BAR INSTABILITY WITH BANK EROSION

Md Jahir UDDIN¹, Norihiro IZUMI², Kazuyoshi HASEGAWA³ and
Yasuharu WATANABE⁴

¹ Student Member of JSCE, PhD Student, Division of Field Engineering for the Environment, Graduate School of Engineering, Hokkaido University (North 13 West 8, Kita-ku, Sapporo, 060-8628, Japan)

² Member of JSCE, PhD, Professor, Faculty of Engineering, Hokkaido University

³ Member of JSCE, Director, Institute of Advanced Technology, Hokkai Suiko Consultant

⁴ Member of JSCE, Professor, Faculty of Engineering, Kitami Institute of Technology

Linear stability analysis of sand fluvial bars with bank erosion is performed with the use of the shallow-water equations. We assume that the average total channel width is constant, and the bank erosion takes place due to increases in the bed shear stress in the vicinity of banks. We obtain analytical solution by the use of asymptotic expansions around the state of no bank erosion (the bank erosion coefficient γ equals 0) in order to clarify the effect of bank erosion on the instability of sand bars with different lateral wavenumbers. We obtain instability diagram for γ evaluated from field and experimental data, and found that bank erosion stabilizes the bed in the ranges of small wavenumbers, and of large wavenumbers and aspect ratios.

Key Words: Bar instability, bank erosion coefficient, perturbation analysis, sand fluvial bars

1. INTRODUCTION

Fluvial single alternate bars and multiple bars are width-scale bedforms arising from deposition and erosion of sediment within active channels. Since the first analytical study on the formation of bars in terms of linear stability analysis performed by Callander¹⁾, a great number of analytical studies have been conducted by researchers in the fields of not only river engineering also geophysics and mathematics. It can be said that analytical study has been well developed by weakly non-linear stability analysis of Colombini et al.²⁾.

While most of the analytical studies so far have postulated straight channels with constant widths and non-erodible banks, Fu-Chun and Tzu-Hao³⁾ have investigated the forced bars induced by the variation of channel width. Their study clarified conditions under which different types of forced bars may form in channels with periodic width variations. In terms of the relation between river bank geometry and river bed topography in nature, however, not only the former affects the latter, also the latter affects the former. There should be a mutual interaction between river bank geometry and river bed topography.

The authors have performed an analytical study on bar instability considering bank erosion⁴⁾, and obtained numerical solutions of the eigenvalue problem involved in the linear stability analysis. Linear stability analysis of the bar formation provides neutral curves dividing the unstable range where sand bars are formed from the stable range where the flat bed is maintained on the streamwise wavenumber-aspect ratio plane. In addition in the analysis, the neutral curve is obtained for each of a variety of differ-

ent lateral wavenumbers, which determine whether single alternate bars or multiple bars. However, their numerical solution cannot give such separable neutral curves.

In this study, we perform linear stability analysis of sand fluvial bars with bank erosion. In order to clarify the effect of bank erosion on the instability of sand bars, we obtain analytical solution by the use of asymptotic expansions around the state of no bank erosion in which the bank erosion coefficient γ vanishes. Because γ is a coefficient representing bank erosion speed, the problem reduces to the original bar instability problem at the lowest order of γ . We evaluated γ from field and experimental data to see effects of bank erosion on bar instability.

2. GOVERNING EQUATIONS

We consider flow in the channel with sinuous bank as shown in Fig. 1. Let $\tilde{x}$ be the longitudinal coordinate, $\tilde{y}$ the lateral coordinate and $\tilde{t}$ time. The St. Venant equations of quasi-steady shallow water flow with a slowly varying erodible bed and bank can be written in terms of the above coordinates in the form

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + F^{-2} \left(\frac{\partial H}{\partial x} + \frac{\partial Z}{\partial x} \right) + \beta C_f \frac{(U^2 + V^2)^{1/2} U}{H} = 0 \quad (1)$$

$$U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + F^{-2} \left(\frac{\partial H}{\partial y} + \frac{\partial Z}{\partial y} \right) + \beta C_f \frac{(U^2 + V^2)^{1/2} V}{H} = 0 \quad (2)$$

$$\frac{\partial UH}{\partial x} + \frac{\partial VH}{\partial y} = 0 \quad (3)$$

$$\frac{\partial Z}{\partial t} + \frac{\partial Q_{bx}}{\partial x} + \frac{\partial Q_{by}}{\partial y} = 0 \quad (4)$$

where (U, V) are the flow velocity components in the x and y directions respectively, H is the flow depth, Z is the bed elevation, Q_{bx} and Q_{by} are sediment transport rate components in the x and y directions, F is the unperturbed Froude number, and β is the aspect ratio defined as the ratio between the channel width and flow depth at the base state flat bed condition. The parameter C_f is the bed friction coefficient, known to be a weak function of the flow depth relative to the roughness height. However, it is assumed to be a constant for simplicity herein.

The following normalization is employed in the above equations:

$$(\tilde{x}, \tilde{y}) = \tilde{B}_n(x, y), \quad (\tilde{U}, \tilde{V}) = \tilde{U}_n(U, V) \quad (5a, b)$$

$$(\tilde{H}, \tilde{Z}) = \tilde{H}_n(H, Z), \quad \tilde{t} = \frac{(1 - \lambda_p)\tilde{H}_n\tilde{B}_n}{\sqrt{R_s g \tilde{d}_s^3}} t \quad (6a, b)$$

$$(\tilde{Q}_b, \tilde{Q}_{bx}, \tilde{Q}_{by}) = \sqrt{R_s g \tilde{d}_s^3} (Q_b, Q_{bx}, Q_{by}) \quad (6c)$$

Here $\tilde{B}_n$, $\tilde{U}_n$ and $\tilde{H}_n$ are the channel width, flow velocity, and flow depth in the base state flat bed condition, R_s is the submerged specific gravity, which is assumed to take a standard value of 1.65, g is gravitational acceleration, λ_p is porosity, and $\tilde{d}_s$ is the sediment diameter.

The bedload components are evaluated with a well-established relation

$$(Q_{bx}, Q_{by}) = Q_b (\cos \phi, \sin \phi) \quad (7)$$

where Q_b is the total bedload rate, determined by the Meyer-Peter and Müller formula in the form

$$Q_b = 8(\theta - \theta_c)^{3/2} \quad (8)$$

where θ is the non-dimensional bed shear stress, and θ_c is the non-dimensional critical bed shear stress, which is assumed to be a constant of 0.047,

The non-dimensional bed shear stress is defined by

$$\theta = \frac{\tilde{T}_b}{\rho R_s g \tilde{d}_s} \quad (9)$$

where $\tilde{T}_b$ is the total bed shear stress, written in the form

$$\tilde{T}_b = \rho C_f (\tilde{U}^2 + \tilde{V}^2) \quad (10)$$

The sine of the direction angle of bedload, $\sin \phi$, is expressed by

$$\sin \phi = \frac{V}{(U^2 + V^2)^{1/2}} - \frac{r}{\beta \theta^{1/2}} \frac{\partial Z}{\partial y} \quad (11)$$

where r is a coefficient which various authors suggest to take as a constant ranging between 0.3 to 0.6. We use $r = 0.3$.

It might be worth while to point out that the non-dimensional bed shear stress in the above equation θ can be described by

$$\theta = \theta_n (U^2 + V^2) \quad (12)$$

where θ_n is the non-dimensional bed shear stress in the base state flat bed condition, which is sometimes called the Shields number.

Fig. 1 Conceptual diagram and coordinates.

Fig. 2 The boundary condition of sediment flux at banks.

3. BOUNDARY CONDITIONS

We assume that the locations of banks are shifted over time. Figure 1 shows the conceptual diagram and coordinate system employed in this analysis. Denoting the y coordinates of right and left banks by R and L respectively, and the total channel width by B , we obtain the relation

$$B = L - R \quad (13)$$

where we assume that the average value of B is constant, but B itself can change in space and time.

The flow cannot penetrate the both banks, so that the velocity components normal to the both banks should vanish respectively, such that

$$\mathbf{U} \cdot \mathbf{e}_{NR} = 0, \quad \mathbf{U} \cdot \mathbf{e}_{NL} = 0 \quad (14a, b)$$

where the quasi-steady assumption has been employed as well as in the shallow water equations. In the above equation, $\mathbf{U}$ is the velocity vector, defined by

$$\mathbf{U} = (U, V) \quad (15)$$

In addition, $\mathbf{e}_{NL}$ and $\mathbf{e}_{NR}$ are unit vectors normal to the left and right banks respectively, expressed by

$$(\mathbf{e}_{NL}, \mathbf{e}_{NR}) = \left(\frac{(-\partial L / \partial x, 1)}{\sqrt{1 + (\partial L / \partial x)^2}}, \frac{(-\partial R / \partial x, 1)}{\sqrt{1 + (\partial R / \partial x)^2}} \right) \quad (16)$$

Figure 2 shows the boundary condition of sediment flux at banks, where the solid thick lines indicate the original bank line and broken lines denote the bank line after erosion. Bank erosion produces sediment supply from the banks to the channel. When the time variation of R is positive, the deposition takes place on the right bank. Therefore, the sediment is absorbed by the right bank, and the negative sediment supply takes place on the right bank. That process is formulated by

$$(M - Z - \beta C_f F^2 x) \frac{\partial R}{\partial t} \cos \psi = -\mathbf{Q}_b(R) \cdot \mathbf{e}_{NR} \quad (17)$$

where M is the bank height defined by $M = \tilde{M}/\tilde{H}_n$, and $Z + \beta C_f F^2 x$ is the deviation of the bed elevation from the original plane bed. Therefore, the relative variation of bank height due to the bed degradation in the vicinity of the bank is taken into account in the above equation. Meanwhile, when the time variation of L is negative, the erosion takes place on the left bank. Therefore, the sediment is added to the river from the left bank, and the positive sediment supply takes place on the left bank. So the process can be formulated by

$$(M - Z - \beta C_f F^2 x) \frac{\partial L}{\partial t} \cos \psi = -\mathbf{Q}_b(L) \cdot \mathbf{e}_{NL} \quad (18)$$

where ψ is the angle between the bank and the y axis, and $\mathbf{Q}_b$ is the bedload vector, described by

$$\mathbf{Q}_b = (Q_{bx}, Q_{by}) \quad (19)$$

4. BANK EROSION MODEL

In this paper, we assume that the river bed and banks are composed of gravel and that bank erosion takes place due to increases in the bed shear stress in the vicinity of banks. If the bed shear stress at the junction between the laterally sloped bank region and the horizontally flat central bed region increases, sediment on the bank region starts to move (Fig. 3). Once sediment starts to move on the bank region, sediment is removed from the bank because it is pulled in the lateral direction by the gravity. This results in bank erosion. It is assumed, therefore, that the bank erosion speed depends on how large the bed shear stress at the junction is compared with the critical bed shear stress. We employ a simple relation of the form

$$(M - Z - \beta C_f F^2 x) \frac{\partial R}{\partial t} \cos \psi = -\gamma [\theta(R) - \theta_n] \quad (20)$$

$$(M - Z - \beta C_f F^2 x) \frac{\partial L}{\partial t} \cos \psi = \gamma [\theta(L) - \theta_n] \quad (21)$$

where θ_n is the non-dimensional bed shear stress in the base state flat bed condition before bars are formed. In addition, γ is the bank erosion coefficient.

5. LINEAR STABILITY ANALYSIS

(1) Asymptotic expansions with A

For performing linear stability analysis, we employ the asymptotic expansions of the form

$$(U, V, H, Z, R, L) = \left(1, 0, 1, -\beta C_f F^2 x, -\frac{1}{2}, \frac{1}{2}\right) + A(U_1, V_1, H_1, Z_1, R_1, L_1) \exp[i(kx - \omega t)] \quad (22)$$

Fig. 3 Channel cross-section.

where A , k and ω are the amplitude, wavenumber and angular frequency of perturbation, respectively.

Substituting the above expansions into (1)–(4), (14), (17), (18), (20) and (21), at $O(A)$, we obtain

$$a_1 U_1(y) + a_2 H_1(y) + a_3 Z_1(y) = 0 \quad (23)$$

$$a_4 V_1(y) + a_5 \frac{dH_1}{dy} + a_5 \frac{dZ_1}{dy} = 0 \quad (24)$$

$$a_6 U_1(y) + \frac{dV_1}{dy} + a_6 k H_1(y) = 0 \quad (25)$$

$$-i\omega Z_1(y) + a_7 U_1(y) + a_8 \frac{dV_1}{dy} - a_9 \frac{d^2 Z_1}{dy^2} = 0 \quad (26)$$

$$\frac{\omega}{k} M V_1\left(\pm \frac{1}{2}\right) = -a_9 \frac{dZ_1\left(\pm \frac{1}{2}\right)}{dy} \quad (27)$$

$$\frac{\omega}{k} M V_1\left(\pm \frac{1}{2}\right) = \mp 2\gamma \theta_n U_1\left(\pm \frac{1}{2}\right) \quad (28)$$

Where the coefficients $a_1 \sim a_9$ are defined by $a_1 = ik + 2\beta C_f$, $a_2 = ikF^{-2} - \beta C_f$, $a_3 = ikF^{-2}$, $a_4 = ik + \beta C_f$, $a_5 = F^{-2}$, $a_6 = ik$, $a_7 = 24ik\theta_n(\theta_n - \theta_c)^{1/2}$, $a_8 = 8(\theta_n - \theta_c)^{3/2}$, and $a_9 = 8(\theta_n - \theta_c)^{3/2} r/\beta\theta_n^{1/2}$.

Eqns. (23)–(26) can be solved with the four boundary conditions (27)–(28) in the following manner. Linearly independent solutions are denoted by

$$(U_1, V_1, H_1, Z_1) = (u, v, h, z) \exp \alpha y \quad (29)$$

Substituting the above equations into (23)–(25), we obtain

$$\mathbf{L}(\alpha) \cdot \mathbf{u} = \mathbf{r}(\alpha) z \quad (30)$$

where

$$\mathbf{L}(\alpha) = \begin{bmatrix} a_1 & 0 & a_2 \\ 0 & a_4 & \alpha a_5 \\ a_6 & \alpha & a_7 \end{bmatrix} \quad (31)$$

$$\mathbf{r}(\alpha) = \begin{bmatrix} -a_3 \\ -\alpha a_5 \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u \\ v \\ h \end{bmatrix} \quad (32a, b)$$

The above equations can be easily solved to show that u , v and h are all expressed in the form of multiples of z , such that

$$u = f_u(\alpha)z, \quad v = f_v(\alpha)z, \quad h = f_h(\alpha)z \quad (33a, b, c)$$

where $f_u(\alpha)$, $f_v(\alpha)$ and $f_h(\alpha)$ are defined by

$$\begin{bmatrix} f_u(\alpha) \\ f_v(\alpha) \\ f_h(\alpha) \end{bmatrix} = \mathbf{L}^{-1}(\alpha) \cdot \mathbf{r}(\alpha) \quad (34)$$

Substituting (33) into (26), we obtain an algebraic equation

$$\mathcal{A}\alpha^4 + \mathcal{B}\alpha^2 + C = 0 \quad (35)$$

The four solutions of the above equation is obtained and among the four solutions, either of the following two solutions with positive signs before the radical sign

$$\sqrt{\frac{-\mathcal{B} + \sqrt{\mathcal{B}^2 - 4\mathcal{A}C}}{2\mathcal{A}}}, \quad \sqrt{\frac{-\mathcal{B} - \sqrt{\mathcal{B}^2 - 4\mathcal{A}C}}{2\mathcal{A}}} \quad (36a, b)$$

is denoted by α_a , and the other α_b . The four solutions are then written in the form

$$\alpha_1 = \alpha_a, \quad \alpha_2 = -\alpha_a, \quad \alpha_3 = \alpha_b, \quad \alpha_4 = -\alpha_b \quad (37)$$

Substituting α obtained in the above into (33), we obtain

$$u_i = f_u(\alpha_i)z_i, \quad v_i = f_v(\alpha_i)z_i, \quad h_i = f_h(\alpha_i)z_i, \quad (i = 1, 2, 3, 4) \quad (38)$$

(2) γ expansion

In order to clarify the structure of solution, all the variables are expanded with the use of γ . That is

$$(U_1, V_1, H_1, Z_1) = (U_{10}, V_{10}, H_{10}, Z_{10}) + \gamma(U_{11}, V_{11}, H_{11}, Z_{11}) \quad (39)$$

Correspondingly, the complex angular frequency ω is also expanded as

$$\omega = \omega_0 + \gamma\omega_1 \quad (40)$$

where, γ is a small parameter.

We substitute the above expansions into the governing equations, and obtain the solution up to $O(\gamma)$. Before proceeding further, we have to specify the phase shift between right and left banks. The phase shift can be arbitrary, but we consider two simple cases, herein, which are commonly observed in the field. One is the case of the both banks being in phase as shown in Figs. 1 and 2, and the other is the case of the both banks being out of phase.

a) The case of both banks being in phase

At first the solution has been done for both banks being in phase, i.e. $R_1 = L_1$ (Fig.1). It is then found, from the boundary conditions that V_1 is an even function and symmetrical with respect to the x axis, while U_1 and Z_1 are odd functions and symmetrical with respect to the origin.

Substituting the (39)–(40) into (23)–(28), we obtain the following results at each order of γ .

At $O(1)$, we obtain

$$a_1 U_{10} + a_2 H_{10} + a_3 Z_{10} = 0 \quad (41)$$

$$a_4 V_{10} + a_5 \frac{dH_{10}}{dy} + a_5 \frac{dZ_{10}}{dy} = 0 \quad (42)$$

$$a_6 U_{10} + \frac{dV_{10}}{dy} + a_6 H_{10} = 0 \quad (43)$$

$$-i\omega_0 Z_{10} + a_7 U_{10} + a_8 \frac{dV_{10}}{dy} - a_9 \frac{d^2 Z_{10}}{dy^2} = 0 \quad (44)$$

$$V_{10}\left(\frac{1}{2}\right) = 0, \quad \frac{dZ_{10}\left(\frac{1}{2}\right)}{dy} = 0 \quad (45a, b)$$

The solutions of the above differential system take the form

$$U_{10} = z_{10,a} f_u(\alpha_a) \sinh \alpha_a y + z_{10,b} f_u(\alpha_b) \sinh \alpha_b y \quad (46a)$$

$$V_{10} = z_{10,a} f_v(\alpha_a) \cosh \alpha_a y + z_{10,b} f_v(\alpha_b) \cosh \alpha_b y \quad (46b)$$

$$H_{10} = z_{10,a} f_h(\alpha_a) \sinh \alpha_a y + z_{10,b} f_h(\alpha_b) \sinh \alpha_b y \quad (46c)$$

$$Z_{10} = z_{10,a} \sinh \alpha_a y + z_{10,b} \sinh \alpha_b y \quad (46d)$$

where the above solutions satisfy the boundary conditions (45) only when $\alpha_a = n\pi i$ ($n = 1, 3, 5$) and $z_{10,b} = 0$, or $\alpha_b = n\pi i$ and $z_{10,a} = 0$. We assume

$$\alpha_a = n\pi i \quad (n = 1, 3, 5, \dots) \quad \text{and} \quad z_{10,b} = 0 \quad (47a, b)$$

Substituting the above equations into (44), we obtain

$$\omega_0 = \frac{a_7}{i} f_u(\alpha_a) - ia_8 \alpha_a f_v(\alpha_a) + ia_9 \alpha_a^2 \quad (48)$$

At $O(\gamma)$, we obtain the following equations

$$a_1 U_{11} + a_2 H_{11} + a_3 F^{-2} Z_{11} = 0 \quad (49)$$

$$a_4 V_{11} + a_5 \frac{dH_{11}}{dy} + a_5 \frac{dZ_{11}}{dy} = 0 \quad (50)$$

$$a_6 U_{11} + \frac{dV_{11}}{dy} + a_6 H_{11} = 0 \quad (51)$$

$$-i\omega_0 Z_{11} + a_7 U_{11} + a_8 \frac{dV_{11}}{dy} - a_9 \frac{d^2 Z_{11}}{dy^2} = i\omega_1 Z_{10} \quad (52)$$

$$\frac{\omega_0}{k} MV_{11} \left(\frac{1}{2}\right) + \frac{\omega_1}{k} MV_{10} \left(\frac{1}{2}\right) = -a_9 \frac{dZ_{11}\left(\frac{1}{2}\right)}{dy} \quad (53)$$

$$\frac{\omega_0}{k} MV_{11} \left(\frac{1}{2}\right) + \frac{\omega_1}{k} MV_{10} \left(\frac{1}{2}\right) = -2\theta_n U_{10} \left(\frac{1}{2}\right) \quad (54)$$

It is found that (52) has the same form as (44) except for the right hand side. The inhomogeneous part has the same form as the solution of homogeneous equation. The solution of the inhomogeneous differential system can be expressed by the combination of the general solution and the special solution in the form

$$U_{11} = u_{11,a} \sinh \alpha_a y + u_{11,b} \sinh \alpha_b y + u_{11,s} y \cosh \alpha_a y \quad (55a)$$

$$V_{11} = v_{11,a} \cosh \alpha_a y + v_{11,b} \cosh \alpha_b y + v_{11,s} y \sinh \alpha_a y \quad (55b)$$

$$H_{11} = h_{11,a} \sinh \alpha_a y + h_{11,b} \sinh \alpha_b y + h_{11,s} y \cosh \alpha_a y \quad (55c)$$

$$Z_{11} = z_{11,a} \sinh \alpha_a y + z_{11,b} \sinh \alpha_b y + z_{11,s} y \cosh \alpha_a y \quad (55d)$$

By substituting the above equations into (49)–(51), we derived some equations where each coefficient contains the terms $\sinh \alpha_a y$, $\sinh \alpha_b y$, $y \cosh \alpha_a y$, $\cosh \alpha_a y$, $\cosh \alpha_b y$, and $y \sinh \alpha_a y$ which are linearly independent, so that each coefficient should vanish. Therefore, we obtain

$$\mathbf{L}(\alpha_a) \cdot \begin{bmatrix} u_{11,a} \\ v_{11,a} \\ h_{11,a} \end{bmatrix} = \mathbf{r}(\alpha_a) z_{11,a} + \begin{bmatrix} 0 \\ -a_5 (h_{11,s} + z_{11,s}) \\ -v_{11,s} \end{bmatrix} \quad (56)$$

$$\mathbf{L}(\alpha_b) \cdot \begin{bmatrix} u_{11,b} \\ v_{11,b} \\ h_{11,b} \end{bmatrix} = \mathbf{r}(\alpha_b) z_{11,b}, \quad \mathbf{L}(\alpha_a) \cdot \begin{bmatrix} u_{11,s} \\ v_{11,s} \\ h_{11,s} \end{bmatrix} = \mathbf{r}(\alpha_a) z_{11,s} \quad (57a, b)$$

Solving (57) by using (34), we obtain

$$\begin{bmatrix} u_{11,b} \\ v_{11,b} \\ h_{11,b} \end{bmatrix} = \begin{bmatrix} f_u(\alpha_b) \\ f_v(\alpha_b) \\ f_h(\alpha_b) \end{bmatrix} z_{11,b}, \quad \begin{bmatrix} u_{11,s} \\ v_{11,s} \\ h_{11,s} \end{bmatrix} = \begin{bmatrix} f_u(\alpha_a) \\ f_v(\alpha_a) \\ f_h(\alpha_a) \end{bmatrix} z_{11,s} \quad (58a, b)$$

Solving (56) by using (58), we obtain

$$\begin{bmatrix} u_{11,a} \\ v_{11,a} \\ h_{11,a} \end{bmatrix} = \begin{bmatrix} f_u(\alpha_a) \\ f_v(\alpha_a) \\ f_h(\alpha_a) \end{bmatrix} z_{11,a} + \begin{bmatrix} g_u(\alpha_a) \\ g_v(\alpha_a) \\ g_h(\alpha_a) \end{bmatrix} z_{11,s} \quad (59)$$

where

$$\begin{bmatrix} g_u(\alpha) \\ g_v(\alpha) \\ g_h(\alpha) \end{bmatrix} = \mathbf{L}(\alpha)^{-1} \begin{bmatrix} 0 \\ -a_5 [f_h(\alpha) + 1] \\ -f_v(\alpha) \end{bmatrix} \quad (60)$$

Substituting the above equations into the Exner equation (52) and eliminating some terms according to the result of (48) at $O(1)$, we obtain the following equation

$$a_7 g_u(\alpha_a) z_{11,s} + a_8 [\alpha_a g_v(\alpha_a) + f_v(\alpha_a)] z_{11,s} - a_9 2\alpha_a z_{11,s} - i\omega_1 z_{10} = 0 \quad (61)$$

We can obtain ω_1 from the above equation by evaluating the relationship between $z_{11,s}$ and z_{10} from the boundary conditions (53) and (54).

b) The case of both banks being out of phase

The solution has been done for both banks being completely out of phase, i.e. $R_1 = -L_1$. In this case, from the boundary conditions we find that V_1 is an odd function and symmetrical with respect to the origin, while U_1 and Z_1 are even functions and symmetrical with respect to the x axis. For evaluating ω_0 and ω_1 in this case, similar calculation has been done (except sin and cos are replaced by cos and sin respectively) as for the case both bank being in phase.

6. RESULTS AND DISCUSSION

(1) Bank erosion coefficient γ

We assume γ is a function of the non-dimensional bed shear stress. The Hasegawa's bank erosion coefficient E_0 which depends on hydraulic factors as well as geotechnical properties of the bank has been discussed by Hasegawa⁵⁾. He determined the values of E_0 for three rivers in Hokkaido (Ishikari, Uryu and Mukawa river) and found that the value ranges from 0.92×10^{-7} to 19.17×10^{-7} . We have compared our γ with E_0 .

According to Hasegawa⁵⁾ (equation (14), pp.750), the rate of bank erosion is expressed by

$$\tilde{\zeta} = E_0 u_B \quad (62)$$

where, $\tilde{\zeta}$ is the bank erosion rate, E_0 is the Hasegawa's bank erosion coefficient and u_B is the excess of near-bank depth-averaged streamwise flow velocity. In our study from (20) and (21), the bank erosion rate can be expressed as the following dimensional form:

$$\tilde{\zeta} = \frac{\tilde{\gamma}(\theta - \theta_n)}{\tilde{M}} \quad (63)$$

where $\tilde{\gamma}$ is the dimensional version of the bank erosion coefficient while the γ itself is non-dimensional, defined by

$$\tilde{\gamma} = \frac{\gamma \sqrt{R_s g \tilde{d}_s^3}}{(1 - \lambda_p)} \quad (64)$$

By using (12) and (22) in (63) and neglecting higher order terms, we obtain

$$\tilde{\zeta} = \frac{2\tilde{\gamma}\theta_n}{\tilde{M}} (AU_1) \quad (65)$$

By rewriting (65), we get

$$\tilde{\zeta} = \frac{2\gamma\theta_n \sqrt{R_s g \tilde{d}_s^3}}{(1 - \lambda_p)\tilde{H}_n \tilde{U}_n} u_B \quad (66)$$

where $AU_1 = u_B/\tilde{U}_n$ and $\tilde{M} = \tilde{H}_n$ is assumed.

By comparing (62) and (66), we obtain

$$\gamma = E_0 \left(\frac{(1 - \lambda_p)\tilde{H}_n \tilde{U}_n}{2\theta_n \sqrt{R_s g \tilde{d}_s^3}} \right) \quad (67)$$

So, it can be said from equation (67) that γ is much larger than E_0 .

By using the average values of E_0 , we calculated our γ by (67) for 116 sets of data obtained from different rivers in Japan and found that it ranges from 4.27×10^{-5} to 4.39×10^{-2} . From Watanabe's experiment we have taken the maximum bank erosion rate $\tilde{\zeta} = 0.65$ cm/min and $u_B = 0.276$ m/s to evaluate γ from (62) and (67), and found the value is 0.08.

(2) Instability diagram of bars with bank erosion

The contours of $\text{Im}[\omega_0 + \gamma\omega_1]$ in the k - β plane for the case $\theta_n = 0.06$, $F = 0.2$, $C_f = 0.01$, $M = 1$, $\gamma = 0$ (dashed line), $\gamma = 0.04$ (red dashed dotted line) and $\gamma = 0.1$ (solid thick line) for different lateral wavenumbers n are shown in Fig. 4. We adopted 0.2 and 0.06 for F and θ_n , respectively, as typical values for gravel bed rivers in the field. However, other choices of those parameters result in qualitatively similar diagrams. All the curves shown in this figure is neutral instability curves. The case $n = 1$ corresponds to single alternate bars, the case $n = 2$ to double row bars, and the cases $n > 2$ to multiple bars. In addition, n of odd numbers correspond to the case of both banks being in phase, and n of even numbers correspond to the case of both banks being out of phase.

In the all cases of lateral wavenumbers, the unstable regions located inside the areas surrounded by the neutral curves generally expands in the direction of increasing k due to bank erosion. That tendency is remarkable especially in the case of single alternate bars ($n = 1$ in Fig. 4). The expansion of unstable region in the direction of increasing k means that the wavelength of bars decreases. The implication is that excessive sediment supplied due to bank erosion causes the increase of bed instability, and the resulting decrease in the bar wavelength. We can also see from this figure that if the value of γ is smaller than 0.04, the effect of bank erosion is negligible at least in the instability diagram.

Fig. 4 The contours of $\text{Im}[\omega_0 + \gamma\omega_1]$ in the k - β plane for the case $\theta_n = 0.06$, $F = 0.2$, $C_f = 0.01$, $M = 1$, $\gamma = 0$ (dashed line), $\gamma = 0.04$ (red dashed dotted line) and $\gamma = 0.1$ (solid thick line) for different lateral wavenumbers n .

Discontinuous changes of neutral curves can be seen in the ranges of large aspect ratios and wavenumbers in Fig. 4. Though we do not show the results, it is found that the discontinuity is not observed in the case of large Froude numbers. From these results, it is suspected that the term at $O(\gamma)$ becomes too large for the γ expansion to be valid. The analysis proposed herein is assumed to be applicable only in the ranges of relatively small aspect ratios and wavenumbers, and large Froude numbers.

(3) Comparison with numerical solutions

The analytical solution obtained in this study is compared with the numerical solution obtained by the authors⁴⁾ here in. In Fig. 5, the contours of ω are shown in the k - β plane for the case $\theta_n = 0.06$, $F = 0.5$, $C_f = 0.01$, $M = 1$ and $\gamma = 0.1$. The analytical solution is shown by the broken lines, and the numerical solution by the solid line.

In the case of the numerical solution, only the maximum growth rate can be calculated, and therefore, the neutral curve corresponds to all the wavenumbers (solid line). In the case of analytical solution, however, neutral curves corresponding to different wavenumbers can be separately obtained (broken lines).

As seen in this figure, the envelope of all the neutral curves of analytical solution agrees well with the neutral curve of numerical solution in the range of large wavenumbers. It is found, however, the agreement is not good in the range of small wave numbers.

Fig. 5 The contours of ω in the k - β plane for the case $\theta_n = 0.06$, $F = 0.5$, $C_f = 0.01$, $M = 1$ and $\gamma = 0.1$, where, the analytical solution is shown by the broken lines, and the numerical solution by the solid line.

7. CONCLUSION

A linear stability analysis of fluvial sand bars relating bank erosion has been conducted. The analysis was performed by the use of the shallow water equations, the exner equation, and the bank erosion equation newly proposed in this study. It was found from the analysis that the tendency of expanded unstable region is remarkable especially in the case of single alternate bars. It is also found from the analysis that bank erosion has an effect to stabilize the bed in the ranges of small wavenumbers, and of large wavenumbers and aspect ratios.

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