be used to accurately estimate the normal and tangential spring constants, \( k \) and \( h \), for the applications in wave propagation studies. Though the equivalent \( k \) and \( h \) can be obtained by experiments like compression test or shear test \((13),(14),(15)\), those may not be suitable for studying wave propagation phenomenon in brick structures. That is because, the \( k \) and \( h \) obtained from static experiments can be different from those obtained from wave speeds; this is often the case with materials like mortar, concrete, etc. Based on the above wave speed predictions, an accurate estimation of \( k \) and \( h \) for any given regular brick packing can be made with a several experimental measurements of wave speeds \((16),(17),(18)\).

For demonstration purposes, let’s consider the wave speed measured by Schullerl et al. \((19)\). Schullerl et al. have used a rectangular brick wall, and measured the wave propagation speeds in different directions. They have used a horizontal input on the left edge of a brick wall and measured the arrival time at the right side edge at different locations. From these observations, they have estimated the wave speeds for different propagation directions. Fig. 4 shows their observations; \( \theta_\xi \) denotes the direction of propagation as shown in Fig. 3. The brick size is about \( 250mm \times 33mm \times 63mm \) and the density is assumed to be \( 1850kg/m^3 \). Based their experimental results and our analytical predictions of wave speeds, we estimated the spring constants \( k \) and \( h \) to be \( 15.5N/mm^3 \) and \( 1.16N/m^3 \). Only the experimental results at \( \theta_\xi = 0^\circ \) and the highest angle are used for this estimation. As seen in Fig. 4, our analytically predicted wave speeds are in a good agreement with the observations. There is a mismatch at the two points between \( \theta_\xi = 30^\circ \) to \( 50^\circ \) range. This is probably due to the measurement errors; according to the simulations in the next section, p-wave amplitudes at these angles are quite weak.

Fig. 4: Comparison of the experimentally obtained wave speeds by Schullerl et al. \((19)\), and predicted wave speeds based on the estimated \( k \) and \( h \).

3. Application of PDS-FEM for brick walls

One advantage of the developed continuum form is that it allows us to use numerical techniques like Finite Element Method (FEM) to simulate brick structures. We developed a Particle Discretization Scheme Finite Element Method (PDS-FEM) extension for simulating brick structures based on the continuumization presented in the previous section. We chose PDS-FEM due to two reasons: the particle nature of the PDS-FEM allows to consider brick arrange-
(1) Particle Discretization Scheme

Particle Discretization Scheme is unique in the sense that it uses characteristic functions of conjugate tessellations, \{\Phi^\alpha\} and \{\Psi^\beta\}, for approximating the function and its derivatives. Originally, PDS-FEM uses characteristic functions of Voronoi and Delaunay tessellations. However, in this study, we use the bricks themselves as the tessellation elements \Phi^\alpha; to be exact, each \Phi^\alpha includes half thickness of the cement layer along the perimeter of the brick (see Fig. 5(a)). The conjugate tessellation \{\Psi^\beta\} is formed with the triangles connecting the centroids of neighboring blocks; see Fig. 5(b). Characteristic functions of \Phi^\alpha, denoted by \phi^\alpha(x), is defined as

\[
\phi^\alpha(x) = \begin{cases} 
1 & \text{if } x \in \Phi^\alpha \\
0 & \text{if } x \notin \Phi^\alpha 
\end{cases}
\]

Replacing \phi^\alpha and \Phi^\alpha with \psi^\beta and \Psi^\beta, respectively, we can obtain the definition of the characteristic function \psi^\beta. According to PDS, \(u\) and \(\nabla u\) are approximated as

\[
u^\alpha = \int_{\Phi^\alpha} u(x,t) \, ds. \quad (9)
\]

The use of characteristic functions \{\phi^\alpha\} introduces discontinuities to the approximation, \(u^\alpha\). However, PDS defines bounded derivatives of \(u^\alpha\) using the characteristic functions of conjugate tessellation to approximate the derivatives. Minimizing the error \(E^\alpha = \int (u^\alpha - u^\alpha)^2 \, ds\), PDS defines the unknown coefficients \(u^\alpha\)'s of \(g^\alpha_j(x)\) as

\[
u^\alpha = \sum_{\alpha} A^\alpha_{ij} u^\alpha, \quad (10)
\]

where \(\alpha \in \{\alpha | \Psi^\beta \cap \Phi^\alpha = \emptyset\}\) denotes the \(\Phi^\alpha\)'s occupying the same space of \(\Psi^\beta\). Further

\[
A^\alpha_{ij} = \frac{2\alpha_3}{\psi^\beta} \int_{\partial \Psi^\beta \cap \partial \Phi^\alpha} n^\alpha \, dl. \quad (11)
\]

\(\psi^\beta\) and \(\Phi^\alpha\) are the areas of the element \(\Psi^\beta\) and \(\Phi^\alpha\). Analytical expression for \(A^\alpha_{ij}\) are given in the appendix.

(2) Formulation of PDS-FEM for brick structure

Based on the following Lagrangian, \(J\), we implemented a PDS-FEM code to numerically study the continuumized block spring system. It is straightforward to show that \(\delta \int J \, dt = 0\) produces the Eq. (6).

\[
J = \frac{1}{2} \int m \ddot{u} \cdot \dot{u} + \theta \cdot I \cdot \dot{\theta} \, ds - \frac{1}{2} \int \nabla u : \nabla u - 2 \theta \cdot q : \nabla u + \theta \cdot d \cdot \dot{\theta} + \nabla \theta : \nabla \theta \, ds \quad (12)
\]

The unknown displacement fields \(u\) and \(\theta\) are approximated using the tessellation, \(\Phi^\alpha\), as

\[
u_i^\alpha \approx \sum_{\alpha} u^\alpha_i \phi^\alpha \quad (13)
\]

\[
\theta_i \approx \sum_{\alpha} \theta^\alpha_i \phi^\alpha. \quad (14)
\]

The derivative of \(u_{i,j}^\alpha\) and \(\theta_{i,j}\) approximated using \{\Phi^\alpha\} as \(u_{i,j}^\alpha \approx \sum_{\beta} u_{i,j}^\beta \psi^\beta\) and \(\theta_{i,j} \approx \sum_{\beta} \theta_{i,j}^\beta \psi^\beta\). Based on the Eq.(10), the unknown coefficients can be expressed as

\[
u_{ij}^\beta = \sum_{\alpha'} A_{ij}^{\beta \alpha'} u_{i}^{\alpha'} \quad (15)
\]

\[
\theta_{ij} = \sum_{\alpha'} A_{ij}^{\beta \alpha'} \theta_{i}^{\alpha'} \quad (16)
\]

Substituting Eq.(7), Eq.(14), Eq.(15), and Eq.(16) to \(J\), we can express \(J\) as

\[
L_{137}
\]
\[ J^d = \frac{1}{2} \sum_{\alpha} \left( m^\alpha \dot{u}_i^\alpha \dot{u}_i^\alpha + I^\alpha_j \dot{\theta}_i^\alpha \dot{\theta}_j^\alpha \right) \]
\[ - \sum_{\beta} \left( \frac{1}{2} u_{ij}^\alpha \kappa_{\beta \alpha \alpha'} u_i^\alpha' \theta_j^\alpha + \frac{1}{2} \theta_j^\alpha \kappa_{\beta \alpha \alpha'} u_i^\alpha' \theta_i^\alpha' \right) + \frac{1}{2} \theta_j^\alpha \kappa_{\beta \alpha \alpha'} \theta_i^\alpha' + \frac{1}{2} \theta_j^\alpha \kappa_{\beta \alpha \alpha'} \theta_i^\alpha' \right), \]  
(17)

where \( m^\alpha = \phi^\alpha \rho \) and

\[
K_{ij}^{\beta \alpha \alpha'} = \int_{s} A_{i}^{\beta \alpha} \ell_{ijkl} A_{k}^{\beta \alpha'} ds \\
\tilde{K}_{ij}^{\beta \alpha \alpha'} = \int_{s} \phi^{\alpha} q_{kl}^{\beta} A_{k}^{\beta \alpha'} ds \\
\tilde{K}_{ij}^{\beta \alpha \alpha'} = \int_{s} \phi^{\alpha} d_{ij}^{\beta} A_{k}^{\beta \alpha'} ds \\
\tilde{K}_{ij}^{\beta \alpha \alpha'} = \int_{s} A_{i}^{\beta \alpha} v_{ijkl}^{\beta} A_{k}^{\beta \alpha'} ds
\]

The explicit expression for these, in matrix form, are given in the appendix. Setting \( J^d dt = 0 \) for arbitrary \( \delta u_i^\alpha \) and \( \delta \theta_j^\alpha \), we can obtain the following set of governing equations. With a suitable time integrator, we can use these equation to study the transient wave propagation in brick structures.

\[
m^\alpha \ddot{u}_i^\alpha = - \sum_{\beta} \left( K_{ij}^{\beta \alpha \alpha'} u_i^\alpha' - \tilde{K}_{ij}^{\beta \alpha \alpha'} \theta_i^\alpha' \right) \]  
(18)

\[
I^\alpha_j \ddot{\theta}_i^\alpha = \sum_{\beta} \left\{ \tilde{K}_{ij}^{\beta \alpha \alpha'} u_i^\alpha' - \left( \tilde{K}_{ij}^{\beta \alpha \alpha'} + \tilde{K}_{ij}^{\beta \alpha \alpha'} \right) \theta_i^\alpha' \right\} \]  
(19)

4. Numerical experiments

To verify the developed PDS-FEM code, we simulated the propagation of translational and rotational waves and compared the numerical results with those of analytical predictions. A brick wall of width 20.3m and height 13.0m shown in Fig.6 was used for the simulations. The domain consists of bricks with 60mm width, 30mm high, and 5mm mortar thickness simulation. The wall thickness is assumed to be 40mm. The density of each block is assumed to be 1850kg/m^3. Averaging Wang et al.’s brick-mortar material properties, \( k \) and \( h \), are determined as 512N/mm^3 and 222N/mm^3, respectively.

The domain is subjected to concentrate in-plane one wave input at center of the domain. In this numerical experiment, we considered 3 cases. First and second cases are with vertical and horizontal translational wave inputs. The third case is with a rotational wave input. In all the cases, following wave form is used (Fig.7).

\[ f(t) = \frac{4}{3\sqrt{3}} A \left( \sin \frac{\omega t}{2} - \frac{1}{2} \sin \omega t \right), \]  
(20)

where \( A \) is the amplitude of the input, \( \omega \) is the input circular frequency. The amplitude is set to be 2mm for vertical and horizontal input. For rotational input, the amplitude is 0.035radian. To obtain narrow waves, so that peaks and valleys of waves are clearly visible, input circular frequency \( \omega \) is set to be \( 1.57 \times 10^4 \)radian/s for vertical and horizontal input, and \( 2.11 \times 10^5 \)radian/s for rotational input.

[Fig.6: Domain for the numerical experiment.]

[Fig.7: Input wave function.]

1) Primary or pressure-wave

Fig.8 shows the distribution of hydrostatic strain, which indicates the amplitude of p-wave, at time 2ms. The thin white line is the theoretically predicted p-wave front.

As is seen, the analytical prediction is in good agreement with numerical results in the regions indicated with letter A. Due to the strong anisotropy, the wave amplitudes are weak in most directions. As an example, when the input wave is oriented in vertical direction, the p-wave amplitude is strong in up and down directions while it is weak in other directions. This is why no wavefront is present in the directions except up and down, in the numerical results.
Fig. 8: Normalized hydrostatic strain and analytical p-wavefront (white line) at 2ms travel time.

of Fig. 8(a). The nearly straight wavefronts in the region C are the shear shock waves generated by the p-wavefront in region A; being an anisotropic medium, deformation due to p-, s- or rotational waves generates each other.

(2) Shear-wave

Fig. 8 shows the distribution of maximum shear strain, which indicates the amplitude of S-wave, at time 2 ms. The thin white line is the theoretically predicted s-wavefronts.

High amplitude s-wave can be observed in the area A. As seen there is a good agreement with the analytical prediction in these regions. High amplitude shear waves propagate in normal directions to the direction of excitation, and amplitudes in other directions are weak. Especially, shear waves in the directions of excitation have extremely small amplitudes. This is why there seems to be a mismatch between numerical and analytical wavefronts in regions except A. Further, the p-wave generated shear deformation is clearly visible in regions C and D. The near straight stripes in region D are the shear shock waves corresponding to those of region C in Fig. 8.

Fig. 9: Normalized maximum shear strain and analytical s-wavefront (white line) at 2 ms travel time.

(3) Rotational-wave

Fig. 10 shows the distribution of the amplitudes of rotational waves generated by the rotational wave input. Unlike the single wave input, the dispersion of the rotational wave occurs. Due to the difficulty of analytically estimate the rotational wave velocity, the double Fast Fourier Transform (FFT) with respect to spatial and time domain is applied. FFT is conducted for two sets of narrow domains oriented horizontally and vertically; shown with yellow lines in Fig. 10.

The double FFT result is shown in Fig. 11. The vertical axis is circular frequency and the horizontal axis is the normalized wave number, where $a_1$ and $a_2$ are the half of the length and height of a brick (Fig. 3). The numerical result is represented by the orange area. Red, green, and yellow lines represented the analytical solutions from the Eq.(6) for p-, s- and r-waves, respectively. It is observed that there is not only r-wave, but also small amplitude p- and s-wave. It is seen that the numerical solution is in good agreement when $|\xi a_1|<0.5$ and $|\xi a_2|<0.5$. In other words, the PDS-FEM is valid for the wavelength is greater than 7 times of the size of bricks.
5. Concluding remarks

We developed an equivalent continuum form for brick structures based on continuo supplementation. The continuum model makes it possible for one to analytically study the characteristic properties of masonry brick walls, and apply numerical techniques used in continuum mechanics to simulate brick structures. As a demonstration, we predicted the variation of wave speeds according to propagation direction, for a single layer brick wall. As shown at the end of section 2, the analytical predictions are on good agreement with experimental observations. Further, we develop a PDS-FEM model for simulating brick structures and verify the developed model comparing numerically obtained wave speeds with that of analytical predictions. It is seen that the numerical solution is in good agreement for the wavelengths longer than 7 times of the size of bricks. A possible extension of this research is to develop simplified models, like beams and shells, for the design of brick structures.

Appendix : Analytic expressions for PDS-FEM in 2D

This appendix presents the explicit expressions for implementing PDS-FEM for brick structures, in 2D settings. In 2D problems, the active DOFs are \{u_1, u_2, \theta_3\}. In deriving the stiffness matrices, we consider the triangle \( \Phi \) formed by connecting the centroids of the bricks \( \Phi_1, \Phi_2, \Phi_3 \) shown in Fig. 12. \( \bar{x} \) is the point at which \( \Phi_1, \Phi_2 \) and \( \Phi_3 \) meet each other.

\[
\Phi^3
\]
\[
\Phi^1
\]
\[
\Phi^2
\]
\[
\bar{x}
\]
\[
x^3
\]
\[
x^2
\]
\[
x^1
\]
\[
y^3
\]
\[
y^2
\]
\[
y^1
\]

![Fig.12: Tessellation of 2D domain](image)

(1) \( A_{i}^{\beta\alpha} \)

From the Fig. 12 we can evaluate \( A_{i}^{\beta\alpha} \) for \( \Phi^\alpha=\Phi^1 \) as

\[
\frac{1}{\psi^3} \int_{\partial\Psi^3 \cap \Phi^1} n^1 dl = -\frac{1}{\psi^3} \left( (x^2-x^1)+(x^3-x^2) \right) \times k
\]

\[
A_{i}^{\beta} = -\frac{1}{\psi^3} (x^3-x^1) \times k,
\]

\( \psi^3 \) is the area of \( \Psi^3 \), while \( \Phi^\alpha \)'s is that of \( \Phi^\alpha \). In vector form, we can write \( A_{i}^{\beta\alpha} \) as

\[
\left\{ \begin{array}{c}
A_{x}^{\beta_1} \\
A_{y}^{\beta_1}
\end{array} \right\} = -\frac{1}{\psi^3} \left\{ \begin{array}{c}
-(x^3-x^1) \\
(x^3-x^1)
\end{array} \right\}
\]
\( A^{\beta 2} \) and \( A^{\beta 3} \) can be obtained by cyclic replacement of \( x^i \)'s. Finally, for the sake of convenience, let's form the following matrix with \( A_i^{\alpha \beta} \):

\[
A^{\beta \alpha} = \begin{bmatrix}
A_1^{\beta \alpha} & 0 \\
0 & A_2^{\beta \alpha} \\
A_2^{\alpha \beta} & 0 \\
0 & A_1^{\alpha \beta}
\end{bmatrix}
\]

Now, we can express the coefficients involving the derivatives (i.e. \( u_{ij}^{\beta} \)'s and \( \theta_{ij}^{\alpha} \)'s) as

\[
\begin{bmatrix}
u_1^{\beta} \\
u_2^{\beta} \\
u_3^{\beta} \\
u_4^{\beta}
\end{bmatrix} = \frac{1}{\bar{\psi}^{\beta}} \begin{bmatrix}
A_{11}^{\beta \alpha} & A_{12}^{\beta \alpha} \\
A_{12}^{\beta \alpha} & A_{22}^{\beta \alpha} \\
A_{21}^{\alpha \beta} & A_{22}^{\alpha \beta} \\
A_{22}^{\alpha \beta} & A_{11}^{\alpha \beta}
\end{bmatrix}^{-1}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
\theta_{31} \\
\theta_{32}
\end{bmatrix} = \frac{1}{\bar{\psi}^{\beta}} \begin{bmatrix}
A_{11}^{\beta \alpha} & A_{12}^{\beta \alpha} & A_{13}^{\beta \alpha} & A_{14}^{\beta \alpha} \\
A_{12}^{\beta \alpha} & A_{22}^{\beta \alpha} & A_{23}^{\beta \alpha} & A_{24}^{\beta \alpha} \\
A_{13}^{\alpha \beta} & A_{23}^{\alpha \beta} & A_{33}^{\alpha \beta} & A_{34}^{\alpha \beta} \\
A_{14}^{\alpha \beta} & A_{24}^{\alpha \beta} & A_{34}^{\alpha \beta} & A_{44}^{\alpha \beta}
\end{bmatrix}^{-1}
\begin{bmatrix}
\theta_{1} \\
\theta_{2} \\
\theta_{3} \\
\theta_{4}
\end{bmatrix}
\]}

(2) Stiffness matrices

a) \( K_{ij}^{\alpha \beta} \)'s

\[
K_{ij}^{\alpha \beta} = \int \phi_{ij} \phi_{kl} A_k^{\alpha \beta} ds \quad \text{can be expressed in matrix form as}
\]

\[
\begin{bmatrix}
K_{11}^{\alpha \beta} & K_{12}^{\alpha \beta} \\
K_{21}^{\alpha \beta} & K_{22}^{\alpha \beta}
\end{bmatrix} = \bar{\psi}^{\beta} \left( A^{\beta \alpha} \right)^T C A^{\beta \alpha}
\]

where

\[
C = \begin{bmatrix}
\phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\
\phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\
\phi_{31} & \phi_{32} & \phi_{33} & \phi_{34} \\
\phi_{41} & \phi_{42} & \phi_{43} & \phi_{44}
\end{bmatrix}
\]

b) \( K_{ij}^{\alpha \beta \alpha'} \)'s

In matrix form \( K_{ij}^{\alpha \beta \alpha'} = \int \psi \phi_{ij} \phi_{kl} A_k^{\alpha \beta} \phi_{l}^{\alpha'} ds \) with \( i,k = \{1,2\} \) and \( j=3 \).

\[
\begin{bmatrix}
K_{11}^{\alpha \beta \alpha'} \\
K_{12}^{\alpha \beta \alpha'}
\end{bmatrix} = \psi^{\alpha} \begin{bmatrix}
\phi_{113} & \phi_{123} \\
\phi_{213} & \phi_{223}
\end{bmatrix} \begin{bmatrix}
A_{1}^{\alpha \beta} \\
A_{2}^{\alpha \beta}
\end{bmatrix}
\]

c) \( K_{ij}^{\alpha \beta \alpha'} \)'s

\[
K_{ij}^{\alpha \beta \alpha'} = \int \psi \phi_{ij} \delta_{jl} \phi_{l}^{\alpha'} ds \quad \text{with } i,j,l=3.
\]

\[
K_{ij}^{\alpha \beta \alpha'} = \phi_{ij} \delta_{jl}^{\alpha'}
\]

and \( K_{33}^{\alpha \beta} = 0 \) if \( \alpha \neq \alpha' \).

d) \( K_{ij}^{\alpha \beta \alpha'} \)'s

\[
K_{ij}^{\alpha \beta \alpha'} = \int \psi A_i^{\beta \alpha} \phi_{ij} \phi_{kl} A_k^{\alpha \beta} ds \quad \text{with } i,k = \{12\} \text{ and } j,l=3.
\]

\[
K_{33}^{\beta \alpha \alpha'} = \psi^{\beta} \begin{bmatrix}
A_{1}^{\beta \alpha} & A_{2}^{\beta \alpha}
\end{bmatrix} \begin{bmatrix}
v_{1313} & v_{1323} \\
v_{2313} & v_{2323}
\end{bmatrix} \begin{bmatrix}
A_{1}^{\gamma \alpha'} \\
A_{2}^{\gamma \alpha'}
\end{bmatrix}
\]

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