Damping Property and Vibration Analysis of Blades with Viscoelastic Layers*

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Abstract
This paper showed the damping effect and the vibration analysis of a shaft-disk-blade system with viscoelastic layers on blades. The focus of the research is on the shaft’s torsional vibration and the blade’s bending vibration. The equations of motion were derived from the energy approach. This model, unlike the previous, used only two displacement functions for layered blades. Then, the assumed-modes method was employed to discretize the equations. The analyses of natural frequencies damping property were discussed afterwards. The numerical results showed the damping effects due to various constraining layer (CL) thickness and viscoelastic material (VEM) thickness. The research also compared FRF’s of the systems with and without viscoelastic layers. It is concluded that both CL and VEM layers promote the damping capability but the marginal effect decreases with their thickness. The CLD treatment also found drop the natural frequencies slightly.

Key words: Damping; Viscoelastic Layers; Rotor Vibration

1. Introduction

Constrained layer damping (CLD) treatment provided an effective way to suppressing vibration and noise in structures. Theoretical works on sandwich beams with a viscoelastic core could be traced back to DiTaranto(1) and Mead and Markus(2) for the axial and the bending vibration of beams. Douglas and Yang(3) modified the theories and applied it to different applications. In last decades, many authors have reported new formulations and techniques, e.g., for beams(4),(5), for plates(6),(7) and for shells(8-11). Johnson(12) summarized the passive damping related works.

Chen and Huang(13) derived the theory for the vibration reduction and optimization of a rectangular plate with partial CLD treatment. They investigated the effects of treatment size and location, CL thickness and stiffness, and VEM thickness. Hu and Huang(14) then developed a generic theory for the constrained layer damping treatment that could be applied to any other commonly encountered geometry, such as cylindrical shells, plates, cones, and beams.

A shaft-disk-blades unit has been for decades a basis in rotor dynamics. The rotor has a peculiar feature that its vibration is unavoidable as long as it is in empowering stroke. Many effects have been put on individual elements such as blade, shaft, disk and supports to suppress vibration. Huang and Ho(15) utilized the concept of structure synthesis and analyzed a whole shaft-disk-blade system rather than separate elements. They discovered that there existed shaft-blade coupling modes and inter-blades coupling modes. The present paper is intended to apply the CLD treatment on the rotating blades and to look into its suppression effects on shaft-torsion and blade-bending coupling vibration.
Nomenclature

- \( q \) generalized vector
- \( \nu \) blade displacements with respect to the Y axes
- \( W_i \) \( i^{th} \) mode shape of the transverse blade
- \( U_i \) \( i^{th} \) mode shape of the axial blade
- \( \phi \) shaft-disk torsional displacement relative to rotation frame
- \( \Phi_i \) \( i^{th} \) mode shape of the shaft-disk
- \( \omega \) natural frequency
- \( \omega^* \) dimensionless natural frequency \( (\omega^* = \omega / \omega_{b0}) \)
- \( \omega_{b0} \) first natural frequency of single cantilever blade
- \( \Omega \) rotational speed
- \( \Omega^* \) dimensionless rotational speed \( (\Omega^* = \Omega / \omega_{b0}) \)

Subscripts

- ( )\(_b\) blade
- ( )\(_c\) CL
- ( )\(_d\) disk
- ( )\(_h\) Host structure
- ( )\(_s\) shaft
- ( )\(_v\) VEM

2. Theoretical analysis

A rotor system composed of shaft, disk and blades is shown in Fig. 1. Energies of the system are first derived and the assumed mode method is employed to discretize the equations of motion.

2.1 Rotating Shaft-Disk

The torsional energies associated with the shaft-disk are:

\[
T_s = \frac{1}{2} \int_0^{L_s} I_s \left( \frac{\partial \phi}{\partial t} + \Omega \right)^2 dZ + \frac{I_d}{2} \left( \frac{\partial \phi}{\partial t} + \Omega \right)^2 |_{Z=Z_d} \\
U_s = \frac{1}{2} \int_0^{L_s} G_r J_r \left( \frac{\partial \phi}{\partial Z} \right)^2 dZ
\]

where \( \phi(Z,t) \) is the torsional displacement of the shaft with respect to a constantly rotating \( (\Omega) \) frame; \( L_s, I_s, \) and \( G_r J_r \) denote the shaft’s length, polar rotary inertia, and torsional rigidity, respectively. \( I_d \) is the disk’s polar rotary inertia.
2.2 Rotating Blades

The blade is assumed to be clamped onto the rigid disk at one end and free at the other end as shown in Fig. 2. The blade is of CLD treatment on one side. The host structure and the constraining layer are assumed isotropic and elastic. The core is viscoelastic material (VEM) with a frequency dependent shear modulus $G'$. The displacements of the host structure and CL, $u_h$ and $u_c$ are expressed as

$$u_x = u_h - d \frac{\partial w}{\partial x} - c_h \left( \frac{\partial^2 u_h}{\partial x^2} \right)$$

$$u_x = u_c - d \frac{\partial w}{\partial x} + c_c \left( \frac{\partial^2 u_c}{\partial x^2} \right) = 0$$

where $c_h = K_h j / G_h$ and $c_c = K_c j / G_c$

![Fig. 2 The viscoelastic layered blade.](image)

![Fig. 3. Coordinate sets and deformation of a blade](image)

Figure 3 shows the rotating blade cantilevered onto a rigid disk. The $(X, Y, Z)$ coordinate system is the inertia frame; $(x, y, z)$ frame rotates at a constant speed $\Omega$. The kinetic and strain energies associated with a blade are

$$T_b = \frac{1}{2} \int_0^l \rho_b A_b \left( \frac{\partial v_h}{\partial t} \right)^2 + (x^2 + v_h^2) \Omega^2 + 2x\Omega \frac{\partial v_h}{\partial t} \right] dx$$

$$U_b = \frac{1}{2} \int_0^l E_b I_b \left( \frac{\partial^2 v_h}{\partial x^2} \right)^2 dx + \int_0^l E_b A_h \left( \frac{\partial u_h}{\partial x} \right)^2 dx + \int_0^l E_c A_c \left( \frac{\partial u_c}{\partial x} \right)^2 dx$$

$$+ \int_0^l \frac{G_c A_c}{2} \left( \frac{\partial \psi}{\partial x} + u_x - u_c \right)^2 dx + \frac{1}{4} \int_0^l \rho_c A_c \Omega^2 (r_c^2 - x^2) \left( \frac{\partial \psi}{\partial x} \right)^2 dx$$

where $\rho_b A_b = \rho_c A_c + \rho_A V_A + \rho_b A_h$, $\rho_b$, $A_b$ and $L_b$ are combined mass density, cross-section area and length. $E_b I_b = E_c I_c$, $E_h$, $I_h$ are Young’s modulus and area moment of inertia. $v_h(x, t)$ is the transverse displacements in $y$ directions. The subscripts $b$, $c$, $v$ stand for blade,
CL and VEM respectively.
The kinematic relations between these displacements are
\[ v_1(x,t) = x\phi(Z,t) + w_1(x,t) \]  
(7)

### 2.3 Equations of Motion

Assumed mode method is adopted to discretize the continuous system, i.e.,
\[ \Phi(Z,t) = \sum_{i=1}^{\infty} \Phi_i(Z)\eta_i(t) = \Phi(Z)\eta(t) \]  
(8)
\[ w_1(x,t) = \sum_{i=1}^{\infty} W_i(x)\xi_i(t) = W(x)\xi(t) \]  
(9)
\[ u_{ik}(x,t) = \sum_{i=1}^{\infty} U_i(x)\phi_{ik}(t) = U(x)\phi_{ik}(t) \]  
(10)
where \( \Phi_i, W_i \) and \( U_i \) are the mode shapes of a torsional shaft, of a transverse blade, and of an axial blade, respectively. These modes are chosen to be
\[ \Phi_i(Z) = \sin\left(\frac{(2i-1)\pi Z}{2L}\right) \]  
(11)
\[ V_i(x) = (\sin \tau_x x - \sin \xi \alpha (\cos \tau_x x - \cosh \xi x) \]  
(12)
is the beam function for blade with
\[ [\cos \tau_i (r_1 - r_2)] \cosh \tau_i (r_1 - r_2) + 1 = 0 \]  
(13)
\[ \alpha_i = \frac{-\sin \tau_i (r_1 - r_2) - \sinh \tau_i (r_1 - r_2)}{\cos \tau_i (r_1 - r_2) + \cosh \tau_i (r_1 - r_2)} \]  
(14)

\( \eta, \xi \) and \( \phi \) are the participation factors. \( m, n \) and \( r \) are the number of modes deemed necessary for required accuracy.

Substitution of the above equations into the energy expressions and employment of the Lagrange equations yields the following discretized equations of motion in matrix notation as
\[ [M]q + ([K]^e + [K]^\Omega - [K]^v)q = 0 \]  
(15)
where \([K]^e\), arising from the elastic deflection, dominates at low rotational speed. \([K]^\Omega\), resulted from rotation, softens the rotor and becomes dominant at high rotational speed. It is also the major role decides the stability of the rotor. \([K]^v\) comes from CLD. The matrices \([M], [K]^e, [K]^\Omega\) and \([K]^v\) are given as:
\[
[M] = \begin{bmatrix}
[M]^e_{\text{trans}} & [M]^e_{\text{trans}} & [M]^e_{\text{trans}} & \cdots & [M]^e_{\text{trans}} \\
[M]^e_{\text{trans}} & [M]^e_{\text{trans}} & [M]^e_{\text{trans}} & \cdots & [M]^e_{\text{trans}} \\
[M]^e_{\text{trans}} & [M]^e_{\text{trans}} & [M]^e_{\text{trans}} & \cdots & [M]^e_{\text{trans}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
[M]^e_{\text{trans}} & [M]^e_{\text{trans}} & [M]^e_{\text{trans}} & \cdots & [M]^e_{\text{trans}} \\
[M]^\Omega_{\text{trans}} & [M]^\Omega_{\text{trans}} & [M]^\Omega_{\text{trans}} & \cdots & [M]^\Omega_{\text{trans}} \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]  
(16)
\[
[K]^e = \begin{bmatrix}
[K]^e_{\text{trans}} & [K]^e_{\text{trans}} & [K]^e_{\text{trans}} & \cdots & [K]^e_{\text{trans}} \\
[K]^e_{\text{trans}} & [K]^e_{\text{trans}} & [K]^e_{\text{trans}} & \cdots & [K]^e_{\text{trans}} \\
[K]^e_{\text{trans}} & [K]^e_{\text{trans}} & [K]^e_{\text{trans}} & \cdots & [K]^e_{\text{trans}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
[K]^e_{\text{trans}} & [K]^e_{\text{trans}} & [K]^e_{\text{trans}} & \cdots & [K]^e_{\text{trans}} \\
[K]^\Omega_{\text{trans}} & [K]^\Omega_{\text{trans}} & [K]^\Omega_{\text{trans}} & \cdots & [K]^\Omega_{\text{trans}} \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]  
(17)
\[
\begin{bmatrix}
K^\alpha \\
K^\beta \\
K^\gamma
\end{bmatrix} =
\begin{bmatrix}
K^\alpha_1 & K^\alpha_2 & \cdots & K^\alpha_N \\
K^\beta_1 & K^\beta_2 & \cdots & K^\beta_N \\
K^\gamma_1 & K^\gamma_2 & \cdots & K^\gamma_N
\end{bmatrix}
\] (18)

\[
\begin{bmatrix}
K_1 \\
K_2 \\
K_3
\end{bmatrix} =
\begin{bmatrix}
K_1^\alpha & K_1^\beta & K_1^\gamma \\
K_2^\alpha & K_2^\beta & K_2^\gamma \\
K_3^\alpha & K_3^\beta & K_3^\gamma
\end{bmatrix}
\] (19)

where
\[
\begin{align*}
K^\alpha &= [K_1] + [K_2]^T \\
K^\beta &= [K_1] + [K_2]^T \\
K^\gamma &= [K_2] + [K_2]^T
\end{align*}
\]

The matrices’s dimensions are \((m + N\times n) \times (m + N\times n)\), where \(N\) denoting the number of blades. The entries of all matrices are given in the Appendix. \(q\) is a generalized vector, i.e.,
\[
q = [\eta^T \ \zeta^T \ \chi^T \ \cdots \ \xi^T]^T
\] (21)

In the usual manner for free vibration analysis, it is assumed the solution is of the form
\[
q = \{d\} e^{\lambda t}, \quad \text{where} \quad \{d\} \quad \text{the undetermined coefficient vector and} \quad \lambda \quad \text{the eigenvalue. Note that}
\]
\[
\lambda = i\omega, \quad i = \sqrt{-1}
\]

Then, yield to be:
\[
\begin{bmatrix}
(K^\gamma + [K^\alpha]) - [K^\gamma] + \lambda[I]\{d\} = [0]
\end{bmatrix}
\] (22)

The characteristic equation is
\[
\begin{bmatrix}
(K^\gamma + [K^\alpha]) - [K^\gamma] + \lambda[I]
\end{bmatrix} = 0
\] (23)

### 2.4 Forced Vibration and Power Spectrum

We now consider FRF due to a uniform harmonic-force \(f = F_0 e^{i\omega t}\). The work due to the force is \(W_f\):
\[
W_f = \int_0^T f \cdot v_s(x, t) dx
\] (24)

Substituting Eqs. (7-9) into Eq. (24) gives
\[
W_f = \sum_{i=1}^N \int_{x_1}^{x_2} F_0 \cdot \Phi_i(Z_0)\eta_i(t)e^{i\omega t} dx + \sum_{i=1}^N \int_{x_1}^{x_2} F_0 \cdot \Phi_i(Z_0)\zeta_i(t)e^{i\omega t} dx
\] (25)

Employment of the Lagrange equations yields the following discretized equations of motion in matrix notation as
\[
[M]\{\ddot{q}\} + \{K^\gamma\} - [K^\gamma] \{\dot{q}\} = \{F\} e^{i\omega t}
\] (26)

where
\[
\{F\} = \sum_{i=1}^N \int_{x_1}^{x_2} F_0 \cdot \Phi_i(Z_0) dx
\] (28)
\[
\{F^i_k\} = \int_{a_i}^{b_i} F_i W_i(x) \, dx, \quad k = 1, 2, \ldots, N
\]  

(29)

where \(N\) denoting the number of blades.

To realize the influences of the CL and VEM thickness on the vibration suppression, we define a dimensionless power dissipation coefficient \(P^*\) as

\[
P^* = 10 \times \log \left( \frac{P_{CLD}}{P_{bare}} \right) \, \text{dB}
\]  

(30)

where \(P_{bare}\) is the power spectrum of the bare system and \(P_{CLD}\) is the one treated with CLD calculated in a bandwidth from 22 to 490 Hz which covers the first three natural frequencies of the system. \(P^*\) plays an index of damping capability.

3. Numerical Results

To be dimensional independent, the numerical results are normalized with respect to the cantilevered blade’s first natural frequency \((\omega_{b1})\), i.e., \(\omega^* = \omega / \omega_{b1}\) and \(\Omega^* = \Omega / \omega_{b1}\). Table 1 lists the geometric and material properties of the illustrated examples. Note that, the length of blades are deliberately elongated in order to magnify the coupling behaviors. Table 2 gives the natural frequencies of individual components, with the other components temporarily removed. Table 2 provides a comparison basis for the effects of component on coupling vibration. These frequencies serve as validation and interpretation of the numerical results as well.

It is first examined how the viscoelastic layers affect the rotor’s characteristics. Fig. 4 shows its frequency changes for a four- and a five-blade system. Note that the abscissa, not drawn in a linear scale, has four reference marks at \(\omega^* = 0.901, 5.879, 1\) and 6.267 respectively denoting the frequencies after and before treatment. It appeared CLD treatment decreased the natural frequencies.

Figure 5 illustrates the mode shapes of a four-blade system. The first x-y plot denotes the shaft’s torsional displacement and the blade’s deflection are illustrated in the following diagrams. Each mode’s natural frequency is specified on its upper right. It is seen that the even number modes are inter-blades modes and the odd number modes, with torsion displacement, are shaft-blade coupling modes.

![Fig. 4. Frequency changes due to viscoelastic layers on blades for a four- and a five-blade rotor](image)
Table 1: Geometric and material properties of the illustrated examples

<table>
<thead>
<tr>
<th>Component</th>
<th>Density: $\rho_s$</th>
<th>7850 kg/m$^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shaft</td>
<td>Shear modulus: $G_s$</td>
<td>82.78 Gpa</td>
</tr>
<tr>
<td></td>
<td>Shaft length: $L_s$</td>
<td>0.6 m</td>
</tr>
<tr>
<td>Disk</td>
<td>Radius: $r_s$</td>
<td>0.02 m</td>
</tr>
<tr>
<td>Disk</td>
<td>Location: $z_d$</td>
<td>0.1 m</td>
</tr>
<tr>
<td></td>
<td>Mass moment of inertia $I_d$</td>
<td>$4.93 \times 10^{-3}$ kg·m$^2$</td>
</tr>
<tr>
<td>Blade</td>
<td>Density: $\rho_{h,c}$</td>
<td>7850 kg/m$^3$</td>
</tr>
<tr>
<td></td>
<td>Young’s modulus: $E_{h,c}$</td>
<td>210 Gpa</td>
</tr>
<tr>
<td></td>
<td>Blade outer end: $r_b$</td>
<td>0.3 m</td>
</tr>
<tr>
<td></td>
<td>Thickness: $h_h$</td>
<td>4 mm</td>
</tr>
<tr>
<td>VEM</td>
<td>Density: $\rho_v$</td>
<td>1340 kg/m$^3$</td>
</tr>
<tr>
<td>VEM</td>
<td>Thickness: $h_v$</td>
<td>0.154 mm</td>
</tr>
<tr>
<td></td>
<td>Kerwin model</td>
<td>$6.0(1+i0.5)$ MPa</td>
</tr>
<tr>
<td></td>
<td>Douglas model</td>
<td>$0.142\left(\frac{\omega}{2\pi}\right)^{0.404}(1+i1.42)$ MPa</td>
</tr>
<tr>
<td>Rotational speed: $\Omega$</td>
<td>0 ~ 1000 Hz</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Natural frequencies (Hz) of shaft-disk, disk and blade

<table>
<thead>
<tr>
<th>Component’s n.f.</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shaft-disk (w/o blades)</td>
<td>553.18</td>
<td>2829.25</td>
<td>5480.73</td>
</tr>
<tr>
<td>Clamped blade (shaft-disk rigid)</td>
<td>83.55</td>
<td>523.61</td>
<td>1466.12</td>
</tr>
<tr>
<td>Viscoelastic layered blade (shaft-disk rigid)</td>
<td>75.29</td>
<td>491.17</td>
<td>1462.36</td>
</tr>
</tbody>
</table>
For a viscoelastic layered blade system, shear modulus is \( G^\nu = G^\mu (1 + i\beta) \). The damping factors are hence calculated and listed in Table 3.

<table>
<thead>
<tr>
<th>Component’s damping factor</th>
<th>( \xi_1 )</th>
<th>( \xi_2 )</th>
<th>( \xi_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.322</td>
<td>0.347</td>
<td>0.752</td>
</tr>
<tr>
<td>5</td>
<td>0.316</td>
<td>0.347</td>
<td>0.684</td>
</tr>
</tbody>
</table>

Figure 6 illustrates how the damping factor of the first modes, varying with the VEM thickness for a four-blade rotor. The x-axial denotes the VEM thickness ratio \( h_v^* = h_v / h_h \). The CL thickness ratio \( h_c^* = h_c / h_h \) were set 0.01, 0.05 and 0.1. It is seen that the damping factor increases with the increase of the VEM thickness. The damping increases rather rapidly at the beginning but slows its rate with further increase. Three curves denote three different CL thicknesses and as anticipated, thicker CL provides better damping ability for its bigger constraining force. Fig. 7 does the reverse way to illustrates how the damping factor of the first modes, varying with the CL thickness. The VEM thickness ratio \( h_v^* \) were set 0.01, 0.05 and 0.1. From Figs 6 and 7 it is noted that damping increased rather exponentially proportional to VEM thickness but not to CL thickness. Fig. 8 illustrates how the frequency of the first three modes, varying with the CL thickness for a four-blade rotor. It is seen that the frequency basically increases with the increase of the CL thickness. The first mode frequency yet tends to decrease because of large mass loading of thicker CL.

Figure 9 shows FRF of a CLD four-blade rotor. Fig. 9(a) shows the torsional response of the shaft. Fig. 9(b) shows the bending response of the blade. The speed is \( \Omega = 10^3 \text{ rpm} \), and \( h_v = 0.354 \text{mm}, \ h_c = 0.176 \text{mm} \). The dashed curve is for no treatment. The long-dashed and solid curves represent the cases with the Douglas \(^1\) and Kerwin \(^{16}\) viscoelastic model respectively. As seen, the presence of CLD significantly reduces the response amplitude. Overall, the significant damping effects appear particularly on the blade’s higher frequency modes. The Douglas viscoelastic system is more obvious.
Fig. 6. Damping factor of the first modes varying with the VEM thickness for a four-blade rotor

Fig. 7. Damping factor of the first modes varying with the CL thickness for a four-blade rotor

Fig. 8. Frequency varying with the CL thickness for a four-blade rotor
Figure 10 shows the power dissipation coefficient varying with the CL thickness for a four-blade rotor. The CL thickness ratio $h_c^*$ were set 0.01, 0.05 and 0.1. From the figure, it is
seen that for a 10% CL thickness, the VEM treatment can reduce the power up to 12dB for a full VEM thickness and 5dB for a 10% VEM thickness.

4. Conclusion

This research explored a shaft-disk-blade system with CLD treatment on blades. The assumed modes method was employed for the analysis. It is arrived at that the frequency is lower due to viscoelastic layers. The even number modes are inter-blades modes and the odd number modes, with torsion displacement, are shaft-blade coupling modes.

The quantitative studies showed that the effects of constrained layer (CL) thickness and viscoelastic material (VEM) thickness for blades. It is noted that damping increased rather exponentially proportional to VEM thickness but not to CL thickness. The FRF studies showed that the presence of constrained-layer damping reduces the vibration amplitude of the system. The significant damping effects appear particularly on the frequency modes. At last, the power dissipation coefficient studies showed that increasing the VEM thickness means the increase of energy absorbing and the increase of CL thickness implies the increase of constrained shear force in VEM. Both enhance the damping capability.

References


Appendix : Matrices Elements

\[
[M_1]_{ij} = \int_0^l I_1 \Phi_i \Phi_j dz + I_3 \Phi_i (Z_d) \Phi_j (Z_d) + \sum_{i=1}^{N} \int_0^l \rho_0 A_i x^2 \Phi_i (Z_d) \Phi_j (Z_d) dx
\]  
(A1)

\[
[M_2]_{ij} = \int_0^l \rho_0 A_i W_i W_j dx
\]  
(A2)

\[
[M_3]_{ij} = \int_0^l \rho_0 A_i x \Phi_i (Z_d) W_j dx
\]  
(A3)

\[
[K_1]_{ij} = \int_0^l G_i j \Phi_i' (Z_d) \Phi_j' (Z_d) dz + \sum_{i=1}^{N} \int_0^l E_i A_i \Phi_i (Z_d) \Phi_j (Z_d) dx
\]  
(A4)

\[
[K_1^O]_{ij} = \frac{1}{2} \sum_{i=1}^{N} \int_0^l \rho \omega^2 \left[ c_i^2 - 3 \lambda \right] \Phi_i (Z_d) \Phi_j (Z_d) dx
\]  
(A5)

\[
[K_2^O]_{ij} = \int_0^l \rho_0 A_i \omega^2 \left[ - W_i W_j + \frac{1}{2} \left[ c_i^2 - 3 \lambda \right] W_i' W_j' \right] dx
\]  
(A6)

\[
[K_3^O]_{ij} = \int_0^l E_i A_i d^2 \Phi_i (Z_d) W_j' dx
\]  
(A7)

\[
[K_4^O]_{ij} = \int_0^l \rho_0 A_i \omega^2 \Phi_i (Z_d) \left[ - x W_i + \frac{1}{2} \left[ c_i^2 - 3 \lambda \right] W_i' \right] dx
\]  
(A8)

\[
[K_5^O]_{ij} = \int_0^l E_i A_i d^2 \Phi_i (Z_d) \left[ - U_j' + C_i U_j'' \right] dx
\]  
(A9)

\[
[K_6^O]_{ij} = \int_0^l \rho_0 A_i \omega^2 \left[ - W_i U_j' + C_i U_j'' \right] \Phi_i (Z_d) \Phi_j (Z_d) dx
\]  
(A10)

\[
[K_7^O]_{ij} = \int_0^l E_i A_i W_i U_j' + E_i A_i C_i U_j'' \left[ C_i U_j''' - U_j'' \right] dx
\]  
(A11)

\[
[K_8^O]_{ij} = \int_0^l \left[ E_i A_i U_i U_j' + E_i A_i C_i U_j'' \left[ C_i U_j''' - U_j'' \right] \right] dx + \frac{G_i A_i}{h^2} C_i \left[ U_i'' U_j' + C_i U_j'' U_j'' - C_i U_j''' U_j'' \right] dx
\]  
(A12)