In this paper, we propose a Dirichlet Neumann domain decomposition finite element/boundary element coupling method with initial Dirichlet data assumed on the FEM/BEM interface approached from the FEM sub-domain. The method accounts for situations where the Neumann boundary conditions are specified on the entire external boundary of the FEM sub-domain. We also investigate the convergence of the method. The theoretical analysis provides an interval from which the relaxation parameter has to be chosen in order to achieve convergence.

**Keywords**: Boundary Element Method; Finite Element Method; Domain Decomposition; Coupling.

## 1. INTRODUCTION

Coupling the boundary element method (BEM) and the finite element method (FEM) is an effective analysis tool, which makes use of their individual merits. Conventionally, coupling is achieved by employing an entire unified equation for the whole domain by altering the formulation of one of the methods to make it compatible with the other. More recently, coupling the BEM and FEM has been achieved through domain decomposition and interface relaxation methods. The advantage is that there will be no need to combine the coefficient matrices of the FEM and the BEM sub-domains, as is required in the conventional coupling methods. Separate computing for each sub-domain and successive renewal of the variables on the interface of the both sub-domains are performed to reach the final convergence.

Gerstle et al. [1] presented a solution method, which is iterative in nature. The sub-domains are analyzed independently by applying trial displacements to degrees of freedom on the interface. The conjugate gradient domain decomposition solver is used to predict a new set of trial interfacial displacements for the next iteration. Their method, however, is only applicable to symmetric BEM formulation. Perera et al. [2] presented a parallel method based on the interface equilibrium of Steklov-Poincare. Kamiya et al. [3] employed the renewal methods known as Schwarz Neumann-Neumann and Schwarz Dirichlet-Neumann methods. It should be noted, however, that the above methods presented in [2,3] are not applicable for problems where Neumann boundary conditions are specified on the entire external boundary of the FEM sub-domain. Kamiya and Iwase [4] introduced an iterative analysis using conjugate gradient and condensation, which again has the same limitation of being applicable only to symmetric BEM formulation. Lin et al. [5] and Feng and Owen [6] presented a method similar to the Schwarz Dirichlet-Neumann method. The method is based on assigning an arbitrary displacement vector to the interface of the BEM sub-domain. Then, the energy equivalent nodal forces of the obtained interface tractions are treated as boundary conditions for the FEM sub-domain to solve for the interfacial displacements. The procedure is iterated until convergence is achieved. Elleithy and Al-Gahtani [7] presented an overlapping domain decomposition method for coupling the FEM and BEM. The domain of the original problem is subdivided into a FEM sub-domain, a BEM sub-domain, and a common region, which is modeled by both methods. The method overcomes situations where the Neumann boundary conditions are specified on the entire external boundary of the FEM sub-domain. Elleithy et al. [8,9] investigated the convergence of the Dirichlet Neumann domain decomposition coupling method.

To summarize, coupling the FEM and BEM through the available Dirichlet Neumann domain decomposition coupling methods may not be suited for certain problems, where the Neumann boundary conditions are specified on the entire external boundary of the FEM sub-domain. In this work, we apply the initial Dirichlet boundary data on the FEM/BEM interface (approached from the FEM sub-domain). We also investigate the convergence of the method and determine the factors, which affect solution convergence. For cases where the Neumann boundary conditions are specified on the entire external boundary of the BEM sub-domain, one may use the method presented in references [5,6].

## 2. DOMAIN DECOMPOSITION COUPLING METHOD

Consider the two 2-D regions of Figure 1, which are governed by Laplace equation, i.e., \( K \nabla^2 u = 0 \) in \( \Omega \).
where $K_i$ is the material property in the sub-domain $\Omega_i$ and $u$ is the potential. Boundary conditions are such that the potential $u$, the flux $q = K\nabla u$ or their combination is prescribed at each point on the boundary. The decomposed portions are modeled using the BEM and FEM. The corresponding boundary integral equation for the BEM sub-domain is given by:

$$[H][u] = [G][q] \in \Gamma_b$$  \hspace{1cm} (1)

where $u$ and $q$ are column matrices containing the boundary nodal values for the potential and the flux. $H$ and $G$ are influence coefficient matrices. For the FEM sub-domain, the assembled element equations are given by:

$$[K][u] = [f] \in \Omega_f$$  \hspace{1cm} (2)

where $K$ is the stiffness matrix for the system, and $u$ and $f$ are the nodal potentials and integrated flux vectors respectively. Now, let us define the following potential vectors:

- $u^I_b$: interface potentials, approached from the BEM sub-domain
- $u^I_F$: non-interface known potentials in the BEM sub-domain
- $u^U_F$: non-interface unknown potentials in the BEM sub-domain
- $u^I_F$: interface potentials, approached from the FEM sub-domain
- $u^U_F$: non-interface known potentials in the FEM sub-domain
- $u^U_F$: non-interface unknown potentials in the FEM sub-domain

Equations (1) and (2) may be partitioned as follows:

$$
\begin{bmatrix}
H_{11} & H_{12} & H_{13} \\
H_{21} & H_{22} & H_{23} \\
H_{31} & H_{32} & H_{33}
\end{bmatrix}
\begin{bmatrix}
u^I_b \\
u^U_b \\
u^U_F
\end{bmatrix} =
\begin{bmatrix}
G_{11} & G_{12} & G_{13} \\
G_{21} & G_{22} & G_{23} \\
G_{31} & G_{32} & G_{33}
\end{bmatrix}
\begin{bmatrix}
u^I_b \\
u^U_b \\
u^U_F
\end{bmatrix}
$$

(3)

and

$$
\begin{bmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{bmatrix}
\begin{bmatrix}
u^I_F \\
u^U_F \\
u^U_F
\end{bmatrix} =
\begin{bmatrix}
f^I_F \\
f^U_F \\
f^U_F
\end{bmatrix}
$$

(4)

At the interface, the compatibility and equilibrium conditions should be satisfied, i.e.,

$$
\begin{bmatrix}
{u^I_b} \\
{f^I_F} \end{bmatrix} + [M]\{q^I_b\} = 0 \in \Gamma_i$$  \hspace{1cm} (5)

$$
\begin{bmatrix}
{u^I_F} \\
{q^I_b} \end{bmatrix} \in \Gamma_i$$  \hspace{1cm} (6)

where, $M$ is the converting matrix, which depends on the interpolation functions used to represent the flux on the interface. The Dirichlet-Neumann domain decomposition coupling method with the initial Dirichlet data assumed on the FEM/BEM interface (approached from the FEM sub-domain) may be described as follows:

Set initial values $u^I_{F,0} = \overline{u}$.

For $n = 0, 1, 2, \ldots$ do until convergence

- Solve Equation (4) and get $f^{I,n}$
- Solve Equation (6) and get $q^{I,n}$
- Solve Equation (3) and get $u^I_{F,n}$

Apply $u^I_{F,n+1} = (1 - \alpha) u^I_{F,n} + \alpha u^I_{F,n}$

where $\alpha$ is a relaxation parameter to ensure and/or accelerate convergence.

3. CONVERGENCE OF THE COUPLING METHOD

In this section we will investigate the convergence of the domain decomposition coupling method depicted in the previous section. We will show the conditions under which the iterations,

$$u^I_{F,n+1} = (1 - \alpha) u^I_{F,n} + \alpha u^I_{F,n}$$  \hspace{1cm} (7)

will converge to the true value of $u^I$.

Eliminating the unknowns $(u^U_b, q^U_b)$ and rearranging, Equation (3) can be written as:

$$
\begin{bmatrix}
H_{33} - [-G_{31} & H_{32}]^{-1} & [-G_{11} & H_{12}]^{-1}H_{13} \\
G_{33} - [-G_{31} & H_{32}]^{-1}G_{13} \\
-H_{31} & G_{32}
\end{bmatrix}
\begin{bmatrix}
u^I_F \\
u^U_F \\
u^U_F
\end{bmatrix} =
\begin{bmatrix}
H_{21} & H_{22} & H_{23}
\end{bmatrix}
\begin{bmatrix}
u^I_b \\
u^U_b \\
u^U_F
\end{bmatrix}
$$

(8)

Figure 1: Domain Decomposed into Finite and Boundary Element Sub-domains
\[
\begin{bmatrix}
K_{33} - K_{32}K_{22}^{-1}K_{23}
\end{bmatrix}
\begin{bmatrix}
u_k
\end{bmatrix}
= 
\begin{bmatrix}
\{f_F^k
\}
\end{bmatrix}
+ 
\begin{bmatrix}
K_{32}K_{22}^{-1}K_{21} - K_{31}
\end{bmatrix}
\begin{bmatrix}
u_{F^k}
\end{bmatrix}
= \begin{bmatrix}
K_{32}K_{22}^{-1}K_{23}
\end{bmatrix}
\begin{bmatrix}
u_F^k
\end{bmatrix}
\quad (9)
\]

Eliminating \(q_F^k\) from Equations (8) and (9) using Equation (6) gives:

\[u_F^k = C u_{F^k} + c\quad (10)\]

where,

\[C = 
\begin{bmatrix}
H_{33} - [G_{31} & H_{32}]^{-1} [G_{11} & H_{12} & H_{13}]^{-1} [H_{13}]
\end{bmatrix}
\]

\[M^{-1} \begin{bmatrix}
K_{33} - K_{32}K_{22}^{-1}K_{23}
\end{bmatrix}\]

and

\[c = 
\begin{bmatrix}
H_{33} - [G_{31} & H_{32}]^{-1} [G_{11} & H_{12} & H_{13}]^{-1} [H_{13}]
\end{bmatrix}
\]

\[\begin{bmatrix}
G_{33} - [G_{31} & H_{32}]^{-1} & G_{11} & H_{12} & H_{13} & [G_{13}]
\end{bmatrix}^{-1}
\]

\[M^{-1} \begin{bmatrix}
K_{32}K_{22}^{-1}K_{21} - K_{31}
\end{bmatrix}\begin{bmatrix}
u_F^k
\end{bmatrix}
= \begin{bmatrix}
K_{32}K_{22}^{-1}K_{23}
\end{bmatrix}\begin{bmatrix}
u_F^k
\end{bmatrix}
\]

\[+ \begin{bmatrix}
-H_{11} & G_{12}
-H_{21} & G_{22}
\end{bmatrix}\begin{bmatrix}
\begin{bmatrix}u_F^{k}
\end{bmatrix}
\end{bmatrix}
\]

Substituting for \(u_{F^k}\) in Equation (7), using Equation (10), we get:

\[u_{F^k}^{1} = \left(1 - \alpha \right) I + \alpha C \begin{bmatrix}u_{F^k}
\end{bmatrix} + \alpha c\quad (11)\]

Now, Equation (11) is an iterative method of the form:

\[X_{n+1} = D_n X_n + d\quad (12)\]

which converges if and only if the set of eigenvalues \(\sigma(D_n)\) of the matrix \(D_n\) are contained in the unit ball \(B(0,1)\) in the complex plane [10].

Following the same procedures given in a previous investigation [8,9], one may conclude that the domain decomposition coupling method will converge if \(x_i < 1\), \(i = 1,2,\ldots,N\) and if we choose:

\[\alpha < \min_{i=1,N} \left( \frac{2(1-x_i)}{(1-x_i)^2 + y_i^2} \right)\quad (13)\]

where, \(\lambda_1 = x_1 + iy_1, \ldots, \lambda_N = x_N + iy_N\) are the eigenvalues of \(D\).

Equation (13) gives the upper limits for the choosing the parameter \(\alpha\) in order to achieve solution convergence. Reexamining Equation (11) and by setting \(\alpha = 0\), one can conclude that the coupling method will not converge, as \(D_n\) will be equal to the identity matrix \(I\) and the eigenvalues of the iteration matrix will not be contained in the unit ball \(B\). In order to achieve convergence one has to assign \(\alpha\) a value greater than zero and also has to satisfy Equation (13) for an upper limit of \(\alpha\).

The optimum \(\alpha\) is obtained as:

\[\alpha = -\frac{\text{Re}(I' (\lambda - 1))}{\|I'\|^2}\quad (14)\]

It is interesting to note that in order to assure convergence, Equation (13) should be satisfied, while selecting \(\alpha\). It is dependent on the eigenvalues of matrix the \(C\), which in turn is dependent on \(K, H, G\) and \(M\) matrices. These coefficient matrices are dependent on the material and geometrical properties of the computational sub-domains. Moreover, the conditions for convergence for the Dirichlet Neumann domain decomposition coupling method with the initial Dirichlet data assumed on the FEM/BEM interface (approached from the FEM sub-domain) are different than those given in references [8,9] for the same method but with the initial Dirichlet boundary conditions assumed on the FEM/BEM interface (approached from the BEM sub-domain).

### 4. NUMERICAL EXAMPLE

In this section, a simple numerical example is given to illustrate the issues presented in the Section 3. It should be noted that the BEM/FEM coupling approach is versatile and capable of handling more complex problems than the example presented in this section.

Consider the case of uni-dimensional potential flow in a rectangular domain as shown in Figure 2. The two domains \(\Omega_s\) and \(\Omega_f\) are governed by Laplace equation. The rectangular domain is decomposed into the FEM and BEM sub-domains with \(0 \leq x \leq a\) and \(0 \leq y \leq b\). The boundary conditions are selected such that \(u(0, y) = 0\), \(u(0, y) = 200\) and zero flux elsewhere. The problem is investigated for different values of \(a_s/a_f\) and \(K_s/K_f\) (see, i.e., Figure 2 which shows the discretization for \(a_s/a_f = 1\)). For \(a_s/a_f = 1\), the problem is modeled using 18 linear boundary elements and 40 linear triangular finite elements.

Table 1 shows the optimum value and applicable range of \(\alpha\) determined experimentally for different combinations of \(a_s/a_f\) and \(K_s/K_f\). The upper limits of the parameter \(\alpha\) and the optimum values given by Table 1.
were found to be in good agreement with those of Equations (13) and (14).

The same problem is reinvestigated with Neumann boundary conditions specified on the entire external boundary of the FEM sub-domain as shown in Figure 3. The problem is solved for $\sigma_b/\sigma_F = 1$ and $K_b/K_F = 1$. The results compare well with the theoretical solution and the range of the relaxation parameter is found to be 0.02-1.98. It should be noted that this simple problem are not solvable using coupling method presented in [5,6].

---

Table 1: Applicable Range and Optimum $\alpha$ for Different Values of $\sigma_b/\sigma_F$ and $K_b/K_F$

<table>
<thead>
<tr>
<th>$\sigma_b/\sigma_F$</th>
<th>$K_b/K_F$</th>
<th>Range</th>
<th>Optimum</th>
<th>Optimum</th>
<th>Optimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>1.0</td>
<td>0.02-1.42</td>
<td>0.02-1.66</td>
<td>0.02-1.80</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.7</td>
<td>0.02-0.62</td>
<td>0.02-0.98</td>
<td>0.02-1.32</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>0.34</td>
<td>0.02-0.22</td>
<td>0.02-0.40</td>
<td>0.02-0.66</td>
<td></td>
</tr>
</tbody>
</table>

---

**5. CONCLUSIONS**

A Dirichlet Neumann domain decomposition finite element/boundary element coupling method is presented in this paper. The method is different from the existing ones as it applies the initial Dirichlet data on the FEM/BEM interface but approached from the FEM sub-domain, to account for situations where the Neumann boundary conditions are specified on the entire external boundary of the FEM sub-domain. We also investigated the convergence of the method and determined the factors, which affect solution convergence. The present study will be extended to consider problems having complex geometry and material properties.

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**REFERENCES**


