Free Flexural Vibration of an Elastically Restrained Elliptical Plate Subjected to an In-Plane Force

Kenzo SATO**
** Department of Engineering and Information Science, Faculty of Education and Human Studies, Akita University
1-1 Tegata-Gakuenmachi, akita,010-8502, Japan
E-mail: sato-kenzo@ed.akita-u.ac.jp

Abstract
The present paper is concerned with the free flexural vibration of an elliptical plate subjected to a uniform in-plane force in its middle plane. The edge of the plate treated here is rigidly supported against transverse displacement and restrained elastically against rotation. The rigorous expression of the vibration displacement is obtained in the form of a Mathieu function series in accordance with conventional thin-plate theory, neglecting the effects of shear deformation and rotary inertia. The frequency equation from which the eigenfrequencies can be obtained numerically is derived by applying the orthogonality of the Mathieu function. Numerical values of the lowest dimensionless eigenfrequencies are tabulated and graphed against various dimensionless rotational spring stiffnesses, dimensionless in-plane forces and aspect ratios.

Key words: Free Flexural Vibration, Elliptical Plate, Eigenfrequency, In-Plane Force, Elastic Edge Restraint, Mathieu Function

1. Introduction
The free flexural vibration of a simply-supported or clamped elliptical plate subjected to a uniform in-plane force has already been investigated by the author(1), (2). The vibration problem of a plate with elastic restraint at its edge is very important in the engineering field. In 1976, the author studied the free flexural vibrations of an elliptical plate supported rigidly against transverse displacement and restrained elastically against rotation at its edge(3) but did not take the effect of an in-plane force on its vibrations into consideration.

In the present paper, the author deals with the free flexural vibration of an elastically restrained elliptical plate subjected to the action of a uniform in-plane force distributed in its middle plane. The elastic restraint considered here prevents transverse edge displacement and provides a restoring edge moment proportional to the rotational angle during vibration. On the bases of conventional thin-plate theory, neglecting the effects of shearing force and rotary inertia, the theoretical analysis is rigorously performed in the elliptical cylinder coordinate system. The general solution of the differential equation of motion of the plate is obtained in the form of an infinite Mathieu function series, and the final form of the mathematical solution, completely satisfying the boundary conditions, is obtained by making use of the orthogonality of Mathieu functions. By regarding a circle as a special type of ellipse, the frequency equation for the vibration of an elastically restrained circular plate under the action of a uniform in-plane force is derived, as a special case, from that obtained here for an elastically restrained elliptical plate.

The effect of the uniform in-plane force on the lowest eigenfrequency, which is of great importance in practical application, is calculated numerically and described in diagrams.
2. Equation of Motion and Boundary Conditions

The problem of the free flexural vibration of an elliptical plate subjected to a uniform in-plane tension $P$ along its middle plane as shown in Fig. 1 can be rigorously analyzed by introducing the elliptical cylinder coordinates ($\xi, \eta, z$), which are related to the rectangular coordinates ($x, y, z$) by

$$x = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta, \quad z = z,$$

where $c$ denotes the semifocal length and $c^2 = a^2 - b^2$ using the semimajor and semiminor axial lengths $a$ and $b$, respectively. The differential equation of motion of the plate can be written, using the displacement $w$, the time $t$, the density $\rho$ and the flexural rigidity $D$ which is equal to $Eh^3/[12(1-\nu^2)]$ composed of Young’s modulus $E$, plate thickness $h$ and Poisson’s ratio $\nu$, as

$$\nabla^2 (D \nabla^2 w - P)w + \rho h \partial^2 w/\partial t^2 = 0,$$

where

$$\nabla^2 = \left(1/\rho \right)^2 \left(\partial^2 / \partial \xi^2 + \partial^2 / \partial \eta^2 \right), \quad \varrho^2 = c^2(\cosh 2\xi - \cos 2\eta)/2.$$

The boundary conditions of the plate under consideration are given along its periphery $\xi = \xi_o$ as

$$w \bigg|_{\xi = \xi_o} = 0, \quad M_{\xi} \bigg|_{\xi = \xi_o} = K \frac{\partial w}{\partial \xi} \bigg|_{\xi = \xi_o},$$

where $K$ denotes the rotational spring stiffness, and the bending moment intensity $M_{\xi}$ is given in terms of the coordinates $\xi$ and $\eta$ as

$$M_{\xi} = -\frac{D c^2}{2} \left(\cosh 2\xi - \cos 2\eta \right) \left(\frac{\partial^2 w}{\partial \xi^2} + \nu \frac{\partial^2 w}{\partial \eta^2} \right) - \left(1 - \nu \right) \left(\sinh 2\xi \frac{\partial w}{\partial \xi} - \sin 2\eta \frac{\partial w}{\partial \eta} \right).$$

3. Frequency Equation

Assuming that the elliptical plate under consideration vibrates harmonically with angular frequency $\omega$, substituting $w(\xi, \eta, t) = W(\xi, \eta) \sin \omega t$ into Eq. (2) yields

$$(\nabla^2 + \alpha^2)(\nabla^2 + \beta^2)W = 0,$$

where

$$\frac{\alpha^2}{\beta^2} = \pm \left(\frac{\lambda^2}{4} + \kappa^4 \mp \frac{\lambda}{2}\right)$$

with

$$\kappa^2 = \omega \sqrt{\rho h / D}, \quad \lambda = P / D.$$
The general solution $W(\xi, \eta)$ of Eq.(6) is given by the sum $W = W_a + W_\beta$ of the solutions of two equations

$$\ddot{W}_a / \dot{\xi}^2 + \ddot{W}_\beta / \dot{\eta}^2 + 2q_a (\cosh 2\xi - \cos 2\eta) W_a = 0, \quad \chi = \alpha, \beta, \tag{9}$$

with

$$q_a = k_a^2 = \alpha^2 c^2 / 4, \quad q_\beta = k_\beta^2 = \beta^2 c^2 / 4. \tag{10}$$

From the symmetry of the present system, free flexural vibrations of the elliptical plate are classified into four types of normal vibration modes. In this study, the solution for only the normal vibration mode symmetrical about both axes of the ellipse is given, and the numerical results for only the lowest normal vibration mode, having no nodal curves except the boundary ellipse, will be presented in Section 5. Thus, the solution of Eq.(6) is given, using the unknown constants $c_n^{(1)}$ and $c_n^{(2)}$, $n = 0, 1, 2, \cdots$, as

$$W = \sum_{n=0}^{\infty} \left[ C_n^{(1)} c_{2n}(\xi, q_a) c_{2n}(\eta, q_a) + C_n^{(2)} c_{2n}(\xi, q_\beta) c_{2n}(\eta, q_\beta) \right], \tag{11}$$

in which $c_{2n}$ and $C_{2n}$ denote the Mathieu and modified Mathieu functions of order $2n$, respectively, expressed by

$$c_{2n}(\eta, q_a) = \sum_{l=0}^{\infty} A_{2n}^{(2n)}(q_a) \cos l^{2n} \eta, \quad c_{2n}(\xi, q_a) = \sum_{l=0}^{\infty} A_{2n}^{(2n)}(q_a) \cos 2l\xi, \quad q = q_a, q_\beta. \tag{12}$$

Further calculation of Eq.(11) yields

$$W = \sum_{n=0}^{\infty} \left[ \sum_{l=0}^{\infty} \phi_{2n,2n}(q_a, q_\beta) C_{2n}(\xi, q_a) + C_{2n}(\xi, q_\beta) \right] c_{2n}(q_a, q_\beta), \tag{13}$$

where

$$\phi_{2n,2n}(q_a, q_\beta) = \sum_{l=0}^{\infty} A_{2l}^{(2n)}(q_a) A_{2l}^{(2n)}(q_\beta) \epsilon_l, \quad \epsilon_l = 2 (l = 0), 1 (l \neq 0), \quad l = 0, 1, 2, \cdots. \tag{14}$$

Here, to make $W$ satisfy the second boundary condition in Eq.(4), let us introduce

$$\frac{1}{(\cosh x - \cos \eta)^{\nu}} = \sum_{j=0}^{\infty} L_j(x, \nu) \cos j\eta, \tag{15}$$

where

$$L_j(x, \nu) = 2^\nu \mu_j e^{-(j+1)^2} \Gamma(\nu, j+1, \nu) / \nu, \quad \mu_j = 1(j = 0), 2(j \neq 0), \quad j = 0, 1, 2, \cdots, \tag{16}$$

with the gamma function $\Gamma$ and the hypergeometric function $F$. Then, using

$$\cos 2\eta c_{2n}(\eta, q_\beta) = \sum_{l=0}^{\infty} \psi_{2l,2n}^{(j)}(q_\beta) c_{2n}(\eta, q_\beta), \quad j, n = 0, 1, 2, \cdots, \tag{17}$$

with

$$\psi_{2l,2n}^{(j)}(q_\beta) = \frac{1}{2} \sum_{r=0}^{\infty} A_{2l+1}^{(2n)}(q_\beta) A_{2r}^{(2n)}(q_\beta) \epsilon_{r+j} + A_{2l}^{(2n)}(q_\beta) \epsilon_{r-j}, \tag{18}$$

we have an infinite Mathieu function series

$$\frac{c_{2n}(q_a, q_\beta)}{(\cosh 2\xi - \cos 2\eta)^{1/2}} = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} L_j(2\xi, \nu, 3/2) \psi_{2l,2n}^{(j)}(q_\beta) c_{2n}(q_a, q_\beta). \tag{19}$$

Now, substituting Eq.(13) into the first boundary condition in Eq.(4) and applying the orthogonality of the Mathieu function yields

$$\sum_{n=0}^{\infty} C_n^{(1)} \phi_{2n,2n}(q_a, q_\beta) C_{2n}(q_a, q_a) + C_n^{(2)} C_{2n}(q_a, q_\beta) = 0, \quad n = 0, 1, 2, \cdots, \tag{20}$$
and similarly substituting Eq. (13) into the second boundary condition in Eq. (4) gives, using Eq. (19),

\[
(\sqrt{2}D/c) \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} L_j(2\xi, 3/2)\phi_{22n}(q_0, q_0) (\cosh 2\xi) \sum_{m=0}^{\infty} C_{2m}^{(1)}\phi_{2m+1}(q_0, q_0) \text{Ce}^{''}_{2m}(\xi_0, q_0) \\
+ C_{2n}^{(2)} \text{Ce}^{''}_{2n}(\xi_0, q_0) - \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} C_{2m}^{(1)} \phi_{2m+1}(q_0, q_0) \text{Ce}^{''}_{2m}(\xi_0, q_0) + C_{2n}^{(2)} \text{Ce}^{''}_{2n}(\xi_0, q_0) \phi_{2n+1}(q_0) \\
-(1-\nu) \sinh 2\xi_0 \sum_{m=0}^{\infty} C_{2m}^{(1)} \phi_{2m+1}(q_0, q_0) \text{Ce}^{''}_{2m}(\xi_0, q_0) + C_{2n}^{(2)} \text{Ce}^{''}_{2n}(\xi_0, q_0) \\
+ K \sum_{m=0}^{\infty} C_{2m}^{(1)} \phi_{2m+1}(q_0, q_0) \text{Ce}^{''}_{2m}(\xi_0, q_0) + C_{2n}^{(2)} \text{Ce}^{''}_{2n}(\xi_0, q_0) = 0, \\
b = 0, 1, 2, \ldots .
\]  

Eliminating \( C_{2n}^{(2)} \), \( n = 0, 1, 2, \ldots \), from Eqs. (20) and (21), and using (4)

\[
\text{Ce}^{''}_{2n}(\xi_0, q_0) = [a_{2n}(q_0) - 2q \cosh 2\xi_0] \text{Ce}_{2n}(\xi_0, q_0), \quad q = q_0, q_0,
\]  

we have

\[
\sum_{m=0}^{\infty} C_{2m}^{(1)} e_{mn} = 0, \quad n = 0, 1, 2, \ldots ,
\]  

where

\[
e_{mn} = \text{Ce}_{2m}(\xi_0, q_0) \left( \frac{\gamma c}{\sqrt{2}} \phi_{2m+1}(q_0, q_0) \left[ \text{Ce}_{2m}(\xi_0, q_0) - \text{Ce}_{2m}(\xi_0, q_0) \right] \\
+ \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} L_j(2\xi_0, 3/2) \phi_{2m+1}(q_0, q_0) \phi_{2m+1}(q_0, q_0) \\
\times \left\{ \cosh 2\xi_0 (a_{2m}(q_0) - a_{2m}(q_0)) - 2(k_m^2 - k_m^2) \cosh 2\xi_0 \right\} \\
- (1-\nu) \sinh 2\xi_0 \left\{ \text{Ce}_{2m}(\xi_0, q_0) - \text{Ce}_{2m}(\xi_0, q_0) \right\} \\
- \sum_{l=0}^{\infty} \phi_{2m+1}(q_0, q_0) \phi_{2n+1}(q_0, q_0) \left\{ a_{2m}(q_0) - a_{2m}(q_0) - 2(k_m^2 - k_m^2) \cosh 2\xi_0 \right\} \right). 
\]  

(24)

with

\[
\gamma = K/D.
\]  

(25)

Eliminating \( C_{2n}^{(1)} \), \( n = 0, 1, 2, \ldots \), from Eq. (23), we obtain the frequency equation in the form of an infinite determinantal equation, whose elements are given by Eq. (24), as

\[
A = \begin{bmatrix}
e_{00} & e_{10} & e_{20} & \cdots \\
e_{01} & e_{11} & e_{21} & \cdots \\
e_{02} & e_{12} & e_{22} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix} = 0.
\]  

(26)

Note that when \( K \to 0 \) or \( \infty \), Eq. (26) becomes the frequency equation for a simply-supported\(^{(1)}\) or clamped\(^{(2)}\) elliptical plate, respectively. Furthermore, when \( P \to 0 \), Eq. (26) becomes the frequency equation for an elastically restrained elliptical plate without an in-plane force\(^{(3)}\). Also, when \( \omega \to 0 \), Eq. (26) reduces to the buckling condition of an elastically restrained elliptical plate\(^{(6)}\).

4. Circular Plate

When the two foci of an ellipse become very close, namely, the semifocal length \( c \to 0 \), we have \( k_m, k_m \to 0 \) and \( q_0, q_0 \to 0 \) from Eq. (10). Moreover, as \( c \to 0, \xi \to \infty \) and
2k_2^2 \cosh 2\xi \to a^2b^2, 2k_2^2 \cosh 2\xi \to \beta^2b^2,$ in which $b$ represents the radius of a circular plate. Consequently (4), using the circular cylinder coordinates $(b, \theta, z),$

$$A_{2n}^{(2n)} \to 1/\sqrt{2}(n = u = 0), 1(n = u \neq 0), 0(n \neq u), u = 0, 1, 2, \cdots,$$

$$a_{2n} \to 4n^2, \quad c_2e_0(n, q_a)$$

and $c_2e_0(n, q_b) \to 1/\sqrt{2}(n = 0), \cos 2n\theta(n \neq 0),$ $c_2e_0(n, \phi_a) \to c_2e_0(n, \phi_b),$ $n = 0, 1, 2, \cdots,$

where $J_{2n}$ denotes the Bessel function of order $2n$, the prime $(\prime)$ attached to $J_{2n}$ denotes the $b$-derivative, and $c_2e_0$ is a constant multiplier. Then, the frequency equation Eq.(26) reduces to

$$e_{mn} = 0, \quad i.e.,$$

$$(1 - \nu - yb) [\frac{abJ_{2n+1}(ab)}{J_{2n}(ab)} - \frac{\beta bJ_{2n+1}(\beta b)}{J_{2n}(\beta b)}] - (ab)^2 + (\beta b)^2 = 0, \quad n = 0, 1, 2, \cdots.$$  

(27)

The above equation is the frequency equation of vibration of an elastically restrained circular plate under the uniform in-plane force and its vibration mode shape is of a number $2n$ of nodal diameters. Note that when $P \to 0, \lambda \to 0$ from Eq.(8), and therefore, Eq.(27) reduces to the frequency equation for the case of no in-plane force (3).

5. Numerical Calculation and Discussion

Numerical calculation of the lowest dimensionless eigenfrequency $\kappa^2b^2$ for the vibration mode having no nodal curves within the elliptical plate with an edge-restraint parameter $yb$ was performed for the change in the dimensionless in-plane force $\lambda b^2$. When the in-plane force used in the numerical calculation is negative, it is compressive, but its magnitude is less than the buckling load of each elliptical plate under consideration (3). The value of Poisson’s ratio $\nu$ was taken to be 0.3. In order to determine the actual value of the root of $\kappa = 0$ given by Eq.(26), the convergence must be examined. This was done by means of forming a series of finite and computable determinants $A_1, A_2, \cdots, A_N$ having 1, 2, 3, $\cdots, N$ rows and columns starting at the left hand top corner of the infinite determinant $A$ as

$$A_N = \begin{vmatrix}
  e_{00} & e_{01} & \cdots & e_{0,N-1} \\
  e_{10} & e_{11} & \cdots & e_{1,N-1} \\
  e_{20} & e_{21} & \cdots & e_{2,N-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  e_{N-1,0} & e_{N-1,1} & \cdots & e_{N-1,N-1}
\end{vmatrix}, \quad N = 1, 2, 3, \cdots. \quad (28)$$

The number $N$ was increased until any root of $A_N = 0$ converged to a definite limit. Here, note that to adopt $A_N$ for the purpose of the numerical calculation means to approximate series (13) representing the displacement with the first $N$ terms with respect to $m$ and $n$. As an example, the convergence test of the lowest dimensionless eigenfrequency $\kappa^2b^2$ is presented in Table I.

Table 1 Convergence test of the lowest dimensionless eigenfrequency $\kappa^2b^2$ in the cases of aspect ratios $a/b = 2, 5$ and dimensionless in-plane forces $\lambda b^2 = -2, 2$ ($yb = 1; \nu = 0.3$) 

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>$\lambda b^2$</th>
<th>$N$</th>
<th>$a/b$</th>
<th>$\lambda b^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.319</td>
<td>3.085</td>
<td>4.400</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3.132</td>
<td>2.574</td>
<td>3.957</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3.132</td>
<td>2.589</td>
<td>4.911</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>3.132</td>
<td>2.589</td>
<td>4.911</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3.132</td>
<td>2.589</td>
<td>4.911</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3.132</td>
<td>2.589</td>
<td>4.911</td>
<td></td>
</tr>
</tbody>
</table>
Table 2  Lowest dimensionless eigenfrequency \( \kappa^2 b^2 \) as a function of aspect ratio \( a/b \) and edge-restraint parameter \( yb \) in the cases of dimensionless in-plane forces \( \lambda b^2 = -2, 0 \) and \( 2 \) (\( \nu = 0.3 \))

(a) \( \lambda b^2 = -2 \) (in-plane compression)

<table>
<thead>
<tr>
<th>( a/b )</th>
<th>( 0 )</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>3.571</td>
<td>3.768</td>
<td>5.019</td>
<td>8.011</td>
<td>9.314</td>
<td>9.511</td>
</tr>
<tr>
<td>1.2</td>
<td>2.815</td>
<td>3.009</td>
<td>4.186</td>
<td>6.788</td>
<td>7.854</td>
<td>8.013</td>
</tr>
<tr>
<td>1.3</td>
<td>2.584</td>
<td>2.776</td>
<td>3.924</td>
<td>6.397</td>
<td>7.398</td>
<td>7.547</td>
</tr>
<tr>
<td>1.5</td>
<td>2.282</td>
<td>2.469</td>
<td>3.567</td>
<td>5.863</td>
<td>6.783</td>
<td>6.920</td>
</tr>
<tr>
<td>2.0</td>
<td>1.939</td>
<td>2.117</td>
<td>3.132</td>
<td>5.210</td>
<td>6.048</td>
<td>6.174</td>
</tr>
<tr>
<td>2.5</td>
<td>1.785</td>
<td>1.956</td>
<td>2.930</td>
<td>4.921</td>
<td>5.728</td>
<td>5.849</td>
</tr>
<tr>
<td>3.0</td>
<td>1.683</td>
<td>1.853</td>
<td>2.810</td>
<td>4.759</td>
<td>5.549</td>
<td>5.668</td>
</tr>
<tr>
<td>4.0</td>
<td>1.547</td>
<td>1.719</td>
<td>2.669</td>
<td>4.582</td>
<td>5.354</td>
<td>5.470</td>
</tr>
<tr>
<td>5.0</td>
<td>1.460</td>
<td>1.635</td>
<td>2.589</td>
<td>4.487</td>
<td>5.250</td>
<td>5.365</td>
</tr>
</tbody>
</table>

(b) \( \lambda b^2 = 0 \) (no in-plane force)

<table>
<thead>
<tr>
<th>( a/b )</th>
<th>( 0 )</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>4.935</td>
<td>5.078</td>
<td>6.063</td>
<td>8.752</td>
<td>10.019</td>
<td>10.216</td>
</tr>
<tr>
<td>1.1</td>
<td>4.516</td>
<td>4.653</td>
<td>5.582</td>
<td>8.047</td>
<td>9.175</td>
<td>9.348</td>
</tr>
<tr>
<td>1.2</td>
<td>4.213</td>
<td>4.343</td>
<td>5.227</td>
<td>7.526</td>
<td>8.560</td>
<td>8.718</td>
</tr>
<tr>
<td>1.3</td>
<td>3.987</td>
<td>4.113</td>
<td>4.959</td>
<td>7.132</td>
<td>8.101</td>
<td>8.248</td>
</tr>
<tr>
<td>1.5</td>
<td>3.681</td>
<td>3.799</td>
<td>4.589</td>
<td>6.589</td>
<td>7.477</td>
<td>7.613</td>
</tr>
<tr>
<td>2.0</td>
<td>3.303</td>
<td>3.410</td>
<td>4.119</td>
<td>5.914</td>
<td>6.720</td>
<td>6.844</td>
</tr>
<tr>
<td>2.5</td>
<td>3.122</td>
<td>3.223</td>
<td>3.895</td>
<td>5.608</td>
<td>6.384</td>
<td>6.503</td>
</tr>
<tr>
<td>3.0</td>
<td>3.009</td>
<td>3.107</td>
<td>3.761</td>
<td>5.434</td>
<td>6.194</td>
<td>6.311</td>
</tr>
<tr>
<td>4.0</td>
<td>2.870</td>
<td>2.966</td>
<td>3.606</td>
<td>5.243</td>
<td>5.985</td>
<td>6.100</td>
</tr>
<tr>
<td>5.0</td>
<td>2.787</td>
<td>2.883</td>
<td>3.518</td>
<td>5.139</td>
<td>5.872</td>
<td>5.985</td>
</tr>
</tbody>
</table>

(c) \( \lambda b^2 = 2 \) (in-plane tension)

<table>
<thead>
<tr>
<th>( a/b )</th>
<th>( 0 )</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>5.567</td>
<td>5.677</td>
<td>6.459</td>
<td>8.722</td>
<td>9.827</td>
<td>10.000</td>
</tr>
<tr>
<td>1.3</td>
<td>5.011</td>
<td>5.111</td>
<td>5.813</td>
<td>7.795</td>
<td>8.743</td>
<td>8.890</td>
</tr>
<tr>
<td>1.5</td>
<td>4.678</td>
<td>4.771</td>
<td>5.421</td>
<td>7.239</td>
<td>8.107</td>
<td>8.243</td>
</tr>
<tr>
<td>2.0</td>
<td>4.250</td>
<td>4.332</td>
<td>4.911</td>
<td>6.537</td>
<td>7.325</td>
<td>7.449</td>
</tr>
<tr>
<td>2.5</td>
<td>4.038</td>
<td>4.116</td>
<td>4.664</td>
<td>6.215</td>
<td>6.972</td>
<td>7.092</td>
</tr>
<tr>
<td>4.0</td>
<td>3.752</td>
<td>3.826</td>
<td>4.345</td>
<td>5.824</td>
<td>6.549</td>
<td>6.664</td>
</tr>
<tr>
<td>5.0</td>
<td>3.661</td>
<td>3.734</td>
<td>4.248</td>
<td>5.712</td>
<td>6.428</td>
<td>6.542</td>
</tr>
</tbody>
</table>

(a) and (b) for the in-plane force parameters \( \lambda b^2 = -2 \) (in-plane compression) and \( 2 \) (in-plane tension) in the cases of the edge-restraint parameter \( yb = 1 \) and aspect ratios \( a/b = 2 \) and \( 5 \). We can see from Table 1 that the convergence is very rapid for all cases.

Tables 2(a), (b) and (c) show the relationship between the lowest dimensionless eigenfrequency \( \kappa^2 b^2 \) and the three types of dimensionless in-plane forces, i.e., \( \lambda b^2 = -2 \) (in-plane compression), \( 0 \) (no in-plane force) and \( 2 \) (in-plane tension), in the cases of \( a/b = 1 - 5 \) and \( yb = 0 - \infty \). Although most of the numerical values in Table 2(b) for the case of no in-plane force have already been given in a previous paper(3), in which the dimensionless eigenfrequency was defined as the square root of the dimensionless eigenfrequency used in the present paper, we showed them here to clearly depict the tendency of the change in the eigenfrequency as the in-plane force changes gradually. Naturally, the dimensionless eigenfrequencies for \( yb = 0 \) and \( \infty \) in the tables are identical to those given in Refs. (1) and (2), respectively. For all aspect
Fig. 2 Lowest dimensionless eigenfrequency $\nu^2 b^2$ as a function of dimensionless in-plane force $\lambda b^2$ for edge-restraint parameters $\gamma b = 0$ (simply-supported), 1, 10 and $\infty$ (clamped) in case of $a/b = 2$ ($\nu = 0.3$)

ratios and edge-restraint parameters, the dimensionless eigenfrequency increases with increasing dimensionless in-plane force. Also, we can see that as the aspect ratio $a/b$ increases, it seems that the dimensionless eigenfrequency $\nu^2 b^2$ for a set of $\lambda b^2$ and $\gamma b$ approaches that for the corresponding set in Table A (see Appendix), giving the numerical values of the dimensionless eigenfrequency for the cylindrical bending vibration of a very long rectangular plate with both ends restrained elastically.

In Fig. 2, the change in the lowest dimensionless eigenfrequency $\nu^2 b^2$ is shown in the cases of $a/b = 2$ and $\gamma b = 0, 1, 10, \infty$ as the dimensionless in-plane force $\lambda b^2$ changes continuously in the interval $-2 \leq \lambda b^2 \leq 2$. We can see that with increasing $\lambda b^2$, the dimensionless eigenfrequencies $\nu^2 b^2$ increase monotonically for all edge-restraint parameters $\gamma b$.

6. Concluding Remarks

The free flexural vibration of an elliptical plate subjected to a uniform in-plane force in its middle plane was rigorously analyzed under the condition that the plate edge is restrained rigidly against transverse displacement and elastically against rotation. The displacement expression during vibration was obtained in the form of an infinite Mathieu function series. The frequency equation determining the eigenfrequency for the vibration mode symmetrical about both axes of the ellipse was derived. The frequency equation for a circular plate was also derived rigorously from that obtained here for an elliptical plate as a limiting case when the focal length tends to zero. The lowest dimensionless eigenfrequencies of elliptical plates with six types of edge-restraint parameters were calculated numerically for various dimensionless in-plane forces. Those increase with increasing dimensionless in-plane force and edge-restraint parameter, and decrease with increasing aspect ratio.

The numerical values given in the present paper will also constitute a valuable contribution toward checking very precisely the accuracy of other approximate numerical techniques.

References

Appendix: Free Flexural Vibration of a Very Long Rectangular Plate Restrained Elastically at Its Parallel Edges and Subjected to an Uniform In-plane Force Distributed in Its Middle Plane

The differential equation of motion of a very long rectangular plate with a finite width 2b in its y−direction and subjected to an in-plane tensile force in its middle plane is

\[
\frac{\partial^2}{\partial y^2} \left( D \frac{\partial^2}{\partial y^2} - p \right) w + \rho h \frac{\partial^2 w}{\partial t^2} = 0. \tag{A.1}
\]

Substitution of \( w = W \sin \omega t \) into Eq.(A.1) readily yields

\[
\left( \frac{d^2}{dy^2} + \alpha^2 \right) \left( \frac{d^2}{dy^2} + \beta^2 \right) W = 0, \tag{A.2}
\]

where \( \alpha^2 \) and \( \beta^2 \) are the same as those in expression (7). The solution of Eq.(A.2) is, using the unknown constants \( C_1, C_2, C_3 \) and \( C_4 \),

\[
W = C_1 \sin \alpha y + C_2 \cos \alpha y + C_3 \sinh \beta y + C_4 \cosh \beta y. \tag{A.3}
\]

The boundary conditions are

\[
W\bigg|_{y=\pm b} = 0, \quad -D \frac{d^2 W}{dy^2}\bigg|_{y=b} = K \frac{dW}{dy}\bigg|_{y=-b}, \quad D \frac{d^2 W}{dy^2}\bigg|_{y=-b} = K \frac{dW}{dy}\bigg|_{y=b}. \tag{A.4}
\]

Substituting expression (A.3) into Eq.(A.4) and eliminating \( C_1, C_2, C_3 \) and \( C_4 \) yield the frequency equation

\[
\begin{vmatrix}
\sin ab & \cos ab & \sinh \beta b \\
-\sin ab & \cos ab & -\sinh \beta b \\
\alpha^2 \sin ab - \gamma \alpha \cos ab & \alpha^2 \cos ab + \gamma \alpha \sin ab & -\beta^2 \sinh \beta \beta - \gamma \beta \cosh \beta b \\
\alpha^2 \sin ab - \gamma \alpha \cos ab & -\alpha^2 \cos ab - \gamma \alpha \sin ab & -\beta^2 \sinh \beta \beta - \gamma \beta \cosh \beta b \\
\end{vmatrix} = 0. \tag{A.5}
\]

In the case of the cylindrical bending vibration mode, which is corresponding to the lowest normal vibration mode of the very long rectangular plate, numerical examples of the dimensionless eigenfrequency \( \kappa^2 b^2 \) obtained from Eq.(A.5) are given in Table A.

<table>
<thead>
<tr>
<th>( \lambda b^2 )</th>
<th>0</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1.074</td>
<td>1.279</td>
<td>2.294</td>
<td>4.169</td>
<td>4.901</td>
<td>5.010</td>
</tr>
<tr>
<td>0</td>
<td>2.467</td>
<td>2.564</td>
<td>3.198</td>
<td>4.782</td>
<td>5.486</td>
<td>5.593</td>
</tr>
</tbody>
</table>