Modeling of Breaking Solitary Wave Run Up

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1. INTRODUCTION

The Aceh tsunami in 2004 and also the tsunami on the north east coast of Japan recently clearly showed the massive impact of tsunami waves to coastal area. The tsunami wave carries and transports large amount of sediment, resulting in coastal morphology changes. It is very important to understand the tsunami wave in order to reduce and to minimize the damage. Thus, tsunami modeling study is important to assess the impact of a tsunami wave on coastal area.

Tsunami modeling has been studied widely, by means of laboratory experiment, analytical solution, and numerical simulation. One of the studies that are considered to have significant impact on the field is canonical problems\(^5\). The study covered a series of laboratory experiments measuring the non-breaking and breaking solitary wave run up along with analytical solution for a solitary wave propagating over constant depth and then running up a sloping beach. The study also contributes to numerical study by providing data set for model verification and validation.

Various models have been developed with various methods and approaches. Simplified approaches are often used for practical application. Simplified approach such as found in depth averaged SWE model has been widely used for numerical simulation of solitary wave run up and is known to yield tsunami evolution well\(^3\). Titov and Synolakis\(^4\) have used SWE model to compute a range of real tsunami inundation, including the 1993 Okushiri tsunami.

Realization of breaking wave in SWE model is still a challenge since it has no appropriate term for breaking wave simulation. Other governing equation, such as Boussinesq-type\(^5\), can adopt breaking wave treatment by introducing a constant value of eddy viscosity in the shallower area\(^6\). However, this model requires longer computation time and does not easily adapt wetting and drying (moving boundary) treatments as compare to the SWE model. It is also noted that their performance and accuracy for tsunami run up even for landslide wave does not exceed SWE model\(^7\). Advanced computational method provides a more realistic way to simulate breaking wave. One of the leading method is Volume of Fluid Method, i.e., NEWFLUME\(^8\). Nevertheless, these methods include more terms and far more complex than the SWE.

Certain treatments and numerical schemes can be applied to SWE model in order to achieve stability and accuracy for wave breaking simulation. The Mac Cormack scheme had been widely employed to solve SWE for investigating run up of a uniform bore on a sloping beach\(^5\). Nevertheless, volume conservation problem arises due to discontinuities. Steep gradients of conservative terms may cause numerical error, which can be
Reduced by using artificial viscosity term (i.e. Hansen\(^{10}\)). However, this term must be determined by trial and error procedures. Application of the Mac Cormack finite difference combines with artificial viscosity is given for 2004 Tsunami, Banda Aceh\(^{11}\). Nevertheless, Accuracy improvement is still required.

Recent development in numerical method leads to the application of shock capturing scheme in SWE for wave breaking simulation. Modification of Godunov-type scheme leads to the second order accuracy in space such as Monotonic Upstream Scheme of Conservation Laws (MUSCL) scheme\(^{12}\). The method was further enhanced by being combined with the First Order Centered Scheme (FORCE)\(^{13}\) and Total Variation Diminished (TVD) Runge-Kutta\(^{14}\). Employment of such scheme will be able to enhanced SWE capability for breaking wave simulation.

One of the advantages of SWE model is its flexibility to accommodate various treatments. Conventional SWE model uses Manning approach to assess bed stress term. Nevertheless, in the case of unsteady motion such as wave, Manning approach is not accurate. Bed stress behaviors under wave can not be accurately reproduced by Manning method\(^{15}\). It has been shown that replacing Manning approach with boundary layer approach provides more accurate results\(^{1}\). However, the method has not been tested for breaking wave case, since it has no appropriate treatment to handle shock.

In this study, FORCE-MUSCL scheme is used to solve SWE, with both conventional Manning and coupling method for assessing bed stress. The method is used to simulate breaking solitary wave running up on a sloping beach. Results are compared to experimental data, the original SWE model and also the high end model (NEWFLUME). Moreover, this study also compares bed stress assessment between conventional SWE and newly developed, coupling method SWE.

2. Numerical Modeling

The governing equations for the new method (coupling) are SWE and k-\(\omega\) equation. SWE model is upgraded by replacing the Manning method with a more accurate bed stress assessment using k-\(\omega\) model. Details of the coupling method are given in the previous study\(^{1}\).

1) Shallow Water Equation

The SWE consists of the continuity equation and the momentum equation as follows.

\[
\frac{\partial h}{\partial t} + \frac{\partial (U h)}{\partial x} = 0 \quad (1a)
\]

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \frac{\partial (h + z)}{\partial x} = -g h S_f \quad (1b)
\]

where \(h\) is the water depth, \(U\) is depth averaged velocity, \(t\) is time, \(g\) is gravity, \(z\) is the bed elevation, \(S_f\) is bottom shear stress (bed). Manning equation is commonly used to assess this parameter. The bed stress relation in the conventional Manning method is assumed linear to the square of free-stream velocity as shown bellow.

\[
\frac{\tau_b}{\rho} = gn^2 / R_e^{\frac{1}{3}} \times U \times |U| \quad (2)
\]

where \(\tau_b\) is the bed stress, \(\rho\) is density, \(R_e\) is the hydraulic radius or can be considered as water depth for a very wide channel, and \(n\) is the Manning roughness.

2) k-\(\omega\) Equation

The governing equation for the k-\(\omega\) model is based on the Reynolds-averaged equations of continuity and momentum.

\[
\frac{\partial u_i}{\partial x_i} = 0 \quad (3a)
\]

\[
\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial p}{\partial x_i} + (2\mu S_{ij} - \rho u_i u_j) \quad (3b)
\]

where \(u_i\) and \(x_i\) denote the velocity in the boundary layer and location in the grid, \(u_i^f\) is the fluctuating velocity in the \(x\) (\(i = 1\)) and \(y\) (\(i = 2\)) directions, \(P\) is the pressure, \(\nu\) is the kinematics viscosity, \(\rho\) is the density of the fluid, \(\rho u_i u_j\) is the Reynolds stress tensor, and \(S_{ij}\) is the strain-rate tensor determined in the following equation.

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (4)
\]

The Reynolds stress tensor is given through eddy viscosity by Boussinesq approximation:

\[
-u_i u_j = \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \quad (5)
\]

with \(k\) is the turbulent kinetic energy and \(\delta_{ij}\) is the Kronecker delta. The k-\(\omega\) model equation is given as follows:

\[
\frac{\partial k}{\partial t} + u_i \frac{\partial k}{\partial x_i} = \tau_0 \frac{\partial u_i}{\partial x_i} - \beta k \omega + \frac{\partial}{\partial x_j} \left( \nu + \sigma \nu \right) \frac{\partial k}{\partial x_j} \quad (6a)
\]
\[
\begin{align*}
\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} &= \alpha \frac{\partial}{\partial v} (\frac{\partial v}{\partial x} + \beta \frac{\partial \omega}{\partial x}) = \alpha \frac{\partial}{\partial v} (\frac{\partial v}{\partial x} + \beta \frac{\partial \omega}{\partial x}) \\
(6b)
\end{align*}
\]

The eddy viscosity is given by:

\[
\nu = \frac{k}{\omega} (7)
\]

The values of the closure coefficients are given as \( \beta = 3/40, \beta^* = 0.09, \alpha = 5/9, \) and \( \sigma = \sigma^* = 0.5. \)

Finite difference central scheme is applied to solve the governing equations in time and space. The boundary condition at the bottom is no slip boundary. At the free stream, it is assumed that the velocity gradient, turbulent kinetic energy gradient and the dissipation rate gradient are zero.

(3) Breaking wave modeling (shock capturing)

Shock capturing scheme, FORCE-MUSCL, is used in this study to handle wave breaking condition. The governing equation, Eq.(1), can be rearranged into,

\[
\frac{\partial V}{\partial t} = -F\big(x_i\big) + S_i (8)
\]

where \( L \) notates the right hand side of the equation.

Spatial derivation of flux \( F \) is approximated by conservative difference as

\[
F\big(x_i\big) = \frac{F^+_{i+1/2} - F^-_{i-1/2}}{\Delta x} (9)
\]

where \( \Delta x \) is the cell size.

The conservative variables (\( h \) and \( hU \)) discretization is conducted using the MUSCL scheme. The Surface Gradient Method (SGM) is applied, by calculating free surface elevation instead of depth to accommodate the effect of bed topography.

\[
h = \eta - z_b (10)
\]

where \( \eta \) is the surface elevation.

The following discretization of conservative variable \( h \) is given as an example. The left hand side (-) and right hand side (+) of the interface are evaluated using linear reconstruction.

\[
\eta_{i+1/2} = (\eta)_i + \frac{1}{2} \delta_1 (\eta) - (\eta)_{i-1/2} - \frac{1}{2} \delta_2 (\eta) \quad (11)
\]

The limited slopes from the above are as follows.

\[
\delta_1 (\eta) = \Psi \Delta_1 (\eta); \Delta_2 (\eta) = \frac{\Delta_{i+1/2} (\eta) + \Delta_{i-1/2} (\eta)}{2} \quad (12)
\]

where \( \Psi \) is the slope limiter which is given by a Superbee-type-non-linear slope limiter.\(^3\)}

\[
\Psi = \begin{cases} 
0 & r \leq 0 \\
2r & 0 < r \leq 1/2 \\
1 & 1/2 < r \leq 1 \\
\min(r, \frac{2}{1+r}, 2) & \text{else}
\end{cases} (13)
\]

with

\[
r_i = \frac{\Delta_{i+1/2} (hU)}{\Delta_{i-1/2} (hU)}; r_i = \frac{\Delta_{i+1/2} (\eta)}{\Delta_{i-1/2} (\eta)} \quad (14)
\]

where \( r \) is the ratio of successive jumps in the conservative \( h \)

\[
\Delta_i (\eta) = \frac{\Delta_{i+1/2} (\eta) + \Delta_{i-1/2} (\eta)}{2} \quad (15a)
\]

\[
\Delta_{i+1/2} (\eta) = (\eta)_{i+1} - (\eta)_{i-1}; \Delta_{i-1/2} (\eta) = (\eta)_{i} - (\eta)_{i-1} \quad (15b)
\]

with

\[
F_{i+1/2}^{\text{FORCE}} = \frac{1}{2} \left( F_{i+1/2}^{\text{LF}} + F_{i+1/2}^{\text{LW}} \right) \quad (16)
\]

The depth at the evaluated locations can be calculated using Eq.(10). Same steps are conducted to evaluate conservative variable \( hU \) without SGM.

The numerical flux (F) is evaluated using FORCE scheme which is a combination of Lax-Friedrichs (LF) and Lax-Wendroff (LW) scheme.

\[
F_{i+1/2}^{\text{LF}} = \frac{1}{2} \left( F_{i+1/2}^{-} + F_{i+1/2}^{+} \right) + \frac{1}{2} \Delta x \left( V_{i+1/2}^{-} - V_{i+1/2}^{+} \right) \quad (17)
\]

Lax-Friedrichs flux (FLF) is given as

\[
F_{i+1/2}^{\text{LF}} = \frac{1}{2} \left( F_{i+1/2}^{\text{LF}} + F_{i+1/2}^{\text{LW}} \right) \quad (18a)
\]

\[
F_{i+1/2}^{\text{LW}} = \frac{1}{2} \left( F_{i+1/2}^{-} + F_{i+1/2}^{+} \right) - \frac{1}{2} \Delta x \left( V_{i+1/2}^{-} - V_{i+1/2}^{+} \right) \quad (18b)
\]

and Lax-Wendroff flux (FLW) is given as

\[
F_{i+1/2}^{\text{LW}} = \frac{1}{2} \left( V_{i+1/2}^{-} + V_{i+1/2}^{+} \right) \quad (19a)
\]

\[
- \frac{1}{2} \Delta x \left( F_{i+1/2}^{-} - F_{i+1/2}^{+} \right) \quad (19b)
\]

where, VLW is the intermediate state of conserved variables and \( \Delta t \) is the time step interval.

Time integration is solved using TVD Runge-Kutta.

\[
V_{i}^{(1)} = V_i + \Delta t \times L(V_i) \quad (20a)
\]

\[
V_{i}^{(2)} = 3/4 \times V_i + 1/4 \times V_i^{(1)} + 1/4 \times \Delta t \times L(V_i^{(1)}) \quad (20b)
\]
\[ V^{(i+1)} = 3/4 V^i + 2/3 V^{(2i)} + 2/3 \Delta t \times L(V^{(i+1)}) \] \hspace{1cm} (20c)

Time step is not constant and evaluated using the Courant-Friedrichs-Lewy (CFL) stability given by

\[ \Delta t = \epsilon \frac{\Delta x}{\left( |U_i| + \sqrt{gk} \right)} \rightarrow 0 < \epsilon \leq 1 \] \hspace{1cm} (21)

where \( \epsilon \) is the Courant number.

Wet dry moving boundary condition is applied by giving a minimum depth \( (h_{\text{min}}) \) to all physically dry cell. Water depth below the minimum depth will be assigned with zero momentum (dry).

3. WAVE RUN UP SIMULATION

(1) Scenario

The developed method is verified with the case of run up of non breaking wave from previous study by Synolakis.\(^2\) The run up occurs due to a solitary wave on a sloping beach or commonly known as canonical problem.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{model_sketch_setup.png}
\caption{Model sketch setup for benchmark, Synolakis\(^2\).}
\end{figure}

Several models are used to simulate this scenario. The first one is SWE Equation model using the conventional Manning method, solved using the FORCE MUSCLE scheme. The second one is also based on the conventional SWE, solved with finite difference Mac Cormack scheme. The third model is an advanced numerical model, NEWFULME, which is based on k-\(\varepsilon\) model with VOF method. The last model is the new method, coupling of SWE and k-\(\omega\) model, with FORCE MUSCLE scheme to handle breaking wave.

The model setup for the case is shown in Fig. 1. The incoming wave for the breaking wave case is given by the ratio of the incoming wave height and water depth of 0.3. The incoming solitary wave Reynolds number \( (R_e) \) is 18000, which can be calculated by the following equation

\[ R_e = \frac{U}{
u} \frac{a_m}{\nu} \] \hspace{1cm} (22)

where \( a_m \) is half of stroke of water displacement and \( v \) is viscosity. This values is below the transition condition \( (2 \times 10^5 < R_e < 5 \times 10^5) \). Hence, the condition of the incoming wave is laminar boundary layer. Nevertheless, as the incoming wave propagates, higher Reynolds number might occurs in the shallower area.

(2) Results and Discussion

Wave profile comparison between the experimental data, and numerical methods is shown in Fig. 2. The Mac Cormack method is not able to provide a realistic wave top. It produces a pointy shape wave top, followed by a near flat water surface which can be seen at \( t^* = 15 \) and \( 20 \) (marked with \[ \square \] in Fig. 2(a) and Fig. 2(b)). It also fails to give a realistic profile at the first instances of motion of a breaking wave over a dry bed (marked with \[ \square \] in Fig. 2(c)).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{wave_profile_comparison.png}
\caption{Free surface comparison.}
\end{figure}
top is somewhat lower than the measured data due to numerical dissipation. The NEW FLUME provides very realistic and accurate results. The model is also able to provide information regarding vertical velocity distribution and turbulence. However, the computation time is about 15 times longer than the SWE (Table 1). Furthermore, the VOF method does not provide an exact depth results. The VOF method requires contour plot and therefore, the free surface profile will highly depend on the contouring method and it is not possible to obtain the exact run up height.

The new method performs slightly better than the conventional SWE (FORCE MUSCL). Nevertheless, it should be noted that the new method is able to provide information on known bed stress behavior in the boundary layer, i.e. sign change.

\[
\frac{R}{h_0} = 2.831 (\cot \beta)^{1/2} \times \left( \frac{H}{h_0} \right)^{5/4}
\]

where \( R \) is the estimated run up height and \( \beta \) is the bed slope as shown in Fig. 1. The simulated run up heights are compared with the run up law. Run up height prediction using the new method is shown to have a better match with data set and run up law as shown in Fig. 3 and Table 1.

Further analyses regarding bed stress estimation (conventional method as in Eq.(2) and new method) are conducted. Surface level, free stream velocity and bed stress are recorded at two measurement points (Fig. 1). The results are shown in Fig. 4.

![Run Up Height Comparison](image)

**Fig. 3 Run up height comparison.**

**Table 1 Model performance and accuracy**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mac Cormack</th>
<th>FORCE MUSCL</th>
<th>NEW FLUME</th>
<th>New Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Governing equation</td>
<td>SWE 1D H</td>
<td>SWE 1D H</td>
<td>RANS 2D</td>
<td>SWE 1D H</td>
</tr>
<tr>
<td>dt (dx)</td>
<td>0.013 m</td>
<td>0.013 m</td>
<td>0.013 m</td>
<td>0.013 m</td>
</tr>
<tr>
<td>dy</td>
<td>N/A</td>
<td>N/A</td>
<td>0.001 m</td>
<td>0.0005 m</td>
</tr>
<tr>
<td>Real time*</td>
<td>approx. 10 minutes</td>
<td>approx. 7 minutes</td>
<td>approx. 150 minutes</td>
<td>approx. 70 minutes</td>
</tr>
<tr>
<td>Compiler*</td>
<td>MATLAB</td>
<td>FORTRAN</td>
<td>MATLAB</td>
<td>FORTRAN</td>
</tr>
<tr>
<td>Visualy</td>
<td>Adequate</td>
<td>Very good</td>
<td>Excellent</td>
<td>Very good</td>
</tr>
<tr>
<td>RMSE of run up height*</td>
<td>0.031</td>
<td>0.008</td>
<td>0.01**</td>
<td>0.007**</td>
</tr>
</tbody>
</table>

*Estimated based on measuring scale with only 2 digits accuracy

Implementation of FORCE MUSCL method gives a much better run up height comparison for SWE-type model, while the Mac Cormack accuracy seems to decrease with the increase of wave height. The run up height can be calculated using Eq.(23) as given by Synolakis\(^3\), which is commonly known as the run up law.
The Manning method estimates the bed stress as a function of square of velocity per depth and always in the same direction with velocity ($|U/U|$), as shown in Eq.(2). This relation is not always accurate, particularly for unsteady flow, as has been explained in the previous section. Based on the results, the conventional method tends to estimate lower magnitude in the deep area and higher magnitude in the shallow area. It is also observed that the new method is able to explain the shifting of bed stress peak to velocity and sign exchange, exactly where the Manning method fails since the Manning method calculates bed stress in the direction of velocity.

4. CONCLUSION

SWE model is upgraded by introducing boundary layer approach for bed stress assessment and further enhanced with the implementation of FORCE MUSCL scheme to handle breaking wave. The new method is applied to canonical problem of breaking wave run up. The results show that the implementation of shock capturing scheme has improved the accuracy of the SWE model significantly, although NEWFLUME provides a more accurate surface comparison. Nevertheless, the computation time of NEWFLUME is longer than SWE.

One of the benefits from the new method is its accurate bed stress assessment. The new method is able to reproduce phase shift between the velocity and bed stress along with sign change. The Manning method does not able to assess these phenomenons.

Overall, the proposed calculation method has shown promising results. It has the efficiency of SWE and able to assess bed stress in a more accurate way. Discussion regarding the boundary layer properties can be conducted. Furthermore, the model efficiency is suitable for practical application.

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