A Numerical Calculation Model of Multi Wound Foil Bearing with the Effect of Foil Local Deformation *

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Abstract
Foil bearings are supposed to be one of the best candidates of supporting component for turbo-machineries because of their design simplicity, reduced weight and size, high speed and temperature capability, and easy maintenance. Among various types of foil bearings, multi wound foil bearing (MWFB), which had been designed and fabricated in our lab, is easy to analyze static characteristics even though load capability of which is small compared with other types of foil bearings.

In this study, a theoretical model of MWFB taking account of the effect of the foil deformation is developed to predict its static performance. Reynolds equation is solved using Finite Difference Method (FDM) to yield air pressure distribution, while the elastic deformation equation is solved by Finite Element Method (FEM) to predict the deformation of the foil. Then, the above two equations are coupled by several iterations until the convergence criterion is reached. Based on such calculations, static characteristics of MWFB such as load capacity, torque are presented.

Key words: Multi Wound Foil Bearing (MWFB), Foil Deformation, Static Performance

1. Introduction
Foil bearings, which can be simply defined as a kind of compliant, self-acting hydrodynamic fluid film bearing by functioning ambient air as lubricant, are considered to be the key technology for oil free turbo-machinery application. They satisfy most of the requirements of oil free turbo-machinery, including design simplicity, reduced weight and size, high speed and temperature capability, and reduced maintenance (1). They are supposed to be the best substitutions of the rolling elements bearings and oil-lubricated bearings. However, at present, the applications of foil bearings mainly rely on an experimental build and test development sequence, which results a severe time consuming and money cost. So, there is an urgent requirement to find out an accurate predictive performance analysis method.

H. Heshmat (2) wrote a crucial paper in 1983, in which the elastic equation for foil deflection was substituted into Reynolds’ equation and finite difference formulas for these equations were derived and various structural, geometric and operational variables were discussed including the start of bearing arc, selection of load angle, number of pads and degree of compliance. Then, he and C.P. Roger Ku (3,4) analyzed the deflection of a single
bump, the friction forces between bump foils and the housing or the top foil and the structural stiffness and damping coefficients. In 1994, based on the previous model, they modified the model of bump foil by taking the curvature effect and fixed end effect (the moving direction of bump foil) into consideration. Meanwhile, Peng and Carpino (5,6) paid attention to the effect of misalignment and effects of membrane stresses, and finite element methods were used to predict both the structural deflections and the pressures distribution. In 2004, based on the elasticity equation of foil bearing by Heshmat (1983), ZC Peng, MM Khonsari (7) improved the method to deal with the effect of foil deformation by taking the motion of shaft into account.

However, by present, the effect of foil deformation on static performance is still not well understood, particularly for the MWFB. This paper tries to find a way to predict the static performance of MWFB with consideration of foil elasticity and gas compressibility. A theoretical model of MWFB is developed. Reynolds’ equation is solved using Finite Difference Method (FDM) to yield air pressure distribution, while the elastic deformation equation is solved by Finite Element Method (FEM) to predict the deformation of the foil. Then, the above two equations are coupled by several iterations until the convergence criterion is reached. Based on such calculations, static characteristics of MWFB such as load capacity and torque are presented.

2. Structure of Multi Wound Foil Bearing

The sketch of the Multi Wound Foil Bearing is shown in Fig. 1.a. The components of bearing are including the housing, the shaft and the wound foil with hemispherical projections distributed on one side by adequate interval. It is shown in Fig. 1.b. The foil is made from a phosphor bronze plate of 0.1mm in thickness, 20mm in width with many hemispherical projections whose height is 0.2mm. This plate is manufactured by wet etching process and projections are made by specially designed jig. The foil is wound triply and fixed by the housing with the plate edge bent into housing at the position of 3mm. The bearing radial clearance between shaft surface and bearing surface is designed to be 20μm (8).

3. Solution of Reynolds’ Equation

3.1 Reynolds’ Equation

The air pressure distribution of foil bearing is yielded from the Reynolds’ Equation by taking the air as the ideal compressible gas flow. The dimensionless compressible Reynolds’ Equation under isothermal condition is given as (2):

\[
\frac{\partial}{\partial \theta} \left( \frac{p h^2}{\bar{\theta}} \frac{\partial \bar{p}}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( \frac{p h^3}{\bar{\theta}} \frac{\partial \bar{p}}{\partial z} \right) = \Lambda \frac{\partial}{\partial \theta} \left( \bar{p} h \right)
\]

(1)

where

\[
\bar{p} = \frac{p}{p_a}, \bar{h} = \frac{h}{C}, \bar{z} = \frac{z}{R}, \Lambda = \frac{6 \mu_a \omega}{p_a} \left( \frac{R}{C} \right)^2
\]

(2)

Then, according to Fig. 2, the film thickness can be given as following:
\[ h = 1 + \epsilon \cos(\theta - \theta_0) + \frac{S_1(\theta,z) + S_2(\theta,z)}{C} \]

Where \( \theta_0 \) is the angular position of the minimum film thickness, while \( S_1(\theta,z) \) and \( S_2(\theta,z) \) are the deformations of foils (shown as the dashed line in Fig. 2), calculated from elastic deformation equation and explained in the next section. And \( \epsilon \) indicates the eccentricity ratio of bearing, which is defined as \( \epsilon = e / C \).

### 3.2 Boundary Condition

The boundary conditions for the solution of Reynolds’ Equation are:

\[
\begin{align*}
\text{at } \theta &= \theta_1, & \bar{p} &= \left( \frac{p}{p_a} \right) = 1 \\
\text{at } \theta &= \theta_2, & \bar{p} &= \left( \frac{p}{p_a} \right) = 1 \\
\text{at } \bar{z} &= \pm (L/D), & \bar{p} &= \left( \frac{p}{p_a} \right) = 1
\end{align*}
\]

The foil bearing essentially does not generate subambient pressures and there is no bearing load at the position of \( \theta_3 \), so, the air pressure at the top of foil equals to \( p_a \). In circumference direction, the film pressure will increase due to the relative motion of shaft and foil. At an unknown angular position \( \theta_4 \), the film pressure will decrease to ambient, called Reynolds boundary condition. Here the pressure must fulfill both the ambient pressure and zero pressure gradient boundary condition. For both sides of the bearing are connected with ambient, the pressures at sides are also \( p_a \).

### 3.3 Numerical Procedures

Finite Difference Method (FDM) is used to deal with this partial differential equation. Firstly, the air film is divided into a grid, as shown in Fig. 4. Then, the pressure gradient of a point is defined with the pressures of the four points around it. Then, Eq. (1) can be written in difference formulas as:

\[
\begin{align*}
\bar{p}_{i,j} &= \frac{1}{2\Delta \theta} \left( \bar{p}_{i+1,j} - \bar{p}_{i-1,j} \right) + 3\bar{p}_{i,j} \left( \frac{\partial \bar{p}_{i,j}}{\partial \theta} \right) + \bar{p}_{i,j} \left( \frac{\partial \bar{p}_{i,j}}{\partial z} \right) + \bar{p}_{i,j} \left( \frac{\partial \bar{p}_{i,j}}{\partial x} \right) + \bar{p}_{i,j} \left( \frac{\partial \bar{p}_{i,j}}{\partial y} \right)
\end{align*}
\]

The above equation is nonlinear and substituted as

\[
\begin{align*}
\bar{p}_{i,j} &= \left( \frac{p_{(i+1,j)} - \bar{p}_{i,j}}{2\Delta \theta} \right) + 3\bar{p}_{i,j} \left( \frac{\partial \bar{p}_{i,j}}{\partial \theta} \right) + \bar{p}_{i,j} \left( \frac{\partial \bar{p}_{i,j}}{\partial z} \right) + \bar{p}_{i,j} \left( \frac{\partial \bar{p}_{i,j}}{\partial x} \right) + \bar{p}_{i,j} \left( \frac{\partial \bar{p}_{i,j}}{\partial y} \right)
\end{align*}
\]

The Newton-Raphson method is used to linearize Eq. (5) \((*)\),

\[
\begin{align*}
f_{i,j}^{(a)} + \frac{\partial f_{i,j}^{(a)}}{\partial p_{i-1,j}} \left( p_{(i+1,j)} - p_{(i,j)} \right) + \frac{\partial f_{i,j}^{(a)}}{\partial p_{i,j}} \left( p_{(i+1,j)} - p_{(i,j)} \right) + \frac{\partial f_{i,j}^{(a)}}{\partial p_{i+1,j}} \left( p_{(i+1,j)} - p_{(i,j)} \right)
\end{align*}
\]

The above equation is nonlinear and substituted as

\[
\begin{align*}
f_{i,j}^{(a)} + \frac{\partial f_{i,j}^{(a)}}{\partial p_{i-1,j}} \left( p_{(i+1,j)} - p_{(i,j)} \right) + \frac{\partial f_{i,j}^{(a)}}{\partial p_{i,j}} \left( p_{(i+1,j)} - p_{(i,j)} \right) + \frac{\partial f_{i,j}^{(a)}}{\partial p_{i+1,j}} \left( p_{(i+1,j)} - p_{(i,j)} \right) = 0
\end{align*}
\]
where, \( n \) stands for the old value, while \( n+1 \) stands for the new value.

4. Solution of Elastic Deformation Equation

4.1 Geometry Model of Foil and Assumptions

The foil is wound triply with bottom layer contacting with the housing and top layer supporting the shaft. So, the foil can be divided into three parts, which are named by their position as top foil, mid foil and bottom foil. For easy understanding, force distribution analysis is done with the case that the three foils are unwrapped as shown in Fig. 3. The foils are separated by projections on the surface of bottom foil and mid foil and the projections are placed at the middle position of two projections on the other layer in circumferential direction.

![Fig. 3 Geometry model and Force analysis of foils](image)

In Fig. 3, \( p-p_o \) is the air pressure forcing on the top foil, \( F_{tm}^{ij} \) and \( F_{bm}^{ij} \) stand for the force acting on mid foil from bottom foil and top foil at the projection \( (i, j) \), while \( F_{mt}^{ij} \) stands for the counterforce of \( F_{tm}^{ij} \) on top foil.

In solving the deformation of foils, the following assumptions are made:
- For the bottom foil contacts with the housing, it is assumed to have no deformation.
- The projections are considered to be rigid points, which means the projections have no size and deformation. In Fig. 3, the projections are enlarged for easier understanding. In fact, they are small protuberances on foil without size.
- The start edge and end edge of the mid foil are fixed, while the start edge of top foil is fixed and end edge is free, as shown in Fig. 3.
- The foil is a plate without thickness, which means the displacements at two surfaces equal to the displacement of middle section.
- The foil has only bended deformation, for the effect of foil extension on the film thickness can be neglected comparing to the effect of bended deformation.

4.2 Elastic Deformation Equation

Based on the principle of virtual work, virtual strain energy equals to virtual work done by external forces \( \{F\}^e \). After knowing, we can formulate the strain energy by multiplying the strain with stress of element, if we impose an arbitrary (virtual) nodal displacement \( \{\delta\}^e \) and relevant virtual strain \( \{\kappa\}^e \):

\[
\left(\{\delta\}^e\right)^T \{F\}^e = \iint \{\kappa\}^e [M] \{\delta\}^e [D] [B] \{\delta\}^e d\theta dz \quad (8)
\]

For the arbitrariness of \( \{\delta\}^e \), the above equation can be transformed into

\[
\{F\}^e = [K]^e \{\delta\}^e \quad (9)
\]

with the stiffness matrix for element (see appendix)

\[
[K]^e = \iint [B]^T [D] [B] d\theta dz \quad (10)
\]

4.3 Mesh Generation and Boundary Condition

With the geometry of foil and assumptions given above, the foil is meshed by using
rectangular element with 4 nodes and 12 degrees of freedoms as shown in Fig. 4. It must be ensured that there is a node on every projection during meshing, since the projections suffered the interaction forces between foils.

![Fig. 4 The mesh generation of foil model](image)

In Fig. 4, the solid points stand for the projections. To make sure the projections are all at the corners of the elements, the positions of nodes, we mesh the foils as above. And every section may have $n \times m$ elements, whose values can be changed to reach various calculation precisions. The integers in the figure stand for the serial number of element and node (the ones in brackets) under the condition of $n = m = 1$.

The boundary conditions of the foil FEM model are as following:

\[
\begin{align*}
\text{For fixed edges:} & \quad w_{|_{\theta=0}} = 0 \\
\text{For projections:} & \quad w_{|_{z=0}} = 0
\end{align*}
\]

where, $w$ indicates the displacements of nodes in vertical direction of foil surface.

4.4 Film Thickness with the Effect of Foil Deformations

The film thickness was given by Eq. (3), which means the film thickness is due to the eccentricity as well as the deformation of foils under the imposed hydrodynamic pressures. Since the top foil contacts with the gas film, the deformation of it, $S_1(\theta, z)$, can be added to film thickness directly, while the deformation of mid foil affect the film thickness by changing the position of projections, which support the top foil. To obtain the variety of film thickness ($S_2(\theta, z)$) caused by the deformation of mid foil, we should find a relationship between the displacements of the top foil and the projections on mid foil. Fig. 5 shows a small segment of top foil supported by four projections. The displacements of four projections are assumed to be $y_1, y_2, y_3,$ and $y_4$. A and B are the distances between every two projections in $\theta$ and $z$ direction. Then, the variety of film thickness caused by the deformation of mid foil can be written as

\[
S_2(\theta', z') = \frac{z'}{B} \left( y_4 - y_1 \right) + \frac{\theta'}{A} \left( y_3 - y_4 \right) - \frac{\theta'}{A} \left( y_2 - y_1 \right) + \frac{\theta'}{A} \left( y_2 - y_1 \right) + y_1
\]

where, $\left( \theta', z' \right)$ is the position related to the first projection shown as the projection with displacement of $y_1$ in Fig. 5.
5. Coupling Solution and Result

5.1 Coupling Solution

To obtain the effect of the foil deformations on air pressure distribution, an appropriate iterative method is used to couple the Reynolds’ Equation producing the air pressure distribution with the elastic deformation equation yielding the foil deformation. The logic chart is shown in Fig.6. The convergence criterion is \(|(p-p_{old})/p|<0.01\), where the \(p\) and \(p_{old}\) stand for the new value and old value of the air pressure.

![Fig.6. MWFB program logic chart with the effect of foil deformation](image)

5.2 Calculation Result

To detect the effect of foil deformation, we compare the air pressure distribution with and without foil deformation under the same operation condition. Table 1 shows the geometrical and lubricant data of air film, while Table 2 shows the geometrical and material parameters of foils. For easy understanding, the air film is cut at the start position \(\theta_s\) and unwrapped in circumferential direction. Fig. 7 shows the dimensionless air pressure distributions with and without the effect of foil deformation under operating conditions of the rotational speed \((w)\) at 50 Krpm, the eccentricity ratio \((\epsilon)\) at 0.8 and the angular position of minimum film thickness \((\theta_0)\) at 40 degree.

Table 1 The geometrical and lubricant data of air film

<table>
<thead>
<tr>
<th>Shaft radius ((R))</th>
<th>10 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bearing Length ((L))</td>
<td>20 mm</td>
</tr>
<tr>
<td>Radial clearance ((C))</td>
<td>0.02 mm</td>
</tr>
<tr>
<td>Absolute viscosity ((\mu_0))</td>
<td>(1.73<em>10^{11} ) N</em>s/mm²</td>
</tr>
<tr>
<td>Ambient pressure ((p_a))</td>
<td>(101.3*10^{-3} ) N/mm²</td>
</tr>
</tbody>
</table>

Table 2 The geometrical and material parameters of foils

<table>
<thead>
<tr>
<th>Number of projections in axial</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of projections in circumferential direction</td>
<td>15(top) or 16(mid)</td>
</tr>
<tr>
<td>Young’s Modulus ((E))</td>
<td>(9.8*10^6) N/mm²</td>
</tr>
<tr>
<td>Poisson’s ratio ((\nu))</td>
<td>0.43</td>
</tr>
<tr>
<td>Thickness of foil ((t))</td>
<td>0.1 mm</td>
</tr>
</tbody>
</table>
As shown in Fig. 7, the air pressure in over ambient region becomes lower if the effect of foil deformation is taken into account. The pressure reaches its maximum value a little before the minimum clearance takes place. The maximum value will decrease from 3.0145 to 1.9734. The change of air pressure is due to the foil deformations, which increase the film thickness. The deformations of top foil and mid foil under this operating condition are shown in Fig. 8.

(a) Air pressure distribution of a rigid foil  
(b) Air pressure distribution with the effect of foil deformation

Fig. 7 The dimensionless air pressure distributions with and without the effect of foil deformation

(a) The deformation of top foil  
(b) The deformation of mid foil

Fig. 8 The deformations of top foil and mid foil
For top foil, the edge connecting with mid foil has no displacement, shown as the fixed edge in Fig. 8(a). The positions of the projections supporting the top foil are also assumed to have no displacements in the calculation, for the displacements of them are not contained in the top foil and considered in the model of mid foil and added to film thickness by Eq. (12). In Fig. 8(a), the equally spaced points with no displacement indicate the projections. Beside these points, all of the else regions of top foil are found to have minus deformations, for the direction of gas pressure is downwards in Fig. 8(a). The deformations of the section near the position of minimum film thickness are noted to be bigger comparing to the ones of other areas. That is because the gas pressure reaches the maximum value at this section as shown in Fig. 7.

For mid foil, the two edges jointing with the other foils are both assumed to be fixed, as pointed out in §4.1, which are denoted as fixed edges in Fig. 8(b). It is easily detected that the deformations at the positions of projections are scraggy with the bulges and hollows appearing alternately, in which the bulges indicate the projections supporting mid foil, while the hollows show the ones supporting top foil. The reason of this fact can be explained as that the mid foil suffers two sets of forces with contrary direction comparing to each other from the projections on the mid foil and button foil as shown in Fig. 3 and the action of them make the mid foil deform towards opposite directions at the position of projections, shown in Fig. 8(b). Actually, the deformations of mid foil at every projection are not changeless, and the maximum value of deformations appears at the position of maximum air pressure, which is not shown clearly in Fig. 8(b) for the differences are too small. The reason can be given as that the counterforce of the supporting force from the top foil because maximal at this position following air pressure as known in the discussion of Fig. 8(a).

After knowing the deformations of both top foil and mid foil, we now can evaluate the film thickness with the effect of foil deformation by using Eqs. (3), (12). The dimensionless film thickness under the mentioned operating condition is shown in Fig. 9. The foil deformations affect the film thickness by not only shifting it in thickness direction but also forming some local peaks.

5.3 Calculation of Static Performance Parameters
According to reference (2), the static performance parameters can be obtained by integral of air pressure \( p(\theta, z) \). The dimensionless load capacity is given as following:

\[
\overline{W} = \frac{W}{p_o R^2} = \sqrt{F_x^2 + F_y^2}, \tan \phi_z = \frac{F_x}{F_y}
\]  

(13)
where, \( \overline{F_x} \) and \( \overline{F_y} \) are the dimensionless loads in horizontal direction and vertical direction, which can be calculated from

\[
\begin{align*}
\overline{F_x} &= \frac{F_x}{p_i R^2} = \int_{-L/D}^{L/D} \int_0^{360} \left( \tilde{p} - 1 \right) \sin(\theta) d\theta d\tilde{z} \\
\overline{F_y} &= \frac{F_y}{p_i R^2} = \int_{-L/D}^{L/D} \int_0^{360} \left( \tilde{p} - 1 \right) \left( -\cos(\theta) \right) d\theta d\tilde{z}
\end{align*}
\]

The dimensionless torque on the journal by the viscous friction force of air can be written

\[
\overline{T} = \frac{T}{p_i C R^2} = \int_{-L/D}^{L/D} \int_0^{360} \left( \frac{h}{2} \left( \frac{\partial p}{\partial \theta} \right) + \frac{\Lambda}{\partial h} \right) d\theta d\tilde{z}
\]

where

\[
\begin{align*}
\tilde{h} &= 1 + \varepsilon \cos(\theta - \theta_0) + \frac{S_1(\theta, z) + S_2(\theta, z)}{C} \quad \text{for} \quad 0 < \theta < \theta_2 \\
\tilde{h} &= h_2 \quad \text{for} \quad \theta_2 < \theta < 360
\end{align*}
\]

5.4 Calculation Result of Static Performance Parameters and Discussion

Fig. 10, Fig. 11 and Fig. 12 are plotted to show the dimensionless minimum film thickness, the dimensionless load capacity and the dimensionless torque with and without the effect of foil deformation at different rotational speed and eccentricity ratios with assumed \( \theta_0 \) of 40 degree. The dashed lines indicate the results with the effect of foil deformation, while the solid ones show the results with the rigid foil.

In Fig. 10, there is only one solid line. That is because \( \tilde{h}_{\text{min}} \) is just a function of eccentricity ratio if the elasticity of foil is ignored, as shown in Eq. 3. However, the film thickness will increase for the effect of foil deformation, as shown in Fig. 10 and a larger increase is produced by a higher rotational speed. This is due to the fact that a higher rotational speed brings a larger foil deformation by causing a higher hydrodynamic pressure, which finally makes a larger \( \tilde{h}_{\text{min}} \). Referring to Fig. 10, another fact can be noticed. That is, for the elastic foil bearings, the change of the minimum film thickness with the rotational speed will rise with the increase of eccentricity ratio. That means the elasticity of foil will make the foil bearings rotate more safely at high rotational speed and high eccentricity ratio, for its larger minimum film thickness, which decreases the possibility of contact between shaft and foil during rotation.

As shown in Fig. 11, like the normal bearings, the load capacity of foil bearings rises
with the increase of the rotational speed and the eccentricity ratio. The load capacity of bearings becomes smaller at certain rotational speed and eccentricity ratio if the effect of foil deformation is considered. The difference between them becomes larger with the increase of eccentricity ratio at the same rotational speed, and also a bigger decrease in load capacity is caused by a higher rotational speed at a certain eccentricity ratio. This can be explained by the fact that the increase of foil deformation with larger hydrodynamic pressure produced by higher rotational speed or higher eccentricity ratio will uplift the film thickness, which finally decrease the air pressure and the load capacity of foil bearing. In Fig. 11, it is noted that the region covered by dashed lines is smaller than that covered by solid lines. It means that the changes in load capacity with rotational speed and eccentricity ratio decrease if the foil is treated as elastic. In another word, the load capacity of foil bearings becomes less sensitive to the changes of rotational speed and eccentricity ratio due to the compliancy of foil.

Fig.12 presents the variation of dimensionless torque for the foil bearings. There is a decrease in torque if the foil deformation is considered, which means that the elasticity of foil will increase the film thickness and finally reduce the torque of bearing. And the higher eccentricity ratio is, the bigger the decrease will be. This, of course, is due to the change of air pressure with the foil deformation, and a higher eccentricity ratio will cause a larger deformation of foils and a bigger decrease in air pressure.

6. Conclusion

Numerical calculation model of MWFB taking account of the effect of the foil local deformation and gas compressibility is developed to predict its static performance. The air pressure distribution and foil deformation are yielded from the coupling solution of Reynolds’ Equation and elastic deformation equation. And the static performance is calculated from the appropriate integral of air pressure. The following conclusions can be obtained from the results.

i) The air pressure in over ambient region decreases if the effect of foil deformation is taken into account.

ii) The film thickness will rise for the effect of foil deformation and a larger increase is produced by a higher rotational speed.

iii) The load capacity and torque become smaller by the effect of foil deformation. And the differences will increase with the eccentricity ratio and rotational speed.

References

(7) ZC Peng, MM Khonsari, Hydrodynamic analysis of compliant foil bearings with

Appendix

The stiffness matrix for element:

\[
[K]=\frac{E t^3}{360a b(1-\mu^2)}
\]

\[
\begin{bmatrix}
k_1 & k_4 & k_2 & \text{sym.}
-k_5 & -k_6 & k_3 & k_1
-k_7 & k_10 & k_{11} & k_1
-k_{10} & 0 & k_4 & k_2
-k_{11} & 0 & k_9 & k_5 & k_6 & k_3
-k_{12} & -k_{15} & k_{16} & -k_{20} & k_{21} & k_1
-k_{15} & k_{13} & 0 & k_{20} & k_{18} & 0 & -k_4 & k_2
-k_{16} & 0 & k_{14} & k_{21} & 0 & k_{19} & k_5 & -k_6 & k_3
-k_{17} & -k_{20} & -k_{21} & -k_5 & -k_{16} & k_7 & -k_{10} & -k_{11} & k_1
-k_{20} & k_{18} & 0 & k_{15} & k_{13} & 0 & -k_{10} & k_8 & 0 & -k_4 & k_2
-k_{21} & 0 & k_{19} & k_{16} & 0 & k_{14} & k_{11} & 0 & -k_9 & -k_5 & k_6 & k_3
\end{bmatrix}
\]

\[
k_1 = 21 - 6\mu + 30\frac{b^2}{a^2} + 30\frac{a^2}{b^2}
\]
\[
k_2 = 8b^2 - 8\mu b^2 + 40a^2
\]
\[
k_3 = 8a^2 - 8\mu a^2 + 40b^2
\]
\[
k_4 = 3b + 12\mu b + 30\frac{a^2}{b}
\]
\[
k_5 = 3a + 12\mu a + 30\frac{b^2}{a}
\]
\[
k_6 = 30\mu ab
\]
\[
k_7 = -21 + 6\mu - 30\frac{b^2}{a} + 15\frac{a^2}{b^2}
\]
\[
k_8 = -8b^2 + 8\mu b^2 + 20a^2
\]
\[
k_9 = -2a^2 + 2\mu a^2 + 20b^2
\]
\[
k_{10} = -3b - 12\mu b + 15\frac{a^2}{b}
\]
\[
k_{11} = 3a - 3\mu a + 30\frac{b^2}{a}
\]
\[
k_{12} = 21 - 6\mu - 15\frac{b^2}{a} - 15\frac{a^2}{b^2}
\]
\[
k_{13} = 2b^2 - 2\mu b^2 + 10a^2
\]
\[
k_{14} = 2a^2 - 2\mu a^2 + 10b^2
\]
\[
k_{15} = -3b + 3\mu b + 15\frac{a^2}{b}
\]
\[
k_{16} = -3a + 3\mu a + 15\frac{b^2}{a}
\]
\[
k_{17} = -21 + 6\mu + 15\frac{b^2}{a} - 30\frac{a^2}{b^2}
\]
\[
k_{18} = -2b^2 + 2\mu b^2 + 20a^2
\]
\[
k_{19} = -8a^2 + 2\mu a^2 + 20b^2
\]
\[
k_{20} = 3b - 3\mu b + 30\frac{a^2}{b}
\]
\[
k_{21} = -3a - 12\mu a + 15\frac{b^2}{a}
\]

Nomenclature

\[C\] = radial clearance
\[D\] = diameter of shaft or bearing
\[E\] = modulus of elasticity
\[F\] = force
\[t\] = thickness of bump foil
\[w\] = displacement in vertical direction
\[x, y\] = horizontal and vertical coordinate
\[z, \theta\] = axial and angular coordinate
$F$ = dimensionless force
$L$ = the length of bearing
$R$ = radius of journal bearing
$S_1(\theta, z), S_2(\theta, z)$ = the deformation of top foil and mid foil

$\bar{z} = z / R$
$\theta_0$ = angular position of $h_{min}$
$\theta_2$ = end of hydrodynamic pressure
$\theta_1$ = start of the foil

$\bar{F}$ = dimensionless force
$L$ = the length of bearing
$R$ = radius of journal bearing
$S_1(\theta, z), S_2(\theta, z)$ = the deformation of top foil and mid foil

$\bar{z} = z / R$
$\theta_0$ = angular position of $h_{min}$
$\theta_2$ = end of hydrodynamic pressure
$\theta_1$ = start of the foil

$T$ = torque
$U$ = linear velocity
$W$ = load on bearing
$\bar{W} = W / p_a R^2$
$e$ = eccentricity

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$\bar{h} = h / C$
$m, n$ = number of element
$p$ = air pressure
$p_a$ = ambient pressure
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$\nu$ = Poisson ratio
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$\omega$ = shaft angular speed
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$B$ = the geometry matrix of element
$D$ = the elasticity matrix
$K^e$ = the stiffness matrix of element
$M$ = stress of element