Time Series Analysis for Jump Phenomena in Random Responses of a Duffing Oscillator Using Continuous Wavelet Transforms*

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Abstract

Stochastic jump phenomena in the random responses of a Duffing oscillator subjected to nonwhite random excitation are investigated. The stochastic jump phenomena correspond to the existence of multiple stationary responses, which differ in the phase angle between excitation and response as well as the amplitude of responses. The response suddenly switches to other stationary response in the long sample function. The purpose of this paper is to propose the criterion that is able to distinguish between two states of the response when the band width of the random excitation becomes broader. In this study, we propose the product of each wavelet transform of the excitation and the response to estimate the phase angle between them. Numerical examples show that the product corresponds to the frequency decomposed phase angle between the excitation and the response, and successfully identifies the states of the response.

Key words : Duffing Oscillator, Stochastic Jump Phenomena, Random Excitation, Time Series Analysis, Continuous Wavelet Transforms

1. Introduction

The stochastic jump phenomena correspond to the existence of multiple stationary responses, as well as the nonlinear oscillator subjected to harmonic excitation. The Duffing oscillator subjected to the harmonic excitation has two stable and one unstable stationary responses in some frequency range⁽¹⁾. These responses differ in the phase angle between the excitation and the response, as well as the amplitude. In the case of the stochastic excitation, there are two states of the response, and the jump between the states occurs suddenly and multiple times in the long sample function⁽²⁾⁻⁽⁵⁾. These two states correspond to two stable stationary responses to the harmonic excitation. Amplitude and phase angle of each state of response are quite different when the band width of the random excitation is narrow. But when the band width of the excitation becomes broader, the excitation and the response contain many frequency components and the difference between two states become obscure.

The purpose of this paper is to propose the criterion that is able to distinguish between two states of the response when the band width of the random excitation becomes broader. In this study, we propose the product of each wavelet transform of the excitation and the response to estimate the phase angle between them. Numerical examples show that the product corresponds to the frequency decomposed phase angle between the excitation and the response, and successfully identifies the states of the response.

2. Equation of Motion and Random Excitation

Consider a nondimensional single-degree-of-freedom Duffing oscillator, which is governed by

$$\ddot{x} + 2\zeta\dot{x} + (x + \varepsilon x^3) = f(t) \quad (1)$$

Fig. 1 Power spectra of random excitations

Fig. 2 Stationary amplitude for harmonic excitation

Fig. 3 Stationary phase angle for harmonic excitation

where $f(t)$ is Gaussian stationary nonwhite noise process with 0 mean and autocorrelation function $R(v)$. In this study, the autocorrelation function $R(v)$ is assumed as

$$R(v) = R_0 e^{-\alpha|v|} \cos \rho v. \quad (2)$$

R_0 and ρ represent the intensity and the dominant frequency of the excitation respectively. The power spectrum of the excitation is derived from the autocorrelation function Eq. (2).

$$S(\omega) = \frac{R_0}{\pi} \frac{\alpha (\alpha^2 + \rho^2 + \omega^2)}{(\alpha^2 + \rho^2 - \omega^2)^2 + 4\alpha^2 \omega^2} \quad (3)$$

For convenience, non-dimensional parameters A and B are introduced. These parameters are defined as

$$A = \frac{\alpha}{\zeta} \quad (4)$$

$$B = \frac{\rho}{\sqrt{1 - \zeta^2}}. \quad (5)$$

These parameters A and B represent the ratio of band width and dominant frequency of $f(t)$ to those of frequency response function of the corresponding linear oscillator ($\varepsilon = 0$), respectively.

In the numerical examples, these values are used,

$$\zeta = 0.05, \quad \varepsilon = 0.1, \quad R_0 = 1.0, \quad B = 1.5 \quad (6)$$

Fig. 4 Sample functions of stochastic jump ($A = 0.01$)

and two cases of band width of the random excitation are considered.

- (1) $A = 0.01$
- (2) $A = 1.0$

The power spectra of the random excitations are shown in Fig. 1.

3. Deterministic Jump Phenomena to Harmonic Excitation

In this section, the excitation $f(t)$ in the equation of motion Eq. (1) is replaced to harmonic excitation as

$$f(t) = \varepsilon F \cos \Omega t \quad (7)$$

and the parameters are set as

$$\zeta = \varepsilon \mu, \quad \Omega = 1 + \varepsilon \sigma.$$

Using the method of multiple scales, these relations for stationary responses are obtained⁽¹⁾

$$\left. \begin{aligned} \mu x_a &= -\frac{F}{2} \sin \gamma \\ x_a \sigma - \frac{3x_a^3}{8} &= -\frac{F}{2} \cos \gamma \end{aligned} \right\}, \quad (8)$$

Fig. 5 Sample functions of stochastic jump ($A = 1.0$)

where x_a and γ represent the amplitude of stationary response and the phase angle between the excitation and the response, respectively. From Eq. (8), following equations are derived.

$$\left[\mu^2 + \left(\sigma - \frac{3x_a^2}{8} \right)^2 \right] x_a^2 = \frac{F^2}{4} \quad (9)$$

$$\gamma = \arctan \left(\frac{8\mu}{8\sigma - 3x_a^2} \right) \quad (10)$$

Figures 2 and 3 show the frequency response curves obtained from Eqs. (9) and (10) for $\zeta = 0.05$, $\varepsilon = 0.1$ and $\varepsilon F = 1/\sqrt{2}$. Solid and dashed lines represent stable and unstable responses respectively. It is clear that there are two stable and one unstable stationary responses in some frequency range. These responses differ in the phase angle between the excitation and the response, as well as the amplitude.

4. Stochastic Jump Phenomena to Random Excitations

In this section, the characteristics of the stochastic jump phenomena to the random excitation are explained.

4.1. Random Excitation with Narrow Band Width

Sample functions of stochastic jump are shown in Fig. 4, when the band width of the random excitation is narrow ($A = 0.01$). Figures 4 (a) and 4 (b) show the sample functions of the excitation and the response, respectively. Figure 4 (c) shows the phase angle between the excitation and the response. This phase angle is calculated from the difference of the time that

Fig. 6 Wavelet transform of random excitation ($A = 1.0$)

each sample function becomes zero. When the band width of the random excitation is narrow, two states of the response are quite distinguishable by the amplitude shown in Fig. 4 (b), and the phase angle shown in Fig. 4 (c).

4.2. Random Excitation with Broader Band Width

Sample functions of stochastic jump are shown in Fig. 5, when the band width of the random excitation is broader ($A = 1.0$). Figures 5 (a) - 5 (c) show the sample functions of the excitation and the response, and the phase angle between the excitation and the response, respectively. When the band width of the random excitation becomes broader, the difference between two states of the response become obscure and the jump occurs frequently. These characteristics make it difficult to distinguish two states by the amplitude shown in Fig. 5 (b) as well as the phase angle shown in Fig. 5 (c).

5. Time Series Analysis Using Wavelet Transforms

5.1. Wavelet Transform of Random Process

In the previous section, it is shown that the difference between two states of the response becomes obscure, when the band width of the random excitation becomes broader. In this study, the authors try the continuous wavelet transforms to distinguish two states of the response. The continuous wavelet transform $C_x(a, b)$ of function $x(t)$ is defined as⁽⁶⁾

$$C_x(a, b) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{|a|}} \overline{\psi\left(\frac{t-b}{a}\right)} x(t) dt, \quad (11)$$

where ψ is mother wavelet and $\overline{\psi}$ represents complex conjugate of ψ . The real part of the Morlet wavelet⁽⁶⁾,

$$\psi(t) = \exp\left(-\frac{t^2}{2}\right) \cos 5t \quad (12)$$

provided by the Matlab wavelet toolbox is chosen as the mother wavelet in this study. From Eq. (12), it is clear that the dominant frequency of the Morlet wavelet is 5. The relationship

Fig. 7 Sample functions and wavelet transform (large amplitude, $A = 0.01$)

between the frequency x_f of signal $x(t)$ and scale parameter a of the wavelet transform is derived as

$$x_f = \frac{5}{a}. \quad (13)$$

To examine the characteristics of the wavelet transform of the random process, the sample function and the wavelet transform of random excitation are shown in Figs. 6 (a) and 6 (b) respectively. In the figures of wavelet transforms in this paper, red and blue represent positive and negative values respectively. From Eq. (5), the dominant frequency of the random excitation is

$$\rho = B \sqrt{1 - \zeta^2} \approx 1.5.$$

(a) Sample function of excitation

(b) Sample function of response

(c) Phase angle between excitation and response

(d) Product of wavelet transforms: $C_{fx}(a, b)$

Fig. 8 Sample functions and wavelet transform (small amplitude, $A = 0.01$)

The scale parameter a corresponding to the dominant frequency is obtained as

$$a = \frac{5}{1.5} \approx 3.3.$$

In Fig. 6 (b), there is a red and blue striped region on $a \approx 3.3$. This region shows the dominant component of time and frequency decomposed phase angle and amplitude of the excitation.

5.2. The Product of Wavelet Transforms

Using the phase angle description in the wavelet transform shown in Fig. 6 (b), we propose the product of each wavelet transforms of the excitation f and the response x to estimate

(a) Sample function of excitation

(b) Sample function of response

(c) Phase angle between excitation and response

(d) Product of wavelet transforms: $C_{fx}(a, b)$

Fig. 9 Sample functions and wavelet transform (large amplitude, $A = 1.0$)

frequency decomposed phase angle between them.

$$C_{fx}(a, b) = C_f(a, b) \cdot C_x(a, b) \quad (14)$$

Positive and negative values of this quantity are expected to represent the states that the excitation and the response are in phase and π rad out of phase, respectively. This characteristic of $C_{fx}(a, b)$ is similar to that the product of two sinusoids in phase becomes positive or zero, and the product of two sinusoids π rad out of phase becomes negative or zero.

Figures 7 and 8 show sample functions of excitation and response, the phase angle calculated by the difference of the time that each sample function becomes zero, and $C_{fx}(a, b)$, in the case that the band width of the random excitation is narrow. Figures 7 and 8 respec-

(a) Sample function of excitation

(b) Sample function of response

(c) Phase angle between excitation and response

(d) Product of wavelet transforms: $C_{fx}(a, b)$

Fig. 10 Sample functions and wavelet transform (small amplitude, $A = 1.0$)

tively correspond to two states that large and small amplitudes are dominant. These sample functions are extracted from the long sample functions ($0 \leq t \leq 10000$). Figure 7 (c) shows that the excitation and the response are in phase. In Fig. 7 (d), there is a red and green striped region on $a \approx 3.3$. This region corresponds to the positive value of $C_{fx}(a, b)$, and represents that the excitation and the response are in phase. Similarly, Figs. 8 (c) and 8 (d) show the excitation and the response are π rad out of phase by almost constant value $-\pi$ and a blue and green striped region on $a \approx 3.3$ respectively.

In the case that the band width of the random excitation is broader the results are shown in Figs. 9 and 10. Figures 9 and 10 respectively correspond to two states that large and small

amplitudes are dominant. The phase angles shown in Figs. 9 (c) and 10 (c) continuously fluctuate. This tendency is different to the case that the band width of the random excitation is narrow (Figs. 7 and 8). These fluctuations make it difficult to identify whether the excitation and the response are in phase or π rad out of phase. But in Figs. 9 (d) and 10 (d), there are positive and negative regions on $a \approx 3.3$, respectively. The positive and negative values of $C_{fx}(a, b)$ represent that the excitation and the response are in phase and π rad out of phase, respectively. Using the values of these regions, two states of the response are easily identified from the product of each wavelet transforms of the excitation and the response. These successful determinations are owing to the frequency decomposed phase angle characteristic represented by $C_{fx}(a, b)$.

6. Conclusion

The stochastic jump phenomena in the random response of a Duffing oscillator subjected to nonwhite random excitation are investigated using the wavelet transforms. The stochastic jump occurs between two states of the response, which correspond to large and small amplitudes. These two states also differ in the phase angle between the excitation and the response. To distinguish two states of the response sample function, we propose the product of each wavelet transforms of the excitation and the response. Numerical examples show that the product corresponds to the frequency decomposed phase angle, and it can be used as the criterion to distinguish two states of the response.

References

- (1) Nayfeh, A. H. and Mook, D. T., *Nonlinear Oscillations*, (1979), John Wiley & Sons, New York.
- (2) Rajan, S. and Davies, H. G., Multiple time scaling of the response of a Duffing oscillator to narrow-band random excitation, *Journal of Sound and Vibration*, Vol. 123, No. 3 (1988), pp. 497–506.
- (3) Roberts, J. B., Multiple solutions generated by statistical linearization and their physical significance, *International Journal of Non-Linear Mechanics*, Vol. 26, No. 6 (1991), pp. 945–959.
- (4) Zhu, W. Q., Lu, M. Q., and Wu, Q. T., Stochastic jump and bifurcation of a Duffing oscillator under narrow-band excitation, *Journal of Sound and Vibration*, Vol. 165, No. 2 (1993), pp. 285–304.
- (5) Huang, Z. L., Zhu, W. Q., Ni, Y. Q., and Ko, J. M., Stochastic averaging of strongly non-linear oscillators under bounded noise excitation, *Journal of Sound and Vibration*, Vol. 254, No. 2 (2002), pp. 245–267.
- (6) Daubechies, I., *Ten Lectures on Wavelets*, (1992), Society for Industrial and Applied Mathematics, Philadelphia.