Vibration Analysis of Elevator Rope (2nd Report, Forced Vibration of Rope with Damping)*

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Abstract

Elevator ropes in high-rise buildings are forcibly excited by the displacement of the building induced by wind force. An approximate solution to the forced vibration of a rope with time-varying length and having linear damping is presented here. It is assumed that the rope tension and the moving velocity are constant, and that the damping coefficient of the rope is small. Virtual sources of waves which can be assigned to reflecting waves are used for obtaining the approximate solution. Finite difference analyses of rope vibration are also performed to verify the validity of this approximate solution. The calculated results of the finite difference analyses are in fairly good agreement with the calculated results of the approximate solution. The effects of moving velocity and damping factor on the maximum rope deflection are quantitatively made clear.

Key words: Flexible Structure, Vibration of Continuous System, Forced Vibration, Finite-Difference Method, Damping, Time-Varying Length, Elevator

1. Introduction

As building heights increase, the travel heights and rope lengths of elevators also increase. High-rise buildings have lower natural frequencies than conventional buildings. Therefore, when a high-rise building is swaying due to strong wind, the elevator rope may resonate with the natural frequency of the building and may collide with and damage hoistway equipment.

Some studies (1)-(11), including those on free vibration (1)-(4), resonance problems (5), coupled vibration of the rope and cage (6), and vibration control (10), have been conducted on the lateral vibration of an elevator rope whose length varies with time due to the up-and-down movement of the cage.

For the free vibration of a rope whose tension and movement velocity are constant and whose damping coefficient is zero, Hashimoto (1) and Kotera (2) have obtained an exact solution to the free vibration induced by an initial displacement. Moreover, for a rope whose damping coefficient is small, Kotera (3) has obtained an approximate solution to the free vibration induced by an initial displacement.

For the forced vibration of a rope whose tension and movement velocity are constant and whose damping coefficient of the rope is zero, Hashimoto (4) has obtained an exact solution to the forced vibration of a rope whose winding end is excited sinusoidally, and compared it to finite element analysis. In addition, using the graphical method, the authors
have obtained an exact solution to the forced vibration of a rope having one end or both
ends excited sinusoidally. Moreover, Yamamoto et al. (5) have obtained an approximate
solution, based on the assumption that the moving velocity is very low compared with the
propagation velocity of the transverse wave. However, in the case where the moving
velocity is not very low, no theoretical solution has yet been obtained except for an elevator
rope with both ends moving (11).

This paper deals with an approximate solution, using the graphical method, to the forced
vibration of a rope with one end moving, based on the assumption that the rope tension and
the moving velocity are constant, and that the damping coefficient of the rope is small. In
addition, finite difference analyses of rope vibration are performed to verify the validity of
these approximate solutions. The calculated results of the finite difference analyses are in
fairly good agreement with those of the approximate solutions. The effects of the changing
rate of rope length and the damping factor on the maximum rope displacement are
quantitatively clarified.

2. Analysis

2.1 Fundamental equations

The elevator rope is composed of a main rope and a compensating rope, where the cage
and counterweight are hung from the main rope. And a compensating sheave and
compensating rope compensate for tension of the main rope.

The simulation model shown in Fig. 1 was used to analyze the vibration of the elevator
rope. The \( z \)-axis that goes down/up from the traction sheave/compensating sheave is
positive, and \( u \) is horizontal rope deflection. The differential equation of the elevator rope is
expressed as follows, based on the assumption that the rope is a string:

\[
\rho A \left( \frac{\partial}{\partial t} - V \frac{\partial}{\partial z} \right)^2 u + C \left( \frac{\partial}{\partial t} - V \frac{\partial}{\partial z} \right) u - \frac{\partial}{\partial z} \left( T(z) \frac{\partial u}{\partial z} \right) = 0
\]

where \( \rho A \) is the mass density per unit length, \( V \) the cage velocity, \( T(z) \) the rope
tension, and \( C \) the damping coefficient per unit length.
If $T$ is a constant, Eq. (1) can be rewritten as follows:

$$\left( \frac{\partial}{\partial t} - V \frac{\partial}{\partial z} \right)^2 u + \frac{C}{\rho A} \left( \frac{\partial}{\partial t} - V \frac{\partial}{\partial z} \right) u - a^2 \frac{\partial^2 u}{\partial z^2} = 0$$

(2)

where $a$ is the propagation velocity of the transverse wave ($= \sqrt{T/\rho A}$).

Numerical analysis is performed by using the following difference equations (7)-(8), derived from Eq. (2). An example of lattice point is shown in Fig. 2:

(except for the upper end: element rope length $\Delta z$ is constant)

$$\left(1 + \frac{C}{\rho A} \frac{\Delta t}{2}\right) u_{i,j+1} = 2 \left[ 1 - a^2 \frac{\Delta t^2}{\Delta z^2} \right] u_{i,j} + a^2 \frac{\Delta t^2}{\Delta z^2} u_{i+1,j} + a^2 \frac{\Delta t^2}{\Delta z^2} u_{i-1,j} + \left[ -1 + \frac{C}{\rho A} \frac{\Delta t}{2} \right] u_{i,j-1}$$

(3)

(upper end: element rope length $m$ varies with time)

$$\left(1 + \frac{C}{\rho A} \frac{\Delta t}{2}\right) u_{i,j+1} = 2 \left[ 1 - a^2 \frac{\Delta t^2}{m \Delta z} \right] u_{i,j} + 2a^2 \frac{\Delta t^2}{m(m + \Delta z)} u_{i+1,j} + 2a^2 \frac{\Delta t^2}{m(m + \Delta z)} u_{i-1,j} + \left[ -1 + \frac{C}{\rho A} \frac{\Delta t}{2} \right] u_{i,j-1}$$

(4)

where $\Delta t$ is the time step, $\Delta z$ the element rope length, $m$ the element rope length at the upper end, $u_{i,j}$ the rope displacement, and subscript $i,j$ on $u$ the indexes of a lattice point in space coordinates and time coordinates, respectively.

The response at each time step is calculated by using Eqs. (3) and (4), while checking the element rope length $m$, because the rope length changes. If $V > 0$ and $m$ becomes negative, we diminish this lattice point. In contrast, if $V < 0$ and $m > \Delta z$, we increase one lattice point.

By using coordinate $x$, whose origin is the cage’s position, as shown in Fig. 1(c), Eq. (2) can be rewritten as follows:
\[
\frac{\partial^2 u}{\partial t^2} + \frac{C}{\rho A} \frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0
\]  
(5)

or

\[
\frac{\partial^2 u}{\partial \tau^2} + \zeta \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial \xi^2} = 0
\]  
(6)

where \( \zeta = x / L_0, \tau = at / L_0, \) and \( \zeta = L_0 C / (a \rho A). \)

2.2 Approximate solution

2.2.1 Wave propagation with small damping

When one point of an infinite rope whose damping coefficient is small is subjected to a forced vibration, the wave's amplitude slowly diminishes on account of the exponential factor \((12).\) Its amplitude depends on traveling distance \(x\) and time \(t:\)

\[
u = Ae^{-\frac{Ct}{2\rho a}} \sin\left(2\pi f (t - \frac{x}{a})\right)
\]  
(7)

where \( A \) is the amplitude of the excitation, and \( f \) is the frequency of the excitation.

2.2.2 When the traction sheave is excited

The rope whose sheave end is fixed and whose cage end is moving, as shown in Fig. 1(a) and (b), is equivalent to the rope shown in Fig. 1(c), one end of which is fixed and whose sheave end moves with time. Wave propagation in the rope is shown in Fig. 3, where \( \xi = x / L_0, \tau = at / L_0, \) and \( \alpha = V / a. \)

The exact solution for a rope whose right end is excited sinusoidally with displacement \(\sin(2\pi \xi)\) and whose damping coefficient \( C = 0 \) is expressed as follows \((9):\)

\[
u(\xi, \tau) = \sum_{n=1}^{\infty} \left[ \sin\left(2\pi \left(\frac{1+\alpha}{1-\alpha}\right)^{n-1} (\xi + \tau - \tau_{An})\right) - \sin\left(2\pi \frac{(1+\alpha)^{n-1}}{(1-\alpha)} (\xi - \tau - \tau_{An})\right) \right] \cdot I(\tau + \xi - \tau_{An})
\]  
(8)

\[
\tau_{An} = \frac{1}{\alpha} \left[1 - \frac{(1-\alpha)^n}{(1+\alpha)^{n-1}}\right] \quad (n = 1, 2, 3 \ldots)
\]  
(9)

\[
\tau_{Bn} = \frac{1 + \tau_{An}}{1 + \alpha}
\]  
(10)

where \( I(\tau + \xi - 1) \) is a step function, which is 1 when the value in parentheses is equal to or larger than 0.

When the rope has small damping, the amplitude diminishes exponentially with dependence on the time \((\tau - \tau_0),\) where \((\tau_0)\) is the wave’s start time, as shown in Fig. 3. This time \((\tau - \tau_0)\) is given by:
Thus, when the rope has small damping, the displacement of the rope whose right end is excited by \( \sin(2\pi \tau) \) is obtained by multiplying factors \( A_{1n} \) and \( A_{2n} \) by each term in Eq. (8):

\[
\begin{align*}
A_{1n} &= \exp\left\{-\frac{\xi}{2}\left[\tau - \frac{(1+\alpha)^{n-1}}{(1-\alpha)^{n-1}}(\tau + \xi - \tau_{An})\right]\right\} \\
A_{2n} &= \exp\left\{-\frac{\xi}{2}\left[\tau - \frac{(1+\alpha)^{n-1}}{(1-\alpha)^{n-1}}(\tau - \xi - \tau_{An})\right]\right\}
\end{align*}
\]

(0 \leq \xi \leq 1 - \alpha \tau) \quad (12)

In a similar manner, the approximate solution for a rope whose right end is excited by displacement \( U_J(\tau) \) is obtained as follows:
2.2.3 When the moving cage is excited

Next, we consider the case where the moving cage is excited. In a similar manner, the amplitude diminishes exponentially with dependence on the time \((\tau - \tau_0)\). This time \((\tau - \tau_0)\) is given by:

\[
\begin{align*}
\tau - \tau_0 &= \begin{cases} 
\xi & \text{(first wave from left side)} \\
\tau - \frac{(1 + \alpha)^n}{(1 - \alpha)^n} (\tau - \xi - \tau_{Bn}) & \text{(waves from left side)} \\
\tau - \frac{(1 + \alpha)^n}{(1 - \alpha)^n} (\tau + \xi - \tau_{Bn}) & \text{(waves from right side)}
\end{cases} \\
(n = 1, 2, 3 \cdots)
\end{align*}
\]

(13)

Thus, the displacement of the rope whose left end is excited by displacement \(U_c(\tau)\) is obtained as follows:

\[
u(\xi, \tau) = B_0 U_c(\tau - \xi) \cdot I(\tau - \xi) \\
+ \sum_{n=1}^{\infty} \left[ B_{1n} U_c \left\{ \frac{(1 + \alpha)^n}{(1 - \alpha)^n} (\tau - \xi - \tau_{Bn}) \right\} \cdot I(\tau - \xi - \tau_{Bn}) \\
- B_{2n} U_c \left\{ \frac{(1 + \alpha)^n}{(1 - \alpha)^n} (\tau + \xi - \tau_{Bn}) \right\} \cdot I(\tau + \xi - \tau_{Bn}) \right]
\]

\[
B_0 = \exp \left\{ -\frac{\xi}{2} \right\} \\
B_{1n} = \exp \left\{ -\frac{\xi}{2} \left[ \tau - \frac{(1 + \alpha)^n}{(1 - \alpha)^n} (\tau - \xi - \tau_{Bn}) \right] \right\} \\
B_{2n} = \exp \left\{ -\frac{\xi}{2} \left[ \tau - \frac{(1 + \alpha)^n}{(1 - \alpha)^n} (\tau + \xi - \tau_{Bn}) \right] \right\}
\]

The displacement of the rope both ends of which are excited is obtained by summing Eqs. (14) and (15).

3. Results and discussion

3.1 When one end is excited

Finite difference analyses are performed to verify the validity of the theoretical solution. Figures 4 and 5 show the time history of the rope displacement in the case of \(\alpha = 0.2\) and \(\zeta = 0.2\) when one end is excited by forced displacement \(U_s = u_0 \sin(0.75 \times 2\pi \tau)\) or \(U_c = u_0 \sin(0.75 \times 2\pi \tau)\), while the rope is becoming shorter.
The right side of these figures is the position of the traction sheave as the elevator system. The initial number of divisions, $N_0$, of the rope is 200. The number of terms, $n$, is up to 10 in the calculation of the approximate solution of Eq. (13) and (15). These figures show that the calculated results of the finite difference analyses are in fairly good agreement with those of the approximate solution.

3.2 When both sides of the rope are excited

If the forced displacement is $\mathbf{U}_s = U_s = u_0 \sin(2\pi \tau)$, the rope resonates at the initial rope length $L = L_0$ (second mode) and at rope length $L/2$ (first mode), because the $i$-th natural frequency $f_i$ of the rope is equal to $(i/2L)\sqrt{T/\rho A}$. To exclude the influence of this resonance at the initial rope length, forced displacement $U_s = U_c = u_0 \sin(0.75 \times 2\pi \tau)$ is used, while the rope resonates at rope length $(2/3)L_0$.

Figure 6(a) and (b) shows the relation between changing rate of rope length $\alpha$ and maximum rope deflection $u_{\max}$ and its position $\Delta L$ when both ends are excited by forced displacement $U_s = U_c = u_0 \sin(0.75 \times 2\pi \tau)$. Damping ratio $\zeta = 0.0283$ is used to allow comparison with the results of reference (5). In the case of the approximate solution, the rope moves at the same time as it is excited. On the other hand, in the case of the finite difference analyses, the rope moves from the steady state of excitation with various timings. The maximum rope deflection varies according to the start timing in the range of FDM (Max) and FDM (Min). For example, the relation between cage position and maximum rope deflection is shown in Fig. 7 in the case of the changing rate of rope length $\alpha$ is 0.2. For reference, the maximum rope deflection in the case of $\alpha = 0$ is also
shown by the dotted line in this figure. The initial number of divisions, $N_0$, of the rope is 200 in the calculation of the finite difference analyses. The number of terms, $n$, is up to 300 in the calculation of the approximate solution of Eq. (13) and (15).

The calculated results of the approximate solution (5) based on the assumption that the moving velocity is very low are also shown in Fig. 6. The calculation of the finite difference analyses is fairly well confirmed by the exact solution to free vibration and forced vibration in the case where the damping coefficient of the rope is zero. The calculated results of the approximate solution, considering small damping, are in fairly good agreement with the

![Graph showing the relation between $u_{\text{max}}/u_0$ and $\alpha$](image)

![Graph showing the relation between $\Delta L/(2L_0/3)$ and $\alpha$](image)

![Graph showing response curves of rope](image)
calculated results of the finite difference analyses, as well as those of Yamamoto (5), thus verifying the validity of the approximate solution, considering small damping.

### 3.3 Some factors that affect maximum rope displacement

Considering the elevator rope hitting and twisting the equipment in the hoistway, the maximum rope displacement is the most important factor. Therefore, in the sections that follow, finite difference analyses are used for obtaining the maximum rope displacement, which is the maximum value found in the calculation of many start times from steady state.

#### 3.3.1 Effects of the changing rate of rope length and the damping factor

Figure 8 shows the calculated results for the effects of the changing rate of rope length $\alpha$ and the damping factor $\zeta$ on the maximum rope displacement, when both ends are excited by forced displacement $u_0 \sin(0.75 \times 2\pi t)$. This figure shows that, as the changing rate of rope length and the damping factor become greater, the maximum rope displacement becomes smaller.

#### 3.3.2 Effect of the moving direction of the rope

This figure also shows that the maximum rope displacement for shortening the rope ($\alpha > 0$) is greater than that for elongating the rope ($\alpha < 0$). As the changing rate of rope length $|\alpha|$ becomes greater, this difference also becomes greater.

#### 3.3.3 Effect of the excitation point

The relation between the changing rate of rope length $\alpha$ and the maximum rope displacement $u_{max}/u_0$ is shown in Fig. 9. Figures 8 and 9 show that the maximum rope displacement in the case of both ends excitation is greater than that in the case of one end excitation. The maximum rope displacement when the traction sheave is excited is greater than that when the moving cage is excited in the range of $\alpha > 0.2$, as shown in Fig. 9.
4. Conclusions

In this paper, an approximate solution to the forced vibration of a rope where one end moves with time is presented, based on the assumption that the rope tension and movement velocity are constant, and that the damping coefficient of the rope is small. The following results were obtained:

1. An approximate solution to the forced vibration of a rope is obtained by using the graphical method, in the case of one end or both ends being excited by forced displacement.
2. The approximate solution considering small damping is fairly well confirmed by the calculation of finite difference analyses.
3. The effects of the changing rate of rope length and the damping factor on the maximum rope displacement are quantitatively clarified in the case of both ends being excited sinusoidally.

References