Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme

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We apply a newly developed characteristic-curve finite element scheme to cavity flow problems. The scheme is useful for large scale computation, because P1/P1 element is employed and the matrix of resulting linear system is symmetric. Numerical results of two- and three-dimensional cavity flow problems are presented. Three types of the Dirichlet boundary condition, discontinuous, \( C^0 \) and \( C^1 \) continuous ones, are treated, and the difference of the solutions is discussed.

Key Words: finite element method, characteristic-curve, pressure stabilization, the Navier-Stokes equations, cavity flow problem

1. Introduction

In this paper we present numerical results of cavity flow problems by a newly developed characteristic-curve finite element scheme\(^{(19)}\). The classical cavity flow problem, whose Dirichlet boundary condition is given by a discontinuous function, is well known as a benchmark one for incompressible fluid flows. Many authors solve the problem, such as Crucchaga and Oñate\(^{(5)}\), Ghia et al.\(^{(9)}\), Kondo et al.\(^{(14)}\), Nallasamy and Prasad\(^{(17)}\), Tabata and Fujima\(^{(25)}\) in 2D, Fujima et al.\(^{(7)}\), Iwatsu et al.\(^{(12)}\), Jiang et al.\(^{(13)}\), Ku et al.\(^{(15)}\) in 3D, and so on. We compute the problem in 2D too.

However, we have some doubt on solving the classical cavity flow problem, because the problem has no weak solution. Therefore, we also compute two other cavity flow problems in 2D and 3D, which are regularized by \( C^0 \) and \( C^1 \) continuous functions to be used for the Dirichlet boundary condition.

The characteristic-curve method is based on an approximation of the material derivative along the trajectory of the fluid particle, and is natural from the physical point of view. The method has an advantage that the matrix for the system of linear equations is symmetric, which leads to symmetric linear solvers.

Several characteristic-curve finite element schemes for the Navier-Stokes equations of first and second order in time have been developed by Boukir et al.\(^{(2)}\), the author and Tabata\(^{(18)}\), Pironneau\(^{(20),(21)}\) and Süli\(^{(23)}\). These schemes impose the inf-sup condition\(^{(4),(10)}\) for the finite elements to be used, e.g., P2/P1 element, which requires large memory.

Recently a pressure stabilized characteristic-curve finite element scheme for the Navier-Stokes equations has been proposed by the author and Tabata\(^{(19)}\). The scheme employs a cheap element P1/P1, i.e., velocity and pressure are both approximated by the piecewise linear elements in triangles (2D) or tetrahedra (3D). The P1/P1 element is useful especially in three-dimensional computation. Since the P1/P1 element does not satisfy the inf-sup condition, a pressure stabilization method by Brezzi and Douglas Jr.\(^{(3)}\) is used. The scheme is an implicit and mixed one. It has been shown that the numerical convergence order to an exact solution is first in both time and space in the paper\(^{(19)}\). The scheme has such advantages that the matrix is symmetric and that it is useful for large scale computation. Considering to find a stationary solution of the nonstationary Navier-Stokes equations, we apply the scheme to cavity flow problems.

For a domain \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) we use the Sobolev spaces \( \mathbb{L}^2(\Omega) \) and \( \mathbb{H}^1(\Omega) \), and their subspace

\[
\mathbb{L}^2_0(\Omega) = \left\{ q \in \mathbb{L}^2(\Omega); \int_{\Omega} q \, dx = 0 \right\}.
\]

We denote by \( \langle \cdot, \cdot \rangle \) the \( \mathbb{L}^2(\Omega) \)-inner products in the scalar-, vector-, and matrix-valued function spaces, by \( \| \cdot \|_0 \) their norms and by \( \| \cdot \|_1 \) the norm in \( \mathbb{H}^1(\Omega)^d \). The dual pairing between a space \( X \) and the dual space \( X' \) is denoted by \( \langle \cdot, \cdot \rangle \).

The outline of this paper is as follows. We set cavity flow problems in Section 2. A pressure stabilized characteristic-curve finite element scheme for the Navier-Stokes equations is reviewed in Section 3. In Section 4 we show numerical results of two- and three-dimensional cavity flow problems.

2. Cavity flow problems

Let \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) be a bounded domain and \( \Gamma \equiv \partial \Omega \) be the boundary of \( \Omega \). We consider the stationary Navier-Stokes problem subject to the Dirichlet boundary condition; find \((u, p):
\( \Omega \rightarrow \mathbb{R}^d \times \mathbb{R} \) such that
\[
\begin{cases}
(u, \nabla)u - \frac{2}{Re} \nabla D(u) + \nabla p = f & \text{in } \Omega, \\
\nabla \cdot u = 0 & \text{in } \Omega, \\
u = g & \text{on } \Gamma,
\end{cases}
\]
where \( u \) is the velocity, \( p \) is the pressure, \( Re \) is the Reynolds number, \( f \) is an external force, \( D(u) \) is the strain-rate tensor defined by
\[
D_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (i, j = 1, \ldots, d)
\]
and
\[
[\nabla D(u)]_{ij} = \sum_{j=1}^{d} \frac{\partial D_{ij}(u)}{\partial x_j} \quad (i = 1, \ldots, d).
\]

2.1 Two- and three-dimensional cavity flow problems
For cavity flow problems the domain \( \Omega \equiv (0, 1)^d \) is a unit square or cube and \( f \equiv 0 \). We set two-dimensional cavity flow problems with four Dirichlet boundary conditions.

**Problem 1** (2D). In (1) we take \( Re = 100, 1,000 \) and 5,000, and consider four boundary conditions as follows (see Fig. 1).

\[
g_1(x) = \begin{cases}
1 & \text{if } x_1 \neq 0, 1, x_2 = 1, \\
0 & \text{otherwise},
\end{cases} \quad g_2 = 0, \quad \text{(DC0)}
\]

\[
g_1(x) = \begin{cases}
1 & \text{if } x_2 = 1, \\
0 & \text{otherwise},
\end{cases} \quad g_2 = 0, \quad \text{(DC1)}
\]

\[
g_1(x) = \begin{cases}
4x_1(1-x_1) & \text{if } x_1 = 1, \\
0 & \text{otherwise},
\end{cases} \quad g_2 = 0, \quad \text{(C0)}
\]

\[
g_1(x) = \begin{cases}
\{4x_1(1-x_1)\}^2 & \text{if } x_2 = 1, \\
0 & \text{otherwise},
\end{cases} \quad g_2 = 0. \quad \text{(C1)}
\]

The problem with the boundary condition (DC0) or (DC1) is known as a benchmark one. The difference between the boundary conditions (DC0) and (DC1) is the values of \( g_1 \) at only two corners \((x_1, x_2) = (0, 1)\) and \((1, 1)\). In the cases of the boundary conditions (DC0) and (DC1), there does not exist a weak solution, i.e., \((u, p) \notin H^1(\Omega)^2 \times L^2(\Omega)\). But we set these problems to compare with the preceding results by Ghia et al.\(^{(9)}\) and see the difference of values of \( g_1 \) at the two corners. We can regularize these problems by considering the boundary conditions (C0) or (C1).

Below is three-dimensional cavity flow problems with \( C^0 \) and \( C^1 \) continuous types of the Dirichlet boundary condition.

**Problem 2** (3D). In (1) we take \( Re = 100, 400 \) and 1,000, and consider two boundary conditions as follows (see Fig. 2).

\[
g_1(x) = \begin{cases}
16x_1(1-x_1)x_2(1-x_2) & \text{if } x_1 = 1, \\
0 & \text{otherwise},
\end{cases} \quad g_2 = g_3 = 0. \quad \text{(C0-3D)}
\]

\[
g_1(x) = \begin{cases}
\{16x_1(1-x_1)x_2(1-x_2)\}^2 & \text{if } x_3 = 1, \\
0 & \text{otherwise},
\end{cases} \quad g_2 = g_3 = 0. \quad \text{(C1-3D)}
\]
3. Review of a pressure stabilized characteristic-curve finite element scheme

In this section we review a pressure stabilized characteristic-curve finite element scheme for the Navier-Stokes equations in the paper\(^\text{19}\).

Let \( T \) be a positive constant. We consider the nonstationary Navier-Stokes problem subject to the Dirichlet boundary condition; find \((u, p) : \Omega \times (0, T) \rightarrow \mathbb{R}^d \times \mathbb{R} \) such that

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \frac{2}{Re} \nabla p &= f & \quad & \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= 0 & \quad & \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= g & \quad & \text{on } \Gamma \times (0, T), \\
\n\frac{\partial}{\partial t} u \big|_{\Gamma} &= 0 & \quad & \text{in } \Omega, \text{ at } t = 0.
\end{aligned}
\]

We assume \( f = f(x, t) \) and \( g = g(x, t) \) only in this section to review the scheme for such general functions.

3.1 An idea of a characteristic-curve method

We explain an idea of a characteristic-curve method of first order in time simply. The idea of the characteristic-curve method is to consider the trajectory of the fluid particle and discretize the material derivative term, \( \partial u / \partial t + (u \cdot \nabla) u \), along the trajectory, (see Fig. 3).

Let \( \Delta t \) be a time increment and \( N_T \equiv [T / \Delta t] \) be a total step number. We set \( t^n \equiv n \Delta t \) for \( n \in \mathbb{N} \cup \{0\} \). For a function \( \phi \) on \( \Omega \times (0, T) \) or \( \Gamma \times (0, T) \) and an integer \( n (0 \leq n \leq N_T) \), \( \phi^n \) means \( \phi^n \equiv \phi(\cdot, t^n) \). For a velocity \( w : \Omega \rightarrow \mathbb{R}^d \), we define \( X_1(\omega, \Delta t) : \Omega \rightarrow \mathbb{R}^d \) by

\[
X_1(\omega, \Delta t)(x) = x - w(x) \Delta t.
\]

We use the symbol \( \circ \) to designate the composition of functions, e.g., for a function \( \phi \) defined in \( \Omega \)

\[
(\phi \circ X_1(\omega, \Delta t))(x) = \phi(X_1(\omega, \Delta t)(x)).
\]

Let \( u : \Omega \times (0, T) \rightarrow \mathbb{R}^d \) be a smooth function and \( X : (0, T) \rightarrow \mathbb{R}^d \) be a solution of the ordinary differential equation,

\[
\begin{aligned}
X(t^n) &= u(X(t^n), t^n), \\
X(t^n) &= x,
\end{aligned}
\]

for a point \( x \in \Omega \) and an integer \( n (1 \leq n \leq N_T) \), (see Fig. 3 again). Then, the material derivative of a smooth function \( \phi : \Omega \times (0, T) \rightarrow \mathbb{R} \) at \( t = t^n \) is approximated as follows:

\[
\frac{\partial}{\partial t} \phi(X(t^n), t^n) + u \cdot \nabla \phi(X(t^n), t^n) = \frac{\partial}{\partial t} \phi(X(t^n), t^n) = \frac{\partial}{\partial t} \phi \big|_{X(t^n)} + O(\Delta t),
\]

where we have used the relation,

\[
X(t^{n+1}) = X_1(\omega^{n+1}, \Delta t)(x) = O(\Delta t^2).
\]

For the Navier-Stokes equations, substituting \( u_i (i = 1, \ldots, d) \) into \( \phi \) in (4), we get the approximation of the material derivative of \( u \) at \( t = t^n \),

\[
\left( \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right)(x, t) = u^n - u^{n-1} \cdot X_1(\omega^{n-1}, \Delta t)(x) + O(\Delta t).
\]

(5)

Fig. 3 Trajectory of a fluid particle whose position is \( x \) at \( t = t^n \).

3.2 A finite element scheme

Let \( \mathcal{B}_h \equiv \{ K \} \) be a triangulation of \( \Omega \), where subscript \( h \) represents real length of the triangulation. We define \( \Omega_h \) by

\[
\Omega_h \equiv \bigcup \{ K : K \in \mathcal{B}_h \}
\]

and the boundary \( \Gamma_h \equiv \partial \Omega_h \). For a vector valued function \( g \) on \( \Gamma \) we set finite element spaces,

\[
X_h \equiv \{ v_h \in C^0(\Omega_h)^d : v_h|_{K} \in P_1(K)^d, \forall K \in \mathcal{B}_h \},
\]

\[
M_h \equiv \{ q_h \in C^0(\Omega_h) : q_h|_{K} \in P_1(K), \forall K \in \mathcal{B}_h \},
\]

\[
V_h(g) \equiv \{ v_h \in X_h : v_h(P) = g(P), \forall P \in \Gamma_h \},
\]

\[
V_h = V_h(0), \quad \mathcal{Q}_h = M_h \cap L^2(\Omega_h),
\]

where \( P \) is any nodal point on \( \Gamma_h \). Let \( \Pi_h \) be the interpolation operator from \( C^0(\Omega_h)^d \) to \( X_h \). For an external force \( f \in C^0([0, T])^d \), \( f_h \) means \( \Pi_h f \). For \( u, w \in H^1(\Omega_h)^d \) we define linear forms \( \mathcal{M}_h(u, w; \omega) \) and \( \mathcal{F}_h \) on \( V_h \),

\[
\langle \mathcal{M}_h(u, w; \omega), v_h \rangle = \frac{\partial}{\partial t} \langle u \cdot \nabla w \rangle \big|_{X_1(\omega, \Delta t)} + \langle u \cdot \nabla w \rangle \big|_{X_1(\omega, \Delta t)},
\]

\[
\langle \mathcal{F}_h, v_h \rangle = \langle f_h, v_h \rangle,
\]

and bilinear forms \( a_h, b_h \) and \( \mathcal{C}_h \) on \( H^1(\Omega_h)^d \times H^1(\Omega_h)^d, H^1(\Omega_h)^d \times L^2(\Omega_h) \) and \( H^1(\Omega_h) \times H^1(\Omega_h) \), respectively,

\[
a_h(u, v) = \frac{2}{Re} \langle D(u), D(v) \rangle,
\]

\[
b_h(u, v) = \langle \nabla v, q \rangle,
\]

\[
\mathcal{C}_h(p, q) = - \delta \sum_{K \in \mathcal{B}_h} b_h(K) \langle \nabla p, \nabla q \rangle_K.
\]

Here \( \delta \) is a positive constant, \( b_h \) is the diameter of element \( K \) and \( \langle \cdot, \cdot \rangle_K \) represents the \( L^2 \)-inner product on element \( K \).

We write the scheme for (2) in the paper\(^\text{19}\) again; find \( \{ u^n_h, p^n_h \} \in V_h \times Q_h ; n = 1, \ldots, N_T \) such that, for \( n = 1, \ldots, N_T \),

\[
\langle \mathcal{M}_h(u^n_h, u^{n-1}; \Delta t), v_h \rangle + a_h(u^n_h, v_h) + b_h(v_h, p^n_h)
\]

\[
+ b_h(u^n_h, q_h) + \mathcal{C}_h(p^n_h, q_h) = \langle \mathcal{F}_h, v_h \rangle,
\]

\[
\forall (v_h, q_h) \in V_h \times Q_h,
\]

where \( u^n_0 \) is a function approximating \( u^0 \).
Now, we consider the stability and advantages of the scheme (7). Generally, for time integration, the forward Euler method yields advantages, symmetry of matrix, explicit scheme and so on, and a disadvantage, low stability, e.g., for a constant $C > 0$ a stability condition $\Delta t < C h^2$ is required, and the backward Euler method (or Crank-Nicolson method) has the opposite properties, i.e., the matrix is nonsymmetric because of the nonlinear convection term, and stability is high. The scheme (7) uses backward Euler method, and we can take large $\Delta t$. In fact the Proposition 1 in the paper(19) on the stability holds, and neither a stability condition like $\Delta t \leq C h^2$ nor the CFL condition(21) is assumed in the proposition. Furthermore, the scheme has an advantage of the characteristic-curve method, i.e., the matrix is symmetric and identical, which leads to symmetric linear solvers. For the Navier-Stokes equations there are two types of stabilization method. The one is a pressure-stabilization method by Brezzi and Douglas Jr. and the characteristic-curve method to stabilize nonlinear convection term. The scheme (7) is indeed a upwind type method, and we can expect the numerical solutions, which has been recognized for Examples 1 and 3 in the paper(19). Therefore, if there exists a (sufficiently smooth) unique solution of (2), we can expect the solution by the scheme to converge to the exact one as $h$ and $\Delta t$ go to 0.

We use the CR method(16) with the point Jacobi preconditioner(1) for solving the system of linear equations, which works for our symmetric matrix. In the scheme we have to compute a integral,

$$
\int_{K} u_h^{n-1} \circ X_1 (u_h^{n-1}, \Delta t)v_h \, dx
$$
on triangular elements $K$. The integrand

$$
u_h^{n-1} \circ X_1 (u_h^{n-1}, \Delta t)v_h
$$
is not smooth on $K$. It is known that rough numerical integration causes oscillation even in the case that the stability is theoretically proved for a scheme with exact integration, see the papers by Tabata(24) and Tabata and Fujima(25). The two solutions using numerical integration formula of degree two (2D: three points, 3D: four points) and five (2D: seven points, 3D: fifteen points)(22) are almost same for Examples 1 and 3 in the paper(19). Therefore, in all the following computations we use the numerical integration formula of degree two.

Let $N_{G}$ be the division number of each side of $\Omega$, $(u, p)$ and $(u_h, p_h)$ be the solutions of the problem (2) and the scheme (7), respectively, and $n_t \equiv \lfloor t/\Delta t \rfloor$ be the step number for $t \in \mathbb{N}$. Setting a norm

$$\| (v, q) \|_{H^1 \times L^2} \equiv \frac{1}{\sqrt{Re}} \| |v|_1 + \| q \|_0 \|
$$
in the product space $H^1 (\Omega)^d \times L^2 (\Omega)$, for $t \in \mathbb{N} \setminus \{1\}$ we define $Diff_t$ by

$$Diff_t \equiv \frac{\| (u_h^n, p_h^n) - (u_h^{n-1}, p_h^{n-1}) \|_{H^1 \times L^2}}{\| (u_h^{n-1}, p_h^{n-1}) \|_{H^1 \times L^2}},$$

which represents a difference of the solution at times $t$ and $t-1$. We set $\delta = 0.2$ and 0.05 for 2D and 3D problems, respectively. These values are the same as the ones in the paper(19).

4. Numerical results

In this section we show two- and three-dimensional numerical results of cavity flow problems by the scheme (7). The scheme has numerical convergence order $O(h + \Delta t)$ to exact solutions, which has been recognized for Examples 1 and 3 in

the paper(19). Therefore, if there exists a (sufficiently smooth) unique solution of (2), we can expect the solution by the scheme to converge to the exact one as $h$ and $\Delta t$ go to 0.

Diff_t < 10^{-5}. (9)
Fig. 4 Meshes used for Problem 1, Fine mesh \((NΩ = 256)\), the mesh magnified around the corners \((x_1, x_2) = (0, 1)\) and \((1, 1)\), Coarse mesh \((NΩ = 128)\) and the mesh magnified around the corners (top to bottom).

Table 1 Discretization parameters for meshes in Fig 4.

<table>
<thead>
<tr>
<th>Mesh</th>
<th># of nodes</th>
<th># of elements</th>
<th>(h_{\text{min}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fine</td>
<td>11,470</td>
<td>21,914</td>
<td>2.76 \times 10^{-3}</td>
</tr>
<tr>
<td>Coarse</td>
<td>5,403</td>
<td>10,282</td>
<td>5.52 \times 10^{-3}</td>
</tr>
</tbody>
</table>

Table 2 Values of \(\Delta t\) used for Problem 1.

<table>
<thead>
<tr>
<th>Re</th>
<th>Fine mesh</th>
<th>Coarse mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1/100</td>
<td>1/50</td>
</tr>
<tr>
<td>1,000</td>
<td>1/200</td>
<td>1/100</td>
</tr>
<tr>
<td>5,000</td>
<td>1/800</td>
<td>1/400</td>
</tr>
</tbody>
</table>

The times of convergence are listed in Table 3. Since we have defined \(\text{Diff}_t\) for only \(t \in \mathbb{N} \setminus \{1\}\), the times in the table are integers. For each \(Re\), Figs. 5, 8 and 11, Figs. 6, 9 and 12 and Figs. 7, 10 and 13 show the graphs of \(u_{h1}(0.5, \cdot)\) and \(u_{h2}(\cdot, 0.5)\) of the two stationary solutions on Fine and Coarse meshes, the streamlines and the pressure contour lines of stationary solutions on Fine mesh, respectively. For the boundary conditions (DC0) and (DC1), we plot the results by Ghia et al.\(^{9}\) in the graphs. In the cases of the boundary conditions (C0) and (C1), the graphs by the two stationary solutions are almost same, and the streamlines exhibit the flow patterns well.

In the cases of the boundary conditions (DC0) and (DC1), although there does not exist a weak solution, the numerical solution exists. For \(Re = 100\) and \(1,000\) of (DC0) and \(Re = 100\) of (DC1), the graphs by the two stationary solutions are almost same and are similar to the results by Ghia et al. For \(Re = 5,000\) of (DC0) and \(Re = 1,000\) and \(5,000\) of (DC1), there are differences in the graphs by the two stationary solutions, and the solutions on Fine mesh are more close to the results by Ghia et al. than ones on Coarse mesh. The difference between the boundary conditions is the values of \(g_1\) at only two corners \((x_1, x_2) = (0, 1)\) and \((1, 1)\). However, there are evident differences of the streamlines by the two boundary conditions in the three Figs. 6, 9 and 12. The similar results have been reported by Cruchaga and Oñate\(^{5}\). They have shown the comparison of graphs of \(u_{h1}(0.5, \cdot)\) and \(u_{h2}(\cdot, 0.5)\) for (DC0) and (DC1) with \(Re = 1,000\), \(5,000\) and \(10,000\).

In the Figs. 7, 10 and 13, we can see meaningful pressure contour lines for each flow pattern, although there are oscillations. We think that these oscillations of the pressure become small as \(\Delta t\) and \(h \to 0\), because the numerical convergence of the scheme to the exact solution by means of a norm using \(H^1(\Omega)^d\)-norm for the velocity and \(L^2(\Omega)\)-norm for the pressure has been observed in the paper\(^{19}\). In fact, the oscillations of the pressure on Fine mesh are smaller than ones on Coarse mesh.

Let us study the difference of solutions by the boundary conditions on Fine mesh. Fig. 14 shows graphs of \(u_{h1}(0.5, \cdot)\) and \(u_{h2}(\cdot, 0.5)\) for the four boundary conditions with the results by Ghia et al., which exhibit the size of boundary layers. Now, we focus on the difference of solutions especially by the boundary conditions (DC0) and (DC1). The difference of the graphs for \(Re = 5,000\) is the biggest in Fig. 14. Detailed graphs for the Reynolds number are presented in Fig. 15, where (DC1/4),...
(DC1/2) and (DC3/4) are additional boundary conditions to the Problem 1,

\[
g_1(x) = \begin{cases} 
1 & (x_1 \neq 0, 1, x_2 = 1) \\
1/4 & (x_1 = 0, 1, x_2 = 1), \quad g_2 = 0, \quad (DC1/4) \\
0 & (otherwise) 
\end{cases}
\]

\[
g_1(x) = \begin{cases} 
1 & (x_1 \neq 0, 1, x_2 = 1) \\
1/2 & (x_1 = 0, 1, x_2 = 1), \quad g_2 = 0, \quad (DC1/2) \\
0 & (otherwise) 
\end{cases}
\]

and

\[
g_1(x) = \begin{cases} 
1 & (x_1 \neq 0, 1, x_2 = 1) \\
3/4 & (x_1 = 0, 1, x_2 = 1), \quad g_2 = 0, \quad (DC3/4) \\
0 & (otherwise) 
\end{cases}
\]

respectively, and its graphs are by stationary solutions on Fine mesh by the scheme (7) with the same parameters, whose initial value is the stationary solution for (DC0) to save computational time. We can see the effect of the values of \(g_1\) at two corners, \((x_1, x_2) = (0, 1)\) and \((1, 1)\).

<table>
<thead>
<tr>
<th>Table 3 Convergence times.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Re)</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>(DC0):</td>
</tr>
<tr>
<td>1,000</td>
</tr>
<tr>
<td>5,000</td>
</tr>
<tr>
<td>(DC1):</td>
</tr>
<tr>
<td>1,000</td>
</tr>
<tr>
<td>5,000</td>
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<tr>
<td>(C0):</td>
</tr>
<tr>
<td>1,000</td>
</tr>
<tr>
<td>5,000</td>
</tr>
<tr>
<td>(C1):</td>
</tr>
<tr>
<td>1,000</td>
</tr>
<tr>
<td>5,000</td>
</tr>
</tbody>
</table>

4.2 Three-dimensional cavity flow problems, Problem 2

In this subsection we show numerical results for Problem 2. The finite element subdivision of the domain is constructed by dividing the domain into a union of triangular prisms and further subdividing each triangular prism into three tetrahedra. In this process, a triangular mesh of the two-dimensional domain \(\omega = (0, 1)^2\) by FreeFem++ is used.

Considering the boundary layers, we used nonuniform two meshes in Fig. 16. We call the meshes Fine and Coarse meshes, respectively, whose discretization parameters are shown in Table 4. In three-dimensional case, for all the Reynolds numbers we set \(\Delta t = 1/32\) for Fine mesh and \(\Delta t = 1/24\) for Coarse mesh.

The numerical solutions converged to stationary solutions in the sense of satisfying the inequality (9). The times of convergence are listed in Table 5. Figs. 17, 24 and 31 show the

![Graphs of \(u_{h1}(0.5, \cdot)\) and \(u_{h2}(\cdot, 0.5)\), \(Re = 100\), (DC0), (DC1), (C0) and (C1) (top to bottom).](image)
Fig. 6 Streamlines, $Re = 100$, (DC0), (DC1), (C0) and (C1) (top to bottom).

Fig. 7 Pressure contour lines, $Re = 100$, $\Delta p = 0.01$, (DC0), (DC1), (C0) and (C1) (top to bottom).
Fig. 8 Graphs of $u_{h1}(0.5, \cdot)$ and $u_{h2}(\cdot, 0.5)$, $Re = 1,000$, (DC0), (DC1), (C0) and (C1) (top to bottom).

Fig. 9 Streamlines, $Re = 1,000$, (DC0), (DC1), (C0) and (C1) (top to bottom).
Fig. 10  Pressure contour lines, $Re = 1,000$, $\Delta p = 0.01$, (DC0), (DC1), (C0) and (C1) (top to bottom).

Fig. 11  Graphs of $u_{h1}(0.5, \cdot)$ and $u_{h2}(\cdot, 0.5)$, $Re = 5,000$, (DC0), (DC1), (C0) and (C1) (top to bottom).
Fig. 12 Streamlines, $Re = 5,000$, (DC0), (DC1), (C0) and (C1) (top to bottom).

Fig. 13 Pressure contour lines, $Re = 5,000$, $\Delta p = 0.01$, (DC0), (DC1), (C0) and (C1) (top to bottom).
Fig. 14 Graphs of \( u_h(0.5, \cdot) \) and \( u_h(\cdot, 0.5) \) for the four boundary conditions, (DC0), (DC1), (C0) and (C1), \( Re = 100 \) (top), 1,000 (middle) and 5,000 (bottom).

Fig. 15 Graphs of \( u_h(0.5, \cdot) \) and \( u_h(\cdot, 0.5) \) for the five boundary conditions, (DC0), (DC1/4), (DC1/2), (DC3/4) and (DC1), and its magnified ones (top to bottom), \( Re = 5,000 \).
graphs of $u_{h1}(0.5,0.5,\cdot)$ and $u_{h3}(\cdot,0.5,0.5)$ of the two stationary solutions on Fine and Coarse meshes for the two boundary conditions (C0-3D) and (C1-3D) for each $Re$. The graphs of the two stationary solutions are almost same. Figs. 18–20, 25–27 and 32–34 are projections of velocity vectors on each plane for each $Re$, which exhibit the flow patterns well of these problems. Pressure contour lines on Fine mesh are shown in Figs. 21–23, 28–30 and 35–37. The oscillations of the pressure on Fine mesh are a little smaller than ones on Coarse mesh.

The effect of the two boundary conditions are presented in Fig. 38.

Table 4  Discretization parameters for meshes in Fig 16.

<table>
<thead>
<tr>
<th>Mesh</th>
<th># of nodes</th>
<th># of elements</th>
<th>$h_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fine</td>
<td>172,965</td>
<td>972,288</td>
<td>$5.16 \times 10^{-3}$</td>
</tr>
<tr>
<td>Coarse</td>
<td>74,627</td>
<td>410,688</td>
<td>$7.09 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 5  Convergence times.

<table>
<thead>
<tr>
<th>$Re$</th>
<th>$t$ (in $N$)</th>
<th>Fine mesh</th>
<th>Coarse mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C0-3D): 100</td>
<td>12</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>32</td>
<td>32</td>
<td></td>
</tr>
<tr>
<td>1,000</td>
<td>58</td>
<td>58</td>
<td></td>
</tr>
<tr>
<td>(C1-3D): 100</td>
<td>11</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>33</td>
<td>33</td>
<td></td>
</tr>
<tr>
<td>1,000</td>
<td>53</td>
<td>53</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 16  Meshes used for Problem 2. Fine mesh ($N_{\Omega} = 64$), the mesh magnified around the points $(x_1,x_2,x_3) = (0,0,1)$ and $(1,1,1)$, Coarse mesh ($N_{\Omega} = 48$) and the mesh magnified around the points (top to bottom).

Fig. 17  Graphs of $u_{h1}(0.5,0.5,\cdot)$ and $u_{h3}(\cdot,0.5,0.5)$, $Re = 100$, (C0-3D) (top) and (C1-3D) (bottom).
Fig. 18 Projections of velocity vectors on the plane $x_2 = 0.5$, $Re = 100$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 19 Projections of velocity vectors on the plane $x_1 = 0.5$, $Re = 100$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 20 Projections of velocity vectors on the plane $x_3 = 0.5$, $Re = 100$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 21 Pressure contour lines on the plane $x_2 = 0.5$, $Re = 100$, $\Delta p = 0.0025$, (C0-3D) (top) and (C1-3D) (bottom).
Fig. 22  Pressure contour lines on the plane $x_1 = 0.5$, $Re = 100$, $\Delta p = 0.0025$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 23  Pressure contour lines on the plane $x_3 = 0.5$, $Re = 100$, $\Delta p = 0.0025$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 24  Graphs of $u_{h1}(0.5, 0.5, .)$ and $u_{h3}(., 0.5, 0.5)$, $Re = 400$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 25  Projections of velocity vectors on the plane $x_2 = 0.5$, $Re = 400$, (C0-3D) (top) and (C1-3D) (bottom).
Fig. 26 Projections of velocity vectors on the plane \(x_1 = 0.5\), \(Re = 400\), (C0-3D) (top) and (C1-3D) (bottom).

Fig. 27 Projections of velocity vectors on the plane \(x_3 = 0.5\), \(Re = 400\), (C0-3D) (top) and (C1-3D) (bottom).

Fig. 28 Pressure contour lines on the plane \(x_2 = 0.5\), \(Re = 400\), \(\Delta p = 0.0025\), (C0-3D) (top) and (C1-3D) (bottom).

Fig. 29 Pressure contour lines on the plane \(x_1 = 0.5\), \(Re = 400\), \(\Delta p = 0.0025\), (C0-3D) (top) and (C1-3D) (bottom).
Fig. 30  Pressure contour lines on the plane $x_2 = 0.5$, $Re = 400$, $\Delta p = 0.0025$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 31  Graphs of $u_{h1}(0.5, 0.5, \cdot)$ and $u_{h3}(\cdot, 0.5, 0.5)$, $Re = 1,000$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 32  Projections of velocity vectors on the plane $x_2 = 0.5$, $Re = 1,000$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 33  Projections of velocity vectors on the plane $x_1 = 0.5$, $Re = 1,000$, (C0-3D) (top) and (C1-3D) (bottom).
Fig. 34 Projections of velocity vectors on the plane $x_3 = 0.5$, $Re = 1,000$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 35 Pressure contour lines on the plane $x_2 = 0.5$, $Re = 1,000$, $Δp = 0.0025$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 36 Pressure contour lines on the plane $x_1 = 0.5$, $Re = 1,000$, $Δp = 0.0025$, (C0-3D) (top) and (C1-3D) (bottom).

Fig. 37 Pressure contour lines on the plane $x_3 = 0.5$, $Re = 1,000$, $Δp = 0.0025$, (C0-3D) (top) and (C1-3D) (bottom).
5. Conclusions

We have applied a newly developed characteristic-curve finite element scheme to cavity flow problems. The scheme uses a cheap element P1/P1 with pressure stabilization method, and the matrix of resulting linear system is symmetric and identical. Therefore, the scheme leads to symmetric linear solvers and easy large scale computations. We have solved two- and three-dimensional cavity flow problems with the Reynolds numbers up to 5,000 (2D) and 1,000 (3D). In the two-dimensional case, we have observed the difference of solutions by the three types of the Dirichlet boundary condition, discontinuous, $C^0$ and $C^1$ continuous ones. From the difference of the solutions of the problems with boundary conditions (DC0) and (DC1), we have seen the influence of the discontinuity of $g_1$. For problems with continuous boundary conditions in 2D and 3D, the streamlines and velocity vectors obtained have shown the flow patterns well. These results imply that the scheme can be applied for the practical problem. An improvement for the pressure oscillations by the scheme is a future work.

The computations in this paper were carried out on IBM eServer p5 595 (power 5, 1.9GHz) with IBM XL C/C++ Enterprise Edition V7.0 at Research Institute for Information Technology of Kyushu University.

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References


