1. Introduction

The boundary element method (BEM) has become an efficient tool in solving many engineering problems. This method has been applied extensively also to acoustical problems including the exterior problems in both the infinite and semi-infinite acoustic medium. In the latter case, there are some specific features as compared with the standard BEM formulations. In order to exclude the integration over an infinite plane, one can use the formulations\(^{(1-3)}\) utilizing the half-space Green’s function. Seybert and Soenarko\(^{(1)}\) proposed such a formulation for the acoustic obstacle which is not in contact with the infinite plane by using the Green’s function for two extreme cases: when the infinite plane is perfectly rigid (infinite acoustical impedance or reflection coefficient \(R_H = 1\)), and when this plane is perfectly soft or free (vanishing acoustical impedance or reflection coefficient \(R_H = -1\)). For intermediate cases, such a Green’s function is a good approximation as long as the source point is at least one-half wavelength from the plane\(^{(1)}\). In the other paper, Seybert and Wu\(^{(2)}\) presented modifications for both the Helmholtz integral equation and the free-term coefficient when an acoustical obstacle is sitting on an infinite plane. They have used again in their formulation the Green’s function which satisfies the impedance boundary condition on the infinite plane only in two above mentioned extreme cases. It should be stressed that the formula for the free-term coefficient is incorrect if the Helmholtz integral equation is collocated at a point on the intersection of the surface of the obstacle with the infinite plane. They present also numerical results for several test examples to demonstrate the applicability of their formulation. In all the examples, the obstacle is sitting on a rigid, infinite plane. This is the only case, when their formula for the free-term coefficient is correct.

In this paper, we re-derive the integral representation of the velocity potential for exterior acoustic problems in a semi-infinite medium with arbitrary impedance boundary conditions. In the formulation, we employ the Green’s function\(^{(3)}\) which satisfies the boundary condition on the infinite plane only in conditions. In the formulation, we employ the Green’s function\(^{(3)}\) which satisfies the boundary condition on the infinite plane. Consequently, the integration over the infinite plane is removed exactly. Making use of the asymptotic behaviour of the Green’s function as the source and field points coincide, we can regularize the integral representation before taking the limit of the source point to any point on the boundary surface of the obstacle except the points lying on the infinite plane. Since the Green’s function has a different asymptotic behaviour when the source point lies on the infinite plane, we consider separately the case when the collocation point lies on the intersection of the boundary surface of the obstacle with the infinite plane. To preserve the
validity of the integral representation for exterior problem also in the considered point, we deform the integration surface around this point. Then, in view of the asymptotic behaviour of the kernels, we regularize such an integral representation and finally perform the limit in which the deformation disappears. In this way, we obtain the Helmholtz integral equation with correct value of the free-term coefficient also at a point on the edge contour of the open boundary surface of the obstacle sitting on an infinite plane. From the asymptotic behaviour of the approximate Green’s function, we obtain immediately also the free-term coefficient in the formulation based on the use of such a Green’s function.

2. Boundary Integral Formulation

Consider an exterior boundary value problem in a semi-infinite acoustic domain \( V \) which involves an obstacle \( B \) bounded by a surface \( S_0 \). A Cartesian coordinate system is set up on the reflecting plane \( HS \) with the acoustic domain \( V \) in the \( x_3 > 0 \) half-space (Fig.1). In the theory of linear acoustics with harmonic time dependence, the sound pressure and particle velocity are described by the velocity potential \( \phi \) governed by the Helmholtz differential equation. If a known incident wave \( \phi_i \) strikes the obstacle \( B \), the total velocity potential is given as a superposition

\[
\phi = \phi_i + \phi_r + \phi_d + \phi_s,
\]

where \( \phi_i \) represents the wave reflected from the infinite plane \( HS \) in the absence of the acoustic obstacle \( B \), while \( \phi_d \) and \( \phi_r \) are the waves diffracted and radiated by \( B \), respectively. Recall that \( \phi_i = \phi_i + \phi_s \) represents the wave that is not affected by the obstacle, but this cannot be said about the scattered wave \( \phi_s = \phi_s + \phi_r \). In general, \( \phi_s \) should be determined by solving the acoustic-structural interaction problem. For simplicity, we are now interested in a pure acoustical problem, when the interaction is replaced by prescription of boundary conditions for the total velocity potential and/or its normal derivative on the boundary surface \( S = S_a + S_H \). Denoting the acoustical impedance of the (locally reacting) surface by \( Z(x) \) and the characteristic impedance of the medium occupying \( V \) by \( z_a = \rho_a c \) (where \( \rho_a \) and \( c \) are the mass density and speed of sound, respectively), we may write the boundary conditions of the impedance type as

\[
y \phi + \frac{\partial \phi}{\partial n} = 0
\]

where \( y = ikz_a / Z \) with \( i = \sqrt{-1} \).

Furthermore the scattered field is required to satisfy the Sommerfeld radiation condition

\[
\lim_{r \to \infty} \left( \frac{\partial \phi}{\partial n} + ik \phi \right) = 0.
\]

Denoting by \( G(x,y) \) a Green’s function obeying the Helmholtz differential equation

\[
\nabla^2 G + k^2 G = -4\pi \delta(x - y).
\]

we may write the integral representation of the total velocity potential as

\[
\Delta(y)\phi(y) = \int_{S_a} \left[ \frac{\partial \phi}{\partial n}(\eta)G(\eta, y) - \phi_i(\eta)\frac{\partial G(\eta, y)}{\partial n(\eta)} \right] dS(\eta),
\]

in which \( \eta \) is the unit outward normal vector with respect to \( V \). \( S_a \) is a surface spanned above \( S_H \) at infinity with \( y > 0 \), and

\[
\Delta(y) = \begin{cases} 4\pi, & y \in V \\ 0, & y \in B \end{cases}.
\]

Recall that it is not a trivial task to perform the limit of \( \nu(y)G(y) \) to a boundary point because of the singular behaviour of the normal derivative of the fundamental solution. Before studying this question it is appropriate to make some simplifications.

Bearing in mind the Sommerfeld radiation conditions for the scattered field, one can see that

\[
\int_{S_H} \left( \frac{\partial \phi}{\partial n} G - \phi \frac{\partial G}{\partial n} \right) dS = 0.
\]

Note that \( \phi_i \) and \( \phi_r \) obey the same physical boundary conditions on \( S_H \). Assuming these boundary conditions to be satisfied also by the Green’s function \( G \), one obtains

\[
\int_{S_H} \left( \frac{\partial \phi}{\partial n} G - \phi \frac{\partial G}{\partial n} \right) dS = 0 \quad (7a)
\]

and

\[
\int_{S_H} \left( \frac{\partial \phi}{\partial n} G - \phi \frac{\partial G}{\partial n} \right) dS = 0 \quad (7b)
\]
Now, in view of Eq. (7b) and the Gauss divergence theorem, we may write

$$\int_{S_+} \left( \frac{\partial \phi}{\partial n} G - \phi_i \frac{\partial G}{\partial n} \right) dS = \int_{S_+ + S_+} \left( \frac{\partial \phi}{\partial n} G - \phi_i \frac{\partial G}{\partial n} \right) dS$$

$$= \int_{S_+ + S_+} \left[ G(\nabla^2 + k^2)\phi_i - \phi_i (\nabla^2 + k^2)G \right] dV,$$

(8)

because $S_+ + S_+$ is the boundary of the half-space $V + B + S_+$, i.e. the acoustic domain without any obstacle. Since $\phi_i$ and $G$ obey the homogeneous Helmholtz equation and Eq. (3), respectively, we obtain from (8) the identity

$$\int_{S_+} \left[ \frac{\partial \phi_i}{\partial n}(\eta)G(\eta,y) - \phi_i(\eta) \frac{\partial G(\eta,y)}{\partial n(\eta)} \right] dS(\eta) = 4\pi \phi_i(y)$$

for $y > 0$.

(9)

Finally, in view of Eqs. (6), (7a) and (9), the integral representation (4) becomes

$$\Delta(y)\phi_i(y) = 4\pi \phi_i(y) + \int_{S_+} \left[ \frac{\partial \phi_i}{\partial n}(\eta)G(\eta,y) - \phi_i(\eta) \frac{\partial G(\eta,y)}{\partial n(\eta)} \right] dS(\eta)$$

$$y \in V \cup B.$$  

(10)

In the case of an obstacle sitting on the infinite plane $y_1 = 0$ (Fig. 2), the total boundary of the acoustic domain $V$ is given again by $S_+ + S_+ + S_+$ with $S_+$ being obtained from the infinite plane by cutting a finite part $S$. Thus, Eqs. (4)-(7a) are still valid while in Eqs. (7b) and (8) one should replace $S_+$ by $S_+ + S$. Eventually, one can see that Eq. (10) is valid also if the obstacle is sitting on the infinite plane.

Concluding, the derived integral representation (10) involves the integration over $S_+$ despite the fact that the real physical boundary surface of the acoustic domain $V$ is given by $S_+ + S_+$ ($+S_+$). The integration over $S_+$ disappears because of the assumption that the Green’s function satisfies the boundary condition

$$\gamma H G(\eta,y) + \frac{\partial G(\eta,y)}{\partial n(\eta)} = 0, \eta \in S_+, \ y > 0.$$  

(11)

Such a Green’s function is available when $S_+$ is an infinite plane and $\gamma_H = kZ / \omega H = \text{const.}$. Note that this Green’s function can be written as

$$G = G_0 + \text{non-singular terms depending on } \gamma_H$$  

(12)

where according to Fig.1,

$$G_i(x,y) = \frac{1}{r} e^{-\omega r} + \frac{1}{r} e^{-\omega r},$$

$$r = |x - y|, \ r' = |x' - y|, \ x = x - 2\chi_0 \delta_3,$$

(13)

is the half-space Green’s function obeying the perfectly rigid reflection condition on $S_H$.

The confinement of the integration to $S_+$ in (10) yields some important consequences. Firstly, one can take the limit of $y$ to $S_+$, i.e. $y_1 \rightarrow 0$, in Eq. (10) directly provided that $y \not\in S$, because there are no difficulties owing to singularities. Thus, one can write

$$\bar{\Delta}(y)\phi_i(y) = 4\pi \phi_i(y) + \int_{S_+} \left[ \frac{\partial \phi_i}{\partial n}(\eta)G(\eta,y) - \phi_i(\eta) \frac{\partial G(\eta,y)}{\partial n(\eta)} \right] dS(\eta)$$

with $\bar{\Delta}(y) = \begin{cases} \frac{4\pi}{r}, & y \in V \cup S_+ \setminus S_+ \\ 0, & y \in B \end{cases}.$

(14)

Secondly, the unknowns are localized on $S_+$ and it is...
sufficient to discretize only $S_o$. In order to compute both the densities $\phi$ and $\partial\phi/\partial n$ on $S_o$, one should supplement the boundary condition by another equation that can be obtained from Eq. (14) by taking the limit of the domain point $y$ to a boundary point $\zeta \in S_o$. For this purpose, it is necessary to know the asymptotic behaviour of the integral kernels as $|y - \zeta| \to 0$, because the integrals with strongly singular kernels are not continuous across the boundary. For this purpose, it is necessary to know the asymptotic behaviour of the integral kernels as $0 \to \eta$, because the integrals with strongly singular kernels are not continuous across the boundary.

Firstly, consider the case when $H \not\subset \\Delta \mathcal{S} \cup \mathcal{S}$. Then, according to Eqs. (12) and (13), one can see that $G$ behaves asymptotically like $r^g/1 = r^g$, since $0 \neq *r$ for any $\zeta \in \mathcal{S}$. The properties of the fundamental solution of the Laplace operator are well known, including the integral identity

$$\int_{\zeta} \left[ \frac{\partial (\eta - y)}{\partial n(\eta)} \right] dS(\eta) = \begin{cases} 0, & y \in V \cup S_H \setminus S_o, \\ 4\pi, & y \in B \end{cases}$$

(15)

Hence and from the definition of $\overline{\mathcal{I}}(y)$, we have

$$\int_{\zeta} \frac{\partial (\eta - y)}{\partial n(\eta)} dS(\eta) = 4\pi - \overline{\mathcal{I}}(y), \quad y \in V \cup B \cup S_H \setminus S_o.$$  

(16)

Since $G$ is weakly singular, the integral of $G(\eta, y)$ is continuous across the boundary $S_o$, and the integral of $\partial G/\partial n$ can be rearranged by subtraction and addition technique in such a form that all the integrals would be continuous across $S_o$ too. Thus, we have from Eqs. (14) and (16),

$$\tilde{c}(y) \phi(y) = 4\pi \phi(y) + \int_{\zeta} \frac{\partial (\eta - y)}{\partial n(\eta)} G(\eta, y) dS(\eta) - \int_{\zeta} \left[ 2 (\eta - y) \frac{\partial G(y, \eta)}{\partial n(\eta)} \right] dS(\eta)$$

(17)

where $y \in B \cup V \cup S_H \setminus S_o$.

$$\tilde{c}(y) = 4\pi + \int_{\zeta} \left[ \frac{\partial G(\eta, y)}{\partial n(\eta)} - \frac{\partial G(y, \eta)}{\partial n(\eta)} \right] dS(\eta).$$

(18)

Assuming $\phi$ to be Hölder continuous, all the integrals in (17) and (18) are continuous across $S_o$, and one can employ these equations for any $y \in B \cup V \cup S_H \cup S_o \setminus (S_o \cap S_H)$.

Note that Eqs. (17) and (18) can be further rearranged because of the orthogonality $r, n(\eta)$ at $\eta = \zeta \in S_o$. Hence, the integrand involving $\partial G/\partial n$ is bounded even on singular elements, and we may write

$$c(y) \phi(y) = 4\pi \int_{\zeta} \left[ \frac{\partial G(y, \eta)}{\partial n(\eta)} \right] dS(\eta)$$

(19)

for any $y \in B \cup V \cup S_H \setminus S_o \setminus (S_o \cap S_H)$, with

$$c(y) = 4\pi - \int_{\zeta} \frac{\partial (\eta - y)}{\partial n(\eta)} dS(\eta).$$

(20)

According to Eqs. (16) and (20), one can see that $c(y) = \overline{\mathcal{I}}(y)$ for $y \in B \cup V \cup S_H \setminus S_o \setminus (S_o \cap S_H)$ (Eq. (14) is reproduced), while for $y = \zeta \in S_o$ the integral in Eq. (20) is equal to the solid angle $\Omega(\zeta)$ made by the tangent planes to $S_o$ at $\zeta$. Thus, the free-term coefficient can be rewritten as

$$c(y) = \begin{cases} \overline{\mathcal{I}}(y), & y \in B \cup V \cup S_H \setminus S_o, \\ 4\pi - \Omega(\zeta), & y = \zeta \in S_o \setminus S_H. \end{cases}$$

(21)

If the solid angle is not known a priori, it can be computed numerically by

$$\Omega(\zeta) = \int_{S_S(\zeta)} \left[ \frac{\partial (\eta - \zeta)}{\partial n(\eta)} \right] dS(\eta)$$

(22)

where $S_S(\zeta)$ is a finite patch which is cut on the unit spherical surface centered at $\zeta$ by straight rays outgoing from $\zeta$ and intersecting $S_o$ (or $S_o \cup S_o$, in the case of the obstacle sitting on the infinite plane). The normal vector on $S_S(\zeta)$ is oriented toward $\zeta$.

It remains to investigate the limit in Eq. (14) as $y \to \zeta \in S_o \setminus S_H$. Since $\zeta(\eta) = 0$, we have $r = |\eta - \zeta(\eta)| = |\eta - \zeta(\eta)| = r$ and

$$G_o(\eta, \zeta(\eta)) = \frac{2}{r} e^{-i \omega r}, r = |\eta - \zeta(\eta)|.$$  

(23)

Hence, $G_o(r)$ behaves asymptotically like $2g(r)$ as $r \to 0$. 

In order to investigate the limit behaviour of singular integrals in integral representation as \( y \to \zeta_H \), one could put usually \( y = \zeta_H \) and deform the domain \( B \) into \( B_\varepsilon \) as \( \varepsilon \to 0 \). The original boundary \( S_0 \cup S_1 \) of the domain \( B \) is deformed into \( (S_0 - E_1) + \Gamma_{s} + (S_1 - C) \) and \((S_0 - E_1) + S_1 + \gamma_1 + (S_1 - C)\), respectively, in the first and second approach. Since any integral representation is valid if \( \gamma_i \geq 0 \), only the first deformation of the domain \( B \) is admissible. Thus, we have, from Eq. (14),

\[
4 \pi \phi_y(\zeta_H) = 4 \pi \phi_y(\zeta_H) + \int_{s_{\zeta_H} - E_1} \left[ \frac{\partial \phi_y(\eta) G(\eta, \zeta_H)}{\partial n(\eta)} - \phi_y(\eta) \frac{\partial G(\eta, \zeta_H)}{\partial n(\eta)} \right] dS(\eta) .
\]

(24)

Recall that the integral of the weakly-singular kernel \( G \) is continuous in the limit \( \varepsilon \to 0 \). The limit of the integral of the second term should be evaluated carefully. Since \( \partial G / \partial n \) is nonsingular at \( \zeta_H \), we may write

\[
\lim_{\varepsilon \to 0} \int_{s_{\zeta_H} - E_1} \phi_y(\eta) \frac{\partial G(\eta, \zeta_H)}{\partial n(\eta)} dS(\eta) = \int_{s_{\zeta_H}} \phi_y(\eta) \frac{\partial G(\eta, \zeta_H)}{\partial n(\eta)} dS(\eta) + \lim_{\varepsilon \to 0} \int_{s_{\zeta_H}} \phi_y(\eta) \frac{\partial G(\eta, \zeta_H)}{\partial n(\eta)} dS(\eta) ,
\]

(25)

where the first integral on the r.h.s. exists in the ordinary sense and the second integral gives a finite contribution. Assuming \( \phi_y \) to be Hölder continuous at \( \zeta_H \), one obtains

\[
\lim_{\varepsilon \to 0} \int_{s_{\zeta_H}} \phi_y(\eta) \frac{\partial G(\eta, \zeta_H)}{\partial n(\eta)} dS(\eta) = \phi_y(\zeta_H) \lim_{\varepsilon \to 0} \int_{s_{\zeta_H}} \frac{\partial G(\eta, \zeta_H)}{\partial n(\eta)} dS(\eta)
\]

hence, in view of Eq. (23),

\[
\lim_{\varepsilon \to 0} \int_{\Gamma_{s_{\zeta_H}}} \frac{\partial g(\eta - \zeta_H)}{\partial n(\eta)} dS(\eta) = 2 \phi_y(\zeta_H) \lim_{\varepsilon \to 0} \int_{s_{\zeta_H}} \frac{\partial g(\eta - \zeta_H)}{\partial n(\eta)} dS(\eta) .
\]

(26)

Apparently,

\[
\lim_{\varepsilon \to 0} \int_{s_{\zeta_H}} \frac{\partial g(\eta - \zeta_H)}{\partial n(\eta)} dS(\eta) = - \int_{s_{\zeta_H}} \frac{\partial g(\eta - \zeta_H)}{\partial n(\eta)} dS(\eta) = - \Omega(\zeta_H)
\]

(27)

Finally, in view of Eqs. (24)-(27), one can obtain the integral equation,

\[
c(\zeta_H) \phi_y(\zeta_H) + \int_{s_{\zeta_H}} \left[ \phi_y(\eta) \frac{\partial G(\eta, \zeta_H)}{\partial n(\eta)} - \frac{\partial \phi_y(\eta)}{\partial n} G(\eta, \zeta_H) \right] dS(\eta) = 4 \pi \phi_y(\zeta_H)
\]

(28)

with

\[
c(\zeta_H) = 4 \pi - 2 \Omega(\zeta_H)
\]

(29)

Comparing this result with that of Seybert and Wu (2), one can see a discrepancy. Seybert and Wu (2) employed in their formulation the Green's function

\[
\psi_H = \frac{1}{r} e^{-\kappa r} + \frac{R_H}{r^2} e^{-\kappa r}
\]
which satisfies the impedance boundary condition on $S_H$ only if $R_H = \pm 1$. If we used $\psi_H$ instead of $G$ (as an approximate Green’s function), Eq. (29) should be replaced by

$$c(\zeta_H) = 4\pi - (1 + R_H)\Omega(\zeta_H)$$

(30)

because $\psi_H(\eta, \zeta_H)$ behaves asymptotically like $(1 + R_H)g(r)$ as $r = |\eta - \zeta_H| \to 0$. Note that the expression given by Eq. (30) coincides with the expression presented by Seybert and Wu(2),

$$c = (1 + R_H) \left[ 2\pi - \int_{S_+} \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS \right]$$

(31)

only if $R_H = 1$. Thus, the reason of the discrepancy in formulas (29) and (31) is not in the utilization of an approximate Green’s function. The explanation consists in the fact that replacement of the integration surface $S_o$ by $S_+ - E_{\gamma} + S_{\gamma}$ is not correct. One can replace the closed surface $S_+ - E_{\gamma} + S_{\gamma}$ by another closed surface $(S_+ - E_{\gamma}) + S_{\gamma} + L_{\gamma} + S_{\gamma}$ (Fig. 5), but there is no gain because $\zeta_H$ lies on both the surfaces. Consideration of $\zeta_H$ as an interior point of $B$ would require deformation of $B$ and $S_o + S_{\gamma}$ as shown in Fig. 4, i.e. $S_o + S_{\gamma}$ should be replaced by $(S_+ - E_{\gamma}) + S_{\gamma} + \gamma_{\gamma} + (S_{\gamma} - C_{\gamma})$. Such a deformation, however, is not admissible as discussed above.

Summarizing our results for an exterior acoustic problem in a semi-infinite medium, the integral representation of the velocity potential and the Helmholtz integral equations are given by Eq. (19), with

$$c(y) = \begin{cases} \delta(y), & y \in B \cup V \cup S_H - L_H \\ 4\pi - \Omega(\zeta_H), & y = \zeta_H \in S_H - L_H \\ 4\pi - 2\Omega(\zeta_H), & y = \zeta_H \in L_H \end{cases}$$

(32)

where $L_H = S_o \cap S_H$ is either the edge contour of the open surface $S_o$ or $L_H$ is empty, if $S_o$ is closed. If the Green’s function is approximated by $\psi_H$ (as has been done by Seybert and Wu(2)), the term $2\Omega(\zeta_H)$ in Eq. (32) should be replaced by $(1 + R_H)\Omega(\zeta_H)$.

3. Conclusions

A revision of the free-term coefficient in the Helmholtz integral equation is given for acoustic problems associated with bodies sitting on an infinite plane. Both the integral representation of the velocity potential and the Helmholtz integral equation are written in a unique formula. The free-term coefficients are given for the formulation utilizing either the exact half-space Green’s function or its approximation.

References

