Present and Future of Nonlinear Dynamics According to a Nonlinear Dynamicist*

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Abstract

The present state and future of “nonlinear dynamics” is explained in this review. First, chaotic vibrations of nonlinear beams are used as a material to demonstrate our present understanding of chaos, compared with the situation of the early stage of its research. Second, two topics, microcantilever dynamics of tapping-mode atomic force microscopy and design of Earth to the Moon transfer trajectories of spacecrafts, are chosen for describing the importance and usefulness of “nonlinear dynamics” in new technologies. Moreover, two applications of “nonlinear dynamics” to biped walking robots and nonlinear optimal control are briefly addressed.

Key words : Nonlinear Dynamics, Chaos, Nonlinear Beam, Microcantilever, Atomic Force Microscopy, Earth-to-Moon Transfer, Biped Walking Robot, Nonlinear Optimal Control

1. Introduction

In this review I will try to tell the reader the present state and future of “nonlinear dynamics”. Since the field is very wide and rapidly evolving, my attempt cannot help being restrictive and narrow not only due to limited space but also for lack of my knowledge and ability. In addition, my interest and taste is naturally reflected here so that it becomes also very biased. Moreover, many of the materials are chosen from my research topics mainly because they are easy to touch. Nevertheless, I believe that the reader can understand and admit the drastically changing state and future possibilities of “nonlinear dynamics”.

In the context of mechanical engineering, “nonlinear dynamics” has often been used as a synonym of “nonlinear vibration” or “nonlinear mechanics”, which is rather traditional and even classical. Triggered by the rediscovery of “chaos”, which means in short irregular motions occurring in deterministic systems, “nonlinear dynamics” has attracted extraordinary attention and an enormous number of researches on it have been done in many scientific and engineering disciplines for about three decades; mechanical engineering is not exceptional. A new mathematical theory, “dynamical systems theory”, has also been developed to study “nonlinear dynamics” and “nonlinear phenomena”.

At the early stage of research, “chaos” was considered to be very rare and peculiar behavior. However, we have learned through our research that it is rather common behavior occurring in most nonlinear systems and complicated motions can occur even in simple systems. Many attempts have also been done to use it in engineering applications such as control, communication and mixing⁽¹⁾.

In §2 we describe a story about chaotic vibrations of forced or unforced, buckled or straight nonlinear beams to demonstrate our present understanding of chaos, compared with the early stage situation. A periodically forced, buckled beam is one of early examples of chaos in mechanical engineering^{(2)–(5)}. Subsequently, it was shown that chaotic vibrations can occur in quasi-periodically forced straight beams^{(6),(7)}. Moreover, it was proved theoretically that free vibrations of undamped, buckled and straight beams also include such motions^{(8),(9)}.

Fig. 1 Buckled beam

Fig. 2 Chaotic orbit of Eq. (1) for $\delta = 0.2$, $\gamma = 0.3$ and $\omega = 1$.

Fig. 3 Phase portrait of Eq. (1) when $\delta = \gamma = 0$.

For describing the importance and usefulness of “nonlinear dynamics” in new technologies, we choose topics on microcantilever dynamics in tapping-mode atomic force microscopy (AFM) in §3 and design of Earth-to-Moon transfers of spacecrafts in §4. The tapping-mode AFM is widely used in nanometer-scale characterization of material surfaces, especially for soft materials such as polymers, DNA molecules and proteins^{(10),(11)}. A vibrating microcantilever with a nano-scale tip utilized as a probe interacts with sample surfaces in a very nonlinear manner, so that extensive works by various approaches of “nonlinear dynamics” have been done for understanding the microcantilever dynamics^{(10),(11)}. On the other hand, the traditional method to transfer spacecrafts from the Earth to the Moon is by a Keplerian semi-elliptic trajectory called the *Hohmann transfer* of the two-body dynamics⁽¹²⁾. Since the cost of such transfers is generally high, some attempts based on more realistic models, i.e., the restricted three-body problem and its perturbation, have been made to obtain lower cost ones^{(13)–(17)}. Further two applications of “nonlinear dynamics” to biped walking robots and nonlinear optimal control are briefly addressed in §5. Finally, we give a summary and some comments on the future of “nonlinear dynamics” in §6.

2. Chaotic Vibrations of Nonlinear Beams

2.1. Forced Vibrations of a damped, buckled beam

In 1971 Tseng and Dugundji⁽²⁾ reported numerical and experimental results in which a damped, buckled beam such as shown in Fig. 1 exhibits continuous snap-through motions such as shown in Fig. 2 when it is subjected to periodic forcing. The motions are what we now call *chaos* although the terminology was not used there since it was coined by Li and Yorke⁽¹⁸⁾ in 1975. Holmes⁽³⁾ mathematically proved the occurrence of such chaotic motions in the first-mode equation of the beam. His approach was based on a technique now called Melnikov’s method, which was developed by Melnikov⁽¹⁹⁾ while its original idea can go back to Poincaré⁽²⁰⁾.

The dimensionless first-mode equation for the beam is given by

$$\ddot{x} + \delta \dot{x} - x + x^3 = \gamma \cos \omega t, \quad (1)$$

where x represents the first-mode amplitude, and δ , γ and ω represent the damping constant,

Fig. 4 Straight beam

Fig. 5 Chaotic orbit of Eq. (3).

Fig. 6 Experimental model for a straight beam.
1 - acceleration pick-up; 2 - strain gage.

Fig. 7 Experimental measurement of a chaotic orbit.

forcing amplitude and frequency, respectively. For $\delta, \gamma = 0$, Eq. (1) has a phase portrait shown in Fig. 3. In particular, there exist a saddle equilibrium at the origin and a pair of homoclinic orbits satisfying $\lim_{t \rightarrow \pm\infty} (x(t), \dot{x}(t)) = (0, 0)$.

When $\delta, \gamma \neq 0$, there exists an unstable periodic orbit near the saddle equilibrium, and it may have *transverse homoclinic orbits*, which tend to the periodic orbit as $t \rightarrow \pm\infty$ while its neighbors do not. The existence of transverse homoclinic orbits implies the presence of chaotic orbit^{(21), (22)} such as given in Fig. 2. Roughly speaking, Melnikov's method linearizes Eq. (1) around the unperturbed homoclinic orbits, and detects the existence of transverse homoclinic orbits. See Refs. (21), (22) for the details of the method.

Such chaotic vibrations in a related experimental model were further investigated by Moon and Holmes⁽⁴⁾. Moreover, Holmes and Marsden⁽⁵⁾ extended Melnikov's method to a class of infinite-dimensional systems and proved that chaotic dynamics occurs in the full equation of motion for the buckled beam,

$$\ddot{u} + u'''' + \left[\Gamma - \kappa \int_0^1 (u')^2 d\zeta \right] u'' = -\delta \dot{u} + \gamma \cos \omega t, \quad (2)$$

where u is the transverse deflection, $\Gamma, \kappa > 0$ are the compressive force and stiffness due to membrane effects, and the prime represents the differentiation with respect to the position z .

2.2. Forced Vibrations of a damped, straight beam

We next consider a damped, straight beam such as shown in Fig. 4. Chaotic motions do not seem difficult to find even when subjected to periodic force but the situation is different when subjected to quasiperiodic force. The dimensionless first-mode equation is given by

$$\ddot{x} + \delta \dot{x} + x + \alpha x^3 = \gamma_0 + \gamma_1 \cos \omega_1 t + \gamma_2 \cos \omega_2 t, \quad (3)$$

where $\delta, \alpha, \gamma_j, \omega_j, j = 1, 2$, are positive constants such that $\delta, \gamma_2 \ll \alpha, \gamma_1$ and $\omega_1 \approx \omega_2$. Using the averaging method^{(21), (23)} and Melnikov's method, we can show that in Eq. (3) chaotic motions occur⁽⁶⁾. A numerically computed chaotic orbit of Eq. (3) for $\delta = 0.004, \alpha = 0.0907, \gamma_0 = 0.3603, \gamma_1 = 0.2162, \gamma_2 = 0.18, \omega_1 = 1.168$ and $\omega_2 = 1.058$ is plotted in Fig. 5.

Experimental evidence of such chaotic motions was given in Ref. (7). The experimental apparatus used there is sketched in Fig. 6. A phosphor bronze beam was clamped at both ends and attached to an electromagnetic shaker. The dimensions of the beam were 27.0 cm long, 1.27 cm wide and 0.080 cm thick. The beam with an initial first-mode frequency of 39.3 Hz was subjected to small tension to give a first-mode frequency of 50.1 Hz. Scotch tape was applied to each face of the beam so that the damping ratio increased to 0.002 ($\delta = 0.004$). The quasiperiodic signal was produced by electronically adding outputs of identical signal generators and then sent to the shaker after amplified. The forcing amplitude was measured by acceleration pick-ups attached to the beam base, and the beam motion was measured with a strain gage attached to the beam near one of the ends.

Fig. 8 A numerically computed orbit of the Poincaré map: (a) projection onto the x -plane; (b) projection onto the (y_1, y_3) -plane when it enters near the y -hyper plane.

A chaotic orbit measured in the experiment, which is similar to the numerical one of Fig. 5, is shown in Fig. 7. Here the forcing amplitudes and frequencies were 0.60, 064 (G) and 58.5, 53 (Hz), respectively, which correspond to the parameter values of Fig. 5.

2.3. Free Vibrations of an undamped, buckled beam

Again, we consider a buckled beam as shown in Fig. 1 while no damping and forcing terms are included. The full equation of motion is given by

$$\ddot{u} + u'''' + \left[\Gamma - \kappa \int_0^1 (u')^2 d\zeta \right] u'' = 0, \quad (4)$$

which is the same as Eq. (2) with $\delta, \gamma = 0$. The beam is assumed to be hinged at both ends as in Fig. 1, so that

$$u(0) = u(1) = 0, \quad u''(0) = u''(1) = 0. \quad (5)$$

We also assume that the dimensionless compressive force satisfies $\pi^2 < \Gamma < 4\pi^2$.

Let $n > 0$ be an integer and let

$$u(z, t) = \sum_{l=1}^{n+1} a_l(t) \sin j_l \pi z, \quad (6)$$

where $j_1 = 1$ and $j_l, l = 2, \dots, n+1$, are positive integers such that $1 < j_2 < \dots < j_{n+1}$. Note that Eq. (6) satisfies the boundary condition (5). Substituting Eq. (6) into Eq. (4), we obtain

$$\ddot{a}_l + (j_l \pi)^2 \left[(j_l \pi)^2 - \Gamma + \frac{\kappa}{2} \sum_{r=1}^{n+1} (j_r \pi)^2 a_r^2 \right] a_l = 0, \quad l = 1, \dots, n+1. \quad (7)$$

Conversely, if $a_l(t), l = 1, \dots, n+1$, satisfy Eq. (7), then Eq. (6) gives an *exact*, special solution of Eq. (4). Moreover, after adequate scaling (see Ref. (8)), Eq. (7) is rewritten as

$$\begin{aligned} \dot{x}_1 &= x_2, \quad \dot{x}_2 = x_1 - \left(x_1^2 + \sum_{r=1}^n \beta_r y_r^2 \right) x_1, \\ \dot{y}_l &= y_{n+l}, \quad \dot{y}_{n+l} = -\omega_l^2 y_l - \beta_l \left(x_1^2 + \sum_{r=1}^n \beta_r y_r^2 \right) y_l, \quad l = 1, \dots, n, \end{aligned} \quad (8)$$

where $x_1 = a_1, x_2 = \dot{a}_1, y_l = a_{l+1}, y_{n+l} = \dot{a}_{l+1}$, and $\beta_r = j_{r+1}^2$. Equation (8) is an $(n+1)$ -degree-of-freedom Hamiltonian system with Hamiltonian

$$H(x, y) = \frac{1}{2} \left(-x_1^2 + \sum_{l=1}^n \omega_l^2 y_l^2 \right) + \frac{1}{4} \left(x_1^2 + \sum_{l=1}^n \beta_l y_l^2 \right)^2 + \frac{1}{2} \left(x_2^2 + \sum_{l=1}^n y_{n+l}^2 \right), \quad (9)$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, and the origin is a saddle-center equilibrium having a pair of real eigenvalues ± 1 and n pairs of purely imaginary eigenvalues.

Applying some extensions of Melnikov's method^{(24), (25)}, we can prove that chaotic motions occur in Eq. (8) if for any $l = 1, \dots, n$,

$$\beta_l \neq \frac{m(m+1)}{2} \text{ for } m = 1, 2, \dots \quad (10)$$

Fig. 9 Numerically computed orbits of the Poincaré map of Eq. (12) with $n = 2, \beta_1 = 1, \beta_2 = 4, \omega_1 = 1, \omega_2 = 3.2$ and $H_n = 11$. Fig. (b) is an enlargement of Fig. (a) near the origin.

See Ref. (8) for the details. A numerically computed orbits of the Poincaré map⁽²⁶⁾ for Eq. (8) with $n = 2, \beta_1 = 0.5, \beta_2 = 1.5, \omega_1 = 1, \omega_2 = (\sqrt{5} - 1)/2$ and $H = 0.01$ is shown in Fig. 8. Here the Poincaré section is taken for the five-dimensional hyperplane $\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^4 \mid y_2 = 0, y_4 > 0\}$, and the orbit is sampled when it intersects the Poincaré section. See Ref. (26) for more details.

2.4. Free Vibrations of an undamped, straight beam

We finally consider free vibrations of an undamped, straight beam, for which the full equation is given by

$$\ddot{u} + u'''' - \left[\Gamma + \kappa \int_0^1 (u')^2 d\zeta \right] u'' = 0, \quad (11)$$

where Γ represents the tension. The beam is assumed to be hinged at both ends as in Fig. 3, so that the boundary condition (5) holds. As in §2.3, we obtain *exact* reduced system as

$$\dot{y}_l = y_{l+n}, \quad \dot{y}_{l+n} = -\omega_l^2 y_l - \beta_l \left(\sum_{r=1}^n \beta_r y_r^2 \right) y_l, \quad l = 1, \dots, n \quad (12)$$

which is an n -degree-of-freedom Hamiltonian system with Hamiltonian

$$H(y) = \frac{1}{2} \sum_{l=1}^n \omega_l^2 y_l^2 + \frac{1}{4} \left(\sum_{l=1}^n \beta_l y_l^2 \right)^2 + \frac{1}{2} \sum_{l=1}^n y_{n+l}^2. \quad (13)$$

Using the differential Galois theory for Hamiltonian systems^{(27),(28)} and its application to a special class of two-degree-of-freedom Hamiltonians^{(28),(29)}, we can prove that Eq. (12) is non-integrable in the following sense: The Hamiltonian system does not have the same number of “meromorphic” first complex integrals which are independent and in involution, as their degrees of freedom, when it is regarded as a Hamiltonian system with complex time and coordinates. This also means the nonintegrability of the infinite-degree-of-freedom model (11). See Ref. (9) for the details.

The differential Galois theory is originally for linear differential equations, and determines whether they are integrable or not in the meaning that their general solutions can be expressed by a combination of integrals, exponentials of integrals and algebraic functions. Its version for Hamiltonian systems states that if the Hamiltonian systems are integrable, then the variational (i.e., linearized) equations around their special solutions are integrable in the meaning of the differential Galois theory. Hence, the Hamiltonian systems are non-integrable if the variational equations are non-integrable. See Ref. (28) for more details.

When a Hamiltonian system is non-integrable in this sense, we can often find chaotic motions in the system although theoretical explanations are seldom given (exceptions were presented in Refs. (28), (30), (31)). We also remark that the existence of chaos implies the nonintegrability of the system⁽³²⁾. A numerically computed orbits of the Poincaré map is plotted in Fig. 9. Here the Poincaré section is taken for the three-dimensional hyperplane

Fig. 10 Microcantilever configuration in tapping mode AFM.

Fig. 11 Frequency response curve: (a) linear system; (b) typical nonlinear system.

$\{y \in \mathbb{R}^4 | y_1 = 0, y_3 > 0\}$. The result suggests that chaotic motions occur although they exist only in a very narrow region of the phase space.

3. Microcantilever Dynamics in Tapping-Mode Atomic Force Microscopy

The microcantilever dynamics in tapping mode AFM has attracted much attention in nonlinear dynamics recently. Here we outline the result of Ref. (33). See also Refs. (10), (11) and references therein for this topic.

Figure 10 shows an analytical model for a microcantilever in tapping mode AFM. The base of the beam is excited by a dither piezoelectric actuator, and the van der Waals and Derjaguin-Muller-Toporov (DMT) forces are assumed between a tip of the beam and the sample surface. Based on common sense of “nonlinear vibrations”, the reader may think that the frequency response curve of the beam is such as shown in Fig. 11(a) or Fig. 11(b), depending on whether its response characteristic is linear or nonlinear. However, this is not the case, as explained below.

Let z be the instantaneous tip-sample separation and assume $z = \Delta$ when no tip-sample interaction occurs. The dimensionless first-mode equation of the beam is given by

$$\ddot{\xi} + \delta \dot{\xi} + \xi + f(1 - \xi) = \gamma \cos \omega t, \quad (14)$$

where $\xi = z/\Delta$ and

$$f(\zeta) = \begin{cases} -\frac{\alpha}{\zeta^2} & \text{for } \zeta > a = \frac{a_0}{\Delta}; \\ -\frac{\alpha}{a^2} + \beta(a - \zeta)^{3/2} & \text{for } \zeta \leq a \end{cases} \quad (15)$$

represents the van der Waals force for $\zeta > a$ and the DMT force for $\zeta \leq a$. Here $a = a_0/\Delta$ with a_0 the intermolecular distance. See Ref. (33) for the derivation of (14). Below we take $\alpha = 7.55 \times 10^{-7}$, $\delta = 3 \times 10^{-2}$, $\gamma = 2.09 \times 10^{-2}$, $\beta = 4.55 \times 10^2$ and $a = 4.22 \times 10^{-3}$. The parameter set is the same as in an experiment of Ref. (35). Note that α , δ and γ are small but β is not. Figure 12 shows the phase portrait of (14) for $\alpha, \delta, \gamma = 0$. We see that there exist a center at the origin and a family of periodic orbits around it.

Applying the averaging method^{(21), (23)} and an extended version⁽³⁴⁾ of the subharmonic Melnikov method^{(21), (22)}, we obtain a frequency response curve shown in Fig. 13(a). See Ref. (33) for the details. This theoretical result explains the experimental one of Ref. (35) as well as the numerical one of Fig. 13(b). Furthermore, chaos and degenerate resonance^{(34), (36)} in the model (14) is analyzed in Ref. (37). Thus, a dynamical systems approach succeeded in explaining nonlinear behavior occurring in an advanced technology, AFM.

Fig. 12 Phase portrait of Eq. (14) for $\alpha, \delta, \gamma = 0$.

Fig. 13 Frequency response curve of Eq. (14): (a) theoretical result; (b) numerical result.

4. New-Type Earth-to-Moon Transfers of Spacecrafts

We turn to the design of a spacecraft transfer with low cost and moderate flight time from the Earth to the Moon. The motion of the spacecraft is modeled by the planar circular restricted three-body problem including a perturbation due to the solar gravitation. Below we outline the results of Refs. (16), (17). See also Refs. (14), (15) for this topic.

We adopt the rotating coordinate system in which the Earth and Moon are fixed on the x -axis, as shown in Fig. 14. Here the time and space variables are nondimensionalized such that the Earth's and Moon's masses are $1 - \mu$ and μ , respectively, and the distance between the Earth and Moon and the gravitational constant are the unity. Let ρ be the distance between the Sun and the origin of the coordinate system, and denote the position and mass of the Sun by $(\rho \cos \theta, \rho \sin \theta)$ and m_S , respectively. Letting (x, y) be the position of the spacecraft in the plane, the equations of motion for the spacecraft are given by

$$\begin{aligned}\ddot{x} - 2\dot{y} &= \frac{\partial}{\partial x} \Omega_0(x, y) + \frac{\partial}{\partial x} \Omega_1(x, y, \theta), \\ \ddot{y} + 2\dot{x} &= \frac{\partial}{\partial y} \Omega_0(x, y) + \frac{\partial}{\partial y} \Omega_1(x, y, \theta), \\ \dot{\theta} &= \omega_S,\end{aligned}\tag{16}$$

where ω_S is the angular velocity of the Sun in the rotating coordinate system and

$$\begin{aligned}\Omega_0(x, y) &= \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_E} + \frac{\mu}{r_M}, \\ \Omega_1(x, y) &= \frac{m_S}{r_S} - \frac{m_S}{\rho^2}(x \cos \theta + y \sin \theta)\end{aligned}\tag{17}$$

with

$$\begin{aligned}r_E &= \sqrt{(x + \mu)^2 + y^2}, \quad r_M = \sqrt{(x - 1 + \mu)^2 + y^2}, \\ r_S &= \sqrt{(x - \rho \cos \theta)^2 + (y - \rho \sin \theta)^2}.\end{aligned}\tag{18}$$

Fig. 14 Rotating coordinate system.

Fig. 15 Earth-to-Moon transfer: (a) near the Earth; (b) near the Moon.

We construct a transfer trajectory of the spacecraft from a circle parking orbit of the Earth to a circle parking orbit of the Moon as follows. First, the spacecraft on the Earth parking orbit of a radius r_s is injected so that its velocity is increased by Δv_s . See Figure 15(a). When the spacecraft arrives at the minimum distance r_0 from the Moon, its velocity is first decreased by

$$\Delta v_0 = v_0 - \sqrt{\frac{2\mu r_e}{r_0(r_0 + r_e)}} \quad (19)$$

to inject it into an elliptic orbit, and then by

$$\Delta v_e = \sqrt{\frac{2\mu r_0}{r_e(r_0 + r_e)}} - \sqrt{\frac{\mu}{r_e}} \quad (20)$$

to inject it into the circular orbit of the radius r_e . See Figure 15(b). The total velocity change to transfer from the Earth parking orbit to the Moon parking orbit is $\Delta v = |\Delta v_s| + |\Delta v_0| + |\Delta v_e|$. For $m_s = 0$ we can show that the total velocity change Δv becomes minimum at $r_0 = r_e$ if the first injection Δv_s and the starting and arriving parking orbits are fixed⁽¹⁶⁾. So we search a trajectory of Eq. (16) such as (i) it starts on the Earth parking orbit of the radius r_s and arrives on the Moon parking orbit of the radius r_e ; (ii) its velocities at the starting and arriving points are tangent to the circular orbits in the non-rotating coordinate systems fixed at the Earth and Moon, respectively; and (iii) Δv is as small as possible.

We assume a realistic situation and set $\mu = 0.01215$, $m_s = 3.289 \times 10^5$, $\rho = 389.2$ and $\omega_s = -0.9253$. For $r_s = 0.01703$ and $r_e = 0.004781$ corresponding to the 167 km altitude Earth parking orbit and the 100 km altitude Moon parking orbit, respectively, we show computed low energy transfer trajectories in Fig. 16 and give their performance in Table 1. Here Δv is the total velocity change (it is 3.893 for the Hohmann type transfer⁽¹³⁾), T is the flight time ($T = 1$ corresponds to about 104 hours), and θ_0 is the initial direction from the origin to the Sun. In particular, the necessary on-board fuel for the obtained transfer in Fig. 16(b) is lowered up to 15.3% from the traditional Hohmann type transfer's although their flight time is not so long and $T = 9.993$ (about 43 days) at most. See Refs. (16), (17) for the details. We should also add that a transfer with 25% less energy (compared with the Hohmann type) but about 90 days flight time was obtained using a different method in Ref. (38).

Fig. 16 Low energy Earth-to-Moon transfer trajectories: (a) $\Delta v = 3.855$, $T = 1.058$; (b) $\Delta v = 3.763$, $T = 9.993$. The circles represent the positions of the Earth and Moon.

Table 1 Performance of the low energy Earth-to-Moon transfers for $m_S = 3.289 \times 10^5$. The values in the parentheses give the corresponding ones for $m_S = 0$.

Δv	Δv_s	Δv_e	T	θ_0/π	Fig. 16
3.855 (3.857)	3.063 (3.063)	0.792 (0.794)	1.058 (1.052)	0.5309	(a)
3.763 (3.85)	3.062 (3.062)	0.701 (0.788)	9.993 (9.511)	0.8197	(b)

5. Other Applications of Nonlinear Dynamics

5.1. Biped walking robots

In 1990 McGeer⁽³⁹⁾ showed in numerical simulations and experiments of a physical model that a simple mechanism with two legs can walk stably and regularly down a slight slope without energy input and control. The gait is called *passive walking* and the energy loss by friction and collision is recovered by gravitational potential. Although limited for a small range of downslopes, the idea has been improved and extended to realistic 2D and 3D passive walking robots^{(40), (41)}.

A simple, compass-like model for biped walking robots is shown in Fig. 17. Its dynamics is essentially the same as that of a double pendulum, which exhibits several complicated motions including chaos, and it can be regarded as a piecewisely smooth dynamical system in the words of “nonlinear dynamics”. An analysis of this model explains passive walking. It seems natural to expect that low energy gaits for a wide range of upslopes and downslopes can be obtained by learning the dynamics of passive walking. Some results have been obtained in this direction^{(42), (43)}. We need to explore the dynamics of a double pendulum for extending them further.

5.2. Nonlinear optimal control

We consider nonlinear control systems of the form

$$\dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (21)$$

with the target equilibrium $x = 0$ and the cost function

$$J(x(0), u) = \int_0^\infty (q(x(t)) + \langle u(t), u(t) \rangle) dt, \quad (22)$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ and $q: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f(0) = Dq(0) = 0$ and $q(0) = 0$, and $\langle \xi, \eta \rangle$ represents the inner product of $\xi, \eta \in \mathbb{R}^n$. The optimal control input u realizing the minimum cost function is given by

$$u = -\frac{1}{2}g^T(x)\bar{p}(x), \quad (23)$$

where $p = \bar{p}(x)$ satisfies

$$H(x, p) = \langle p, f(x) \rangle - \frac{1}{2}\langle p, R(x)p \rangle + q(x) = 0 \quad (24)$$

Fig. 17 Compass-like biped model.

with $R(x) = \frac{1}{2}g(x)g^T(x)$ (the superscript “T” denotes the transpose operator), and the origin is asymptotically stable in the system

$$\dot{x} = f(x) - R(x)\bar{p}(x), \quad (25)$$

which is obtained by substituting Eq. (23) into Eq. (21). See, e.g., Ref. (44). A standard technique to solve this optimal control problem is available if $f(x)$ is linear and $g(x)$ is constant, but we do not have such a technique otherwise.

Let us consider the Hamiltonian system

$$\dot{x} = f(x) - R(x)p, \quad \dot{p} = -(Df(x))^T p + \frac{1}{2}(D_x \langle p, R(x)p \rangle)^T - (Dg(x))^T \quad (26)$$

with the Hamiltonian $H(x, p)$. It was shown in Ref. (45) that the stable manifold $W^s(0)$ of the origin in Eq. (26) is represented as

$$W^s(0) = \{(x, p) \mid p = \bar{p}(x)\}. \quad (27)$$

Using this result and calculating the stable manifold $W^s(0)$, we can obtain the optimal control input u . Thus, dynamical systems approaches can play an essential role in solving the nonlinear optimal problems which are important from an engineering point of view. See Refs. (46), (47) for some results in this line.

6. Conclusions and Future

The present state and future of “nonlinear dynamics” has been explained. First, our present understanding of “chaos” was demonstrated for chaotic vibrations of nonlinear beams. Secondly, the importance and usefulness of “nonlinear dynamics” in new technologies was described for microcantilever dynamics of atomic force microscopy and design of Earth to the Moon transfers of spacecrafts. Moreover, two applications of “nonlinear dynamics” to biped walking robots and nonlinear optimal control are briefly discussed to this end.

As described in the above, “nonlinear dynamics” is not just a synonym of “nonlinear vibration” or “nonlinear mechanics” any more. “Dynamical systems theory” is a useful tool for studying and exploiting “nonlinear dynamics” and has a very broad range of applications. It seems obvious that new technologies will be more easily created and developed by investigating and utilizing nonlinearities, which have been ignored in the past. Many new applications are waiting for challenges of “nonlinear dynamicists”. The future of “nonlinear dynamics” must be very fruitful.

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