Forced Vibration Analysis of a Beam Structure with Nonlinear Support Elements
(3rd Report: Application of ITSCM and Stability Analysis Using a Reduction Model to a Three-Dimensional Tree Structure) *

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Abstract
In a previous paper, incremental transfer stiffness coefficient method was developed in order to analyze the periodic steady-state vibrations of a large-scale structure having locally strong nonlinear elements. By this method, the computation cost of this method can be reduced markedly from the iterative computation process of approximate solution. In this paper, an algorithm based on the presented method is formulated to analyze the longitudinal, flexural and torsional coupled vibration of a three-dimensional tree structure supported by nonlinear base support elements. In addition, stability analysis using a reduction model is applied to the periodic solution obtained for the three-dimensional tree structure. The validity of the incremental transfer stiffness coefficient method and the method of stability analysis using the reduction model for three-dimensional tree structures is confirmed by the numerical computational results.

Key words: Nonlinear Vibration, Forced Vibration, Stability, Modal Analysis, Substructure Synthesis Method, Incremental Transfer Stiffness Coefficient Method, Tree Structure

1. Introduction

In the previous paper [1], the authors proposed the incremental transfer stiffness coefficient method (ITSCM) as a high-speed and high-accuracy analysis method for a periodic steady-state vibration generated in nonlinear structures with high degree of freedom (DOF). And this method was applied to the analysis of the in-plane flexural forced nonlinear vibration of a straight-line beam structure. Also, a method of stability analysis using a reduction model was proposed [2] to analyze the steady-state solution obtained with high-speed and high-accuracy.

ITSCM was constructed by combining the harmonic balance method (a powerful method for nonlinear systems) and the transfer stiffness coefficient method (a very efficient method for linear systems proposed by the authors) through the use of the incremental method. In computing the periodic steady-state vibration generated in a nonlinear structure, in which nonlinearity exits locally, by ITSCM, the inner DOFs of the linear elements is...
completely eliminated from the iterative computation process.

Stability analysis using a reduction model allows for the stability of the periodic steady-state solution of large scale nonlinear systems to be analyzed, which is a very difficult or impossible problem when traditional methods are used, at high-speeds and with high-accuracy. This method aims to reduce the dimension of the variational equation for analyzing the stability of the zero solution by applying the concept of modal analysis and the method to extract the dominant modes.

In practice, there are many apparatuses that contain bending points and subsystems such as a multi-joint manipulator or a piping system. Because the need for such apparatuses or structures has been increasing, vibration problems caused by the nonlinearity in the systems have become a big problem. However, the necessary numerical computations cannot be easily achieved with sufficient accuracy and at acceptable computation speeds by traditional analysis methods, because these systems have many DOF in general. Therefore, it is very important to develop an efficient analysis method for such apparatuses. In this paper, ITSCM and stability analysis using a reduction model are extended to a three-dimensional tree structure [3] that is a general model of the problem mentioned above. An analytical algorithm for longitudinal, flexural and torsional coupled vibration is formulated. The validity of these methods is investigated using the results of concrete numerical computations.

2. Analytical model and treatment of nonlinear base support elements

2.1 Analytical model

The analytical model of the three-dimensional tree structure treated herein is shown in Fig. 1. This structure consists of one main system and several subsystems connected to the main system. Each system is divided into multiple linear uniform beam elements at bending points. At these points, the main system and the subsystem is connected, the external forces and moments act, and the shearing force, bending moment and cross-sectional area vary discontinuously. At each dividing point, there is a rigid body supported by a nonlinear spring-damper with respect to translation and rotation. Simply stated, the nonlinearity in the analytical model exists locally at the base support elements, while the uniform beams are assumed to be linear.

The relative left end of the main system is denoted as node 0 and the other divided points are numbered sequentially up to node \( n \) at the right end of the main system. The uniform beam element between node \( j - 1 \) and node \( j \) is labeled the \( j \)-th beam element. The rigid body and the base support element at node \( j \) are called rigid body \( j \) and the \( j \)-th base support element, respectively. For the subsystem, the end connected to the main system is assumed to be the right end, and the other end is assumed to be the left end. Each uniform beam element is subdivided into several identical fundamental elements, which are modeled as linear lumped mass systems. External forces and moments act with a period \( 2\pi \) with respect to dimensionless time \( \tau = \omega t \); where: \( \omega \) is the circular frequency of the external forces and moments, and \( t \) is time. These external forces and moments are assumed to act only on the nodes.

The \( j \)-th local orthogonal coordinate system \( (X_j, Y_j, Z_j) \) is set for the \( j \)-th beam element. The origin of the coordinates is set at node \( j - 1 \). The three axes of the local coordinate system are set parallel to the principal axes of inertia of uniform beam element, and \( X_j \) is aligned to the longitudinal direction of the uniform beam element. For brevity, the setting direction of the \( j \)-th base support element and the principal axes of inertia of rigid body \( j \) are assumed to be parallel to the \( j \)-th local orthogonal coordinate system.

Symbols, subscripts and superscripts are applied as following:
(1) Symbols with and without right superscripts [*] denote physical quantities with respect
to dimensionless time \( \tau \) and the Fourier coefficient of the corresponding physical quantity, respectively.

(2) Symbols with right subscripts \([c]\) and \([s]\) denote the cosine and sine components of the Fourier coefficients, respectively. A right superscript \([k]\) denotes the order of the Fourier coefficient.

(3) Symbols with a right subscript \([j]\) denote the physical quantities with respect to node \( j \) or the \( j \)-th beam element \( (j = 0, 1, \ldots, n) \).

(4) Symbols with a right superscript \([j]\) denote the physical quantities expressed in the \( j \)-th local coordinate system.

(5) Symbols with and without the head mark \([-]\) denote the physical quantities on the left and right sides of node \( j \), respectively.

(6) Symbols with the head mark \[\sim\] denote the physical quantities with respect to the nonlinear base support element or rigid body at a node.

(7) Symbols proceeded by \([\Delta]\) denote incremental quantities of the corresponding physical quantities.

(8) Symbols with right superscripts \([L]\) and \([R]\) denote the left and right side physical quantities of a fundamental element, respectively \([1]\).

(9) Symbols with left superscripts \([t]\) denote the transpose of a vector or matrix.

Also, \( p_0 \) and \( I_p \) denote the \( p \)-dimensional zero matrix and unit matrix, respectively.

2.2 Equation of motion at node \( j \) and incremental expression of equation of motion

The positive direction of translational displacement and angular displacement of the center of mass of the rigid body \( j \), and the forces and moments acting on the rigid body \( j \), are defined as shown in Fig. 2. The displacement vector \(*_j d_{xyz}^j\), force vector \(*_j f_{xyz}^j\), \(*_j f_{xyz}^j\), external force vector \(*_j q_{xyz}^j\) and mass matrix \(*_j m_{jxyz}\) of rigid body \( j \) are defined as:

\[
d_j^j = (x, \theta, \psi, \phi)\]
\[
f_j^j = (F_x^j, \dot{N}_x^j, F_y^j, \dot{N}_y^j, F_z^j, \dot{N}_z^j)\]  
\[
f_j^j = (\dot{F}_x^j, \ddot{N}_x^j, \dot{F}_y^j, \ddot{N}_y^j, \dot{F}_z^j, \ddot{N}_z^j)\]  
\[
q_j^j = (q_x^j, q_y^j, q_z^j, \theta_j^j, \psi_j^j)\]  
\[
m_j = diag(\dot{m}_j, \ddot{m}_j, \dddot{m}_j, \ddot{m}_j)\]  

where \( (x, y, z) \) and \( (\theta, \phi, \psi) \) are the displacements and the angular displacements of the rigid body \( j \), and \( (F_{x,y,z}^j, F_{x,y,z}^j, \dot{F}_{x,y,z}^j) \) and \( (\dot{N}_{x,y,z}^j, \dot{N}_{x,y,z}^j, \ddot{N}_{x,y,z}^j) \) are the forces and moments acting on node \( j \). \( (q_{x,y,z}^j, q_{x,y,z}^j) \) are the external forces and moments acting on node \( j \), and \( \dot{m}_j \) and \( (\dddot{J}_{x,y,z})_j \) are mass and moment of inertia of rigid body \( j \), respectively.

The equations of motion of the rigid body at node \( j \) with respect to translation and rotation, and the linearized equation with respect to the incremental quantities of these
quantities are:
\[
\omega^2 m_j \ddot{d}_j + \ddot{f}_j^* + \ddot{f}_j = q_j^* \tag{5}
\]
\[
\omega^2 m_j \Delta \ddot{d}_j + \Delta \ddot{f}_j^* + \Delta \ddot{f}_j = r_j^* \tag{6}
\]
where \( [\cdot] = d/d\tau \).

The component of the nonlinear reaction forces \( \dot{f}_j^* \) from the base support element are assumed to be a nonlinear function with respect to displacement and velocity (angular displacement and angular velocity) for the corresponding axial direction (axial rotation), because the setting direction of the \( j \)-th base support element is assumed to be parallel to the \( j \)-th local coordinate system. Thus, increments of them \( \Delta \dot{f}_j^* \) can be expressed as:
\[
\Delta \dot{f}_j^* = \dot{V}_j^* \Delta \dot{d}_j^* + \dot{W}_j^* \Delta d_j \tag{7}
\]
\[
\dot{V}_j^* = \frac{\partial \dot{f}_j^*}{\partial \dot{d}_j^*} = \text{diag} \left[ \frac{\partial \dot{F}_{ij}^*}{\partial x_j^*}, \frac{\partial \dot{N}_{ij}^*}{\partial y_j^*}, \frac{\partial \dot{F}_{ij}^*}{\partial z_j^*}, \frac{\partial \dot{N}_{ij}^*}{\partial \phi_j^*} \right] \tag{8}
\]
\[
\dot{W}_j^* = \frac{\partial \dot{f}_j^*}{\partial d_j} = \text{diag} \left[ \frac{\partial \dot{F}_{ij}^*}{\partial x_j^*}, \frac{\partial \dot{N}_{ij}^*}{\partial y_j^*}, \frac{\partial \dot{F}_{ij}^*}{\partial z_j^*}, \frac{\partial \dot{N}_{ij}^*}{\partial \phi_j^*} \right] \tag{9}
\]

2.3 Assumption of approximate solution, and force and moment

In order to obtain a periodic steady-state vibration of period \( 2\pi \) with respect to \( \tau \) using the harmonic balance method, displacement vector \( \dot{d}_j \) and their increments \( \Delta \dot{d}_j \) are approximated using the finite real Fourier series of the \( N \)-th order, thus:
\[
\dot{d}_j = E \dot{d}_j, \quad \Delta \dot{d}_j = E \Delta \dot{d}_j \tag{10}
\]
where \( \dot{d}_j \) and \( \Delta \dot{d}_j \) are the \( 6(2N+1) \)-dimensional displacement amplitude vectors which is comprised of the Fourier coefficients of \( \dot{d}_j^* \) and \( \Delta \dot{d}_j \), and \( E \) is a \( 6 \times (2N+1) \)-dimensional real matrix. They are defined as:
\[
d_j^* = \{ (\dot{d}_0^*, \dot{d}_1^*, \ldots, \dot{d}_N^*, \dot{d}^*_N) \}
\]
\[
\Delta \dot{d}_j = \{ (\Delta \dot{d}_0^*, \Delta \dot{d}_1^*, \ldots, \Delta \dot{d}_N^*, \Delta \dot{d}^*_N) \}
\]
\[
(d^*_i) = \{ \{ x, y, \psi, \phi \} \}_{i=0}^{N}
\]
\[
\Delta (d^*_i) = \{ \{ \Delta x, \Delta y, \Delta \psi, \Delta \phi \} \}_{i=0}^{N}
\]
\[
E = [I_6/2, I_6 \cos \tau, I_6 \sin \tau, \ldots, I_6 \cos N\tau, I_6 \sin N\tau] \tag{11}
\]
Then, the derivatives of \( \dot{d}_j \) and \( \Delta \dot{d}_j \) with respect to \( \tau \) are expressed as:
\[
\dot{d}_j^* = EUd_j, \quad \Delta \dot{d}_j^* = EU \Delta d_j, \quad \Delta \dot{d}_j = EU^2 \Delta d_j \tag{12}
\]
\[
U = \text{Diag} \left[ \begin{array}{cccc}
0_6 & rI_6 & U_2 & \cdots & U_N
\end{array} \right]
\]
\[
U_r = \begin{pmatrix}
0_6 & rI_6 \\
-\tau I_6 & 0_6 \end{pmatrix}, \quad (r = 1, 2, \ldots, N) \tag{13}
\]
where \( \text{Diag} \{ \cdot \} \) denotes a block diagonal matrix.

For a \( 2\pi \) periodic steady-state vibration, the force and moment that acts on the rigid body and their increments also become \( 2\pi \) periodic functions with respect to \( \tau \). Thus, these can also be expressed as the following finite real Fourier series of the \( N \)-th order:
\[
\dot{f}_j^* = E \dot{f}_j, \quad \Delta \dot{f}_j^* = E \Delta \dot{f}_j, \quad \dot{f}_j = Ef_j \tag{14}
\]
\[
\Delta \dot{f}_j = E \Delta \dot{f}_j, \quad \Delta f_j = E \Delta f_j \tag{15}
\]
where \( \dot{f}_j, \dot{f}_j, \Delta \dot{f}_j, \Delta \dot{f}_j \) and \( \Delta f_j \) are \( 6(2N+1) \)-dimensional force amplitude vectors and incremental force amplitude vectors that consist of the corresponding Fourier coefficients and are defined using the same formulation used for \( \dot{d}_j \) in Eq.(10). As they also become \( 2\pi \) periodic functions for a \( 2\pi \) periodic steady-state vibration, \( \dot{V}_j^* \) and
*\hat{j}W\), as defined in Eq. (8), can be expanded as a Fourier series. By using their real Fourier coefficients, the relationship between \(\Delta f_j\) and \(\Delta d_j\) becomes:

\[
\Delta f_j = \hat{V}_j \Delta d_j, \quad \hat{V}_j = \hat{V}_j + \hat{W}_j U
\]

where \(\hat{V}_j\) and \(\hat{W}_j\) are square matrices of the order \(6(2N+1)\) whose elements are the real Fourier coefficients \((\hat{V}_j^k)\), \((\hat{W}_j^k)\), \((\hat{V}_j^k)\), \((\hat{W}_j^k)\) of \(\hat{V}_j^*\) and \(\hat{W}_j^*\), respectively. As their forms have almost the same formulation as in the previous paper [1] (given by replacing elements of matrices by above real Fourier coefficient matrices), the concrete expression is omitted herein for brevity.

When computing the odd harmonic solution, which consists of only odd harmonics, an effective computation can be executed because the dimensions of the vectors and matrices used in the computation process can be reduced to about half their size.

Sub-harmonic vibrations of order \(1/L\) can be analyzed using the same algorithm as that for the ordinary process. That is to say, by replacing the variable transformation \(\omega t = \tau\) with \(\omega t/L = \tau'\), sub-harmonic vibrations of order \(1/L\) can be obtained by computing the results for \(2\pi\) periodic vibrations with respect to \(\tau'\).

2.4 Application of the harmonic balance method

Substituting Eqs. (9), (11), (13) and (14) into Eq. (6) and applying the harmonic balance method to Eq. (6) leads to:

\[
\hat{P}_j \Delta d_j + \Delta \hat{f}_j - \Delta f_j = r_j
\]

\[
\hat{P}_j = \hat{K}_j + \omega^2 \hat{M}_j U^2, \quad \hat{M}_j = \text{Diag}\{\hat{m}_j, \cdots, \hat{m}_j\}
\]

\[
r_j = q_j - \omega^2 \hat{M}_j U^2 d_j - \hat{f}_j + f_j
\]

where \(q_j\) is the \(6(2N+1)\)-dimensional external force amplitude vector which is comprised of the Fourier coefficients of the periodic external force \(q_j^*\), and that is constructed in a same form as \(d_j\) in Eq. (10).

To complete the analytical algorithm, it is necessary to obtain the relationships between \((\hat{f}_j, f_j)\) and \(d_j\), and between \((\Delta \hat{f}_j, \Delta f_j)\) and \(\Delta d_j\). An efficient process for deriving these relations is detailed in the next section.

3. Efficient elimination of the inner DOFs of uniform beam elements

In this section, right subscript \([j]\), which denotes a physical quantity with respect to the \(j\)-th uniform beam, is omitted for simplicity except for in §§ 3.3.

3.1 Modeling of the fundamental element

As shown in Fig. 3, each uniform beam is divided equally into several fundamental elements. Subsequently, each fundamental element is modeled as a bilaterally symmetrical lumped mass system. That is, the inertial properties at the centers of mass of the fundamental elements are equally distributed to each end. These lumped masses are assumed to be connected by massless elastic beams. For this, \(m\) and \(J_{x,y,z}\) are respectively used to denoted the mass and moment of inertia with respect to the principal axis of inertia of the lumped mass, \(l\) and \(A\) are respectively the length and cross-sectional area of the fundamental element, \(GI_x\) is the torsional stiffness with respect to \(x\)-axis rotation, \(EI_{y,z}\) is the bending stiffness

![Fig. 3 Fundamental element and analytical model of uniform linear beam](image-url)
with respect to $y, z$-axis rotation, and $\kappa_{y,z}$ and $G$ are respectively the cross-section coefficient and the modulus of the transverse stiffness. The viscous damping in the uniform beam is also distributed equally to each end as equivalent lumped viscous damping coefficients $c_{xyz}$ and $c_{xyz}$ with respect to translation and rotation, respectively.

### 3.2 Auto and cross dynamic stiffness coefficient matrices, and series transmission rule

Because the uniform beams are assumed to be linear, the steady-state vibration solution at the inner nodes of uniform beam can be separated into each order of the Fourier series. Then, by applying the same method as in the previous paper [1], the relation between the left side complex state variable vectors $d^{L,k}$ and those for the right side $d^{R,k}$ becomes:

$$
F^{R,k} = S^{ka} d^{R,k} + S^{kc} d^{L,k}, \quad F^{L,k} = s^{kv} d^{R,k} + s^{kv} d^{L,k}
$$

where $S^{ka}$ and $s^{kv}$ are the auto dynamic stiffness coefficient matrices, and $S^{kc}$ and $s^{kv}$ are cross dynamic stiffness coefficient matrices, which become:

$$
S^{ka} = \text{Diag}[S^{ka}_x, S^{ka}_y, S^{ka}_z, S^{ka}_{xy}], \quad S^{kc} = \text{Diag}[S^{kc}_x, S^{kc}_y, S^{kc}_z, S^{kc}_{xy}]
$$

Next, the fundamental elements will be rigidly connected in series to reconstruct the uniform beam element. Applying the same method as in the previous paper [1], the series transmission rule to reconstruct "subsystem C" from "subsystem A" and "B" becomes:

$$
S^{cm} = S^{ka} + S^{kc} \quad \text{and} \quad S^{cm} = S^{ka} + S^{kc}
$$

This reconstruction process can be separated into four types of vibration components, that is, longitudinal, torsional and flexural in two planes vibration.

### 3.3 Relation between state variable vectors for both ends of the uniform beam element

By repeating the process described in the last subsection, the inner DOFs of the uniform beam can be completely eliminated, and the relationship between the state variable complex vectors at both ends of the uniform beam can be obtained for each order of the Fourier series. By converting these complex vectors to real form, and by rearranging them for all orders of Fourier coefficients ($k = 0, 1, \sim, n$) to suit the form of $d_j$ and $\bar{f}_j$ as defined in Eqs. (10) and (13), the following equations are obtained:

$$
\bar{f}_j = S^{l}_j d_j + S^{l}_j d_{j-1}, \quad f_{j-1} = s^{l}_j d_j + s^{l}_j d_{j-1}
$$

Equations (20) and (21) express the relationships between $\bar{f}_j$ and $f_{j-1}$, and $d_j$ and between
4. Formulation of the incremental transfer stiffness coefficient method

4.1 Efficient computation of the incremental displacement amplitude vector

The displacement amplitude vector \( d_j \) can be determined by computing \( \Delta d_j \), and by iterating using \( d_j + \Delta d_j \rightarrow d_j \) until the norm of \( \Delta d_j \) falls within a convergence tolerance. As expected, the computation for \( \Delta d_j \) is the primary contributor to the computational burden of the approximate computation process. Herein, an efficient computation process for \( \Delta d_j \), which applies the recurrent computation, is presented. In this section, note that the right superscript denotes the local coordinate system in which physical quantities are expressed.

First, in ITSCM, the relationship between the incremental displacement amplitude vector \( \Delta d_j \) and the incremental force amplitude vectors \( \Delta f_j \) is defined as follows:

\[
\Delta f_j = S_j^T \Delta d_j + s_j^T
\]

(22)

where \( S_j \) are the dynamic stiffness coefficient matrices of order \( 6(2N+1) \), and \( s_j \) denote the incremental force correction vectors of dimension \( 6(2N+1) \).

In ITSCM, the dynamic stiffness coefficient matrix and the force correction vector are computed recurrently from node 0 to node \( n \). Subsequently, the displacement amplitude vector is computed recurrently from node \( n \) to node 0 using the computational result for \( S_j \) and \( s_j \). The concept of corresponding transmission computational rules is that same as in the previous paper [1]. So, only the outline of the computational process for a three-dimensional tree structure is summarized herein.

(a) The case that a subsystem does not exist

The initial values \( S_0 \) and \( s_0 \), and the transmission rule for \( S_j \) and \( s_j \) become:

\[
S_0 = \hat{P}_0, \quad s_0 = -r_0
\]

(23)

\[
S_j = \hat{P}_j + S_j^{\text{corr}} + S_j^{\text{int}} V_j, \quad s_j = -S_j^{\text{corr}} (G_j)^{-1} s_{j-1} - r_j
\]

(24)

The coordinate transformation rule between vectors and matrices expressed in \( j \)-th and \((j+1)\)-th local coordinates is given by using coordinate transformation matrix \( T_j \) as follows:

\[
\Delta d_j = T_j \Delta d_j^{\text{corr}}, \quad \Delta f_j = T_j \Delta f_j^{\text{corr}}, \quad S_j^{\text{corr}} = T_j S_j T_j, \quad s_j^{\text{corr}} = T_j s_j
\]

(25)

Next, the initial values \( \Delta d_0 \), and the transmission rule for \( \Delta d_j \) becomes:

\[
\Delta d_n = -S_n^{-1} s_n
\]

(26)

\[
\Delta d_{j-1} = V_j^T \Delta d_j^{\text{corr}} - (G_j)^{-1} s_{j-1} \quad (j = n, n-1, \ldots, 1)
\]

(27)

(b) Treatment of subsystem

It is also possible to apply the computation process formulated for the main system to the subsystem. In this case, first compute the dynamic stiffness coefficient matrices \( \hat{S}_j \) and incremental force correction vectors \( \hat{s}_j \) at the right end of the subsystem (which is connected node \( j \) of the main system) as follows:

\[
\Delta f_j = \hat{S}_j^T \Delta d_j + \hat{s}_j
\]

(28)

Next, by considering the balance of forces at node \( j \), the transmission rule at node \( j \) of the main system, defined in Eq. (24), is changed as follows:

\[
S_j = \hat{P}_j + S_j^{\text{corr}} + S_j^{\text{int}} V_j + \hat{S}_j
\]

\[
s_j = -S_j^{\text{corr}} (G_j)^{-1} s_{j-1} - r_j + \hat{s}_j
\]

(29)

The recurrent process for the incremental displacement amplitude vectors in the subsystem coincides completely with that for the main system (Eq.(27)).
Using Eqs. (23) to (29), the incremental displacement vector $\Delta d_j$ for all nodes in the system can be computed recurrently and efficiently by using small dimensional matrices.

### 4.2 The computational procedure

The computational procedure for the analysis of nonlinear forced vibrations generated in a three-dimensional tree structure using ITSCM is summarized as follows:

1. Compute the auto and cross dynamic stiffness coefficient matrices for a given circular velocity $\omega$ of the external forces and moments using the process described in §§ 3.2.
2. Assume an initial value of $d_j^0$ for each node in the main system and the subsystems.
3. Compute $\hat{f}_j^0$, $\hat{f}_j$, and $f_j$ from $d_j^0$. Then, $\hat{f}_j^0$ and $f_j$ are obtained from Eq. (20).
4. Compute $\hat{P}_j$ and $\hat{R}_j$ for each node using Eqs. (14) and (16).
5. Compute $\tilde{S}_j$ and $\tilde{s}_j$ in the subsystem recurrently using the process detailed in §§ 4.1. Subsequently, compute $S_j$ and $s_j$ in the main system recurrently.
6. First, compute the correction quantity $\Delta d_j$ for each node in the main system. Subsequently, compute this value for each node in the subsystem.
7. If $\|\Delta d_j\|$ is outside the convergence tolerance, $d_j + \Delta d_j$ is adopted as $d_j$ in the next iteration. Return to step (3).
8. If $\|\Delta d_j\|$ is within the convergence tolerance, the successive computational process may be regarded as convergent. Compute the displacement vectors of the inner nodes of the uniform beam elements if necessary [1].
9. Set a new value for $\omega$, and return to step (1).

In the procedure described above, computation for the inner nodes of the uniform beam elements is completely eliminated from the iterative computational process (steps (3) to (8)), which are the main contributors to the computational burden. Furthermore, by introducing the recurrent computation process, the computation of $\Delta d_j$ (the largest computational burden) is achieved very efficiently with regard to computational load and memory size.

### 5. Stability analysis method using a reduction model

#### 5.1 Fundamental matters and problems of the stability analysis

The investigation into the infinitesimal stability of the periodic solution results in the stability analysis of the zero solution of the variational equation. In stability analysis, all nodes of analytical model including the inner nodes of the uniform beam, which were eliminated in the computation process of the periodic solution, must be considered in theory. That is, the variational equation for all nodes, which is given as follows, should be treated.

$$\omega^2 M \hat{\eta}^* + \omega C \hat{\eta}^* + K \hat{\eta}^* = 0$$

where $\eta^* = [\eta_1(\tau)]$ is the variational displacement vector, which consists of the variation with respect to the translational and angular displacements of all nodes, and whose dimension is $6M$ ($M$ is the total number of DOF of the analytical model). The physical quantities used in this section are expressed in certain standard coordinates, so the right superscript that typically indicates the local coordinate is omitted in this section. With respect to a periodic steady-state vibration, the coefficient matrices $K'$ and $C'$ become periodic functions of period $2\pi$ with respect to $\tau$. Therefore, $K'$ and $C'$ can be expanded into the Fourier series as follows:

$$K' = K'^0 + \sum_{k=1}^{\infty} \left( K_{1k} \cos k\tau + K_{2k} \sin k\tau \right)$$

$$C' = C'^0 + \sum_{k=1}^{\infty} \left( C_{1k} \cos k\tau + C_{2k} \sin k\tau \right)$$

(31)
The stability of the zero solution of Eq. (30) \((\eta^* = 0)\) can be determined by obtaining the characteristic exponents or characteristic multipliers from Eq. (30). However, obtaining these criteria from Eq. (30), whose dimension is huge, tends to be impossible when the traditional method is used. This makes the stability analysis for large-scale nonlinear systems impossible.

### 5.2 Construction of reduction model

In order to overcome the above problem, the authors proposed a stability analysis method that uses a reduction model [2]. This method derives a reduction model for the stability analysis by applying the concept of modal analysis to the variational equation and by adequately extracting the dominant modes for the stability of the solution. The outline of this method is now presented.

In stability analysis using a reduction model, the following two kinds of linear ordinary differential equations with constant coefficients, which are derived from Eq. (30), are used.

(a) **Reduction method using the linear term mode (Mode I method)**

\[
M\ddot{\eta} + K_0\eta = 0
\]  
(32)

This equation is derived from Eq. (30) by substituting 1 for \(\omega\), omitting \(C^\ast\) and extracting linear component \(K_0\) from the nonlinear base support elements.

(b) **Reduction method using the constant term mode (Mode II method)**

\[
M\ddot{\eta} + K^\ast_0\eta = 0
\]  
(33)

This equation is derived from Eq. (30) by substituting 1 for \(\omega\), omitting \(C^\ast\) and extracting \(K^\ast_0\) in Eq. (31).

In the mode I method, the reduction model doesn’t reflect the effect of the steady-state solution because \(K_0\) is determined only from the linear components of the base support element. On the other hand, in the mode II method, the reduction model reflects the effect of the periodic solution because \(K^\ast_0\) is the function with respect to the periodic solution. Therefore, the accuracy of the stability calculated using the mode II method is expected to be superior to the mode I method in general.

The real eigenvalues and the corresponding eigenvectors from the first to up to the \(I(M) \ll M\) can be obtained by applying ordinary procedure of real modal analysis to Eq. (32) or (33). Then, the \(6M \times M\) dimensional modal matrix \(\Phi\) is obtained. \(\Phi\) is normalized with respect to \(M\) as follows:

\[
\Phi_\Omega = \text{diag}[\sigma_1, \sigma_2, \ldots, \sigma_M]
\]
(34)

where \(\sigma_p\) is the eigenvalues of Eq. (32) or (33). By utilizing \(\Phi\) and \(\Omega\), the \(M\)-dimensional vector \(\zeta^\ast\) is newly introduced as follows:

\[
\eta^* = \Phi\Omega^{-1}\zeta^*
\]
(35)

Then, by using \(\zeta^\ast\), the dimension of Eq. (30) is reduced from \(6M\) to \(M\) as follows:

\[
\omega^2\zeta^\ast + \Omega^2[I_M + K^\ast]\zeta^\ast = 0
\]
(36)

### 5.3 Method for extracting the dominant modes

There is a possibility that the modal matrix \(\Phi\) used for the reduction model might involve modes that only slightly affect the stability of the periodic solution. In such cases, the dimension of the model can be reduced further by removing these useless modes and extracting only the dominant modes. Then, considering that the constant term of stiffness matrix \((I_M + \hat{K}^\ast)\) is the unit matrix, if the absolute values of elements of \(\hat{K}^\ast\) were sufficiently smaller than 1, the modes corresponding with these elements can be removed from the modal matrix [2].
5.4 Stability analysis

The problem of computing the characteristic exponents or characteristic multipliers described in §§ 5.1 is expected to be solved, because the dimension of Eq. (36) can be greatly reduced compared to that of Eq. (30) when the reduction method described above is applied. Then, if the real parts of the characteristic exponents are all negative or the absolute values of the characteristic multipliers are all less than 1, the zero solution of Eq. (30) is stable; otherwise it is unstable.

6. Results of numerical computation

6.1 Computational conditions

In order to confirm the validity of the method proposed in the present paper, a numerical computation was executed for a concrete analytical model. For the numerical computation, a 32-bit personal computer with double precision and FORTRAN 77 was used. The algorithm for the odd harmonic solution was also used. In the process of successive approximation of the periodic solution, the 9th order approximate solution was computed, and the convergence tolerance for the relative error of incremental quantities was set to $10^{-8}$ or less. In the stability analysis, the reduction model was constructed by the mode I or mode II method. Subsequently, the characteristic exponents were obtained from the reduction model by the double QR method [4].

6.2 Computational model

The computational model of a three-dimensional tree structure is shown in Fig. 4. The structure consists of seven hollow steel beams measuring 20 mm in outer diameter, 15 mm in inner diameter, 500 mm in length, and 3.0 N·s/m$^2$ in distributed viscous damping for translation of each beam. Each beam is equally divided into $2^3$ fundamental elements. Therefore, the total DOF of the system becomes 342. The base support elements are placed at the nodes as shown in Fig. 4, and they have a piecewise linear spring or a continuous nonlinear spring of order three with respect to translation expressed as follows:

$$
\begin{align*}
1: \hat{F}_u^a &= 15\hat{u} + 10^5|u| + 10^5(|u|^3) \\
2: \hat{F}_u^a &= 15\hat{u} + 10^4|u| + 10^5(|u|^3) \\
3: \hat{F}_u^a &= 15\hat{u} + k_F u, \quad k_F = \begin{cases} 
10^3; & |u| < 5 \text{ mm} \\
10^5; & |u| \geq 5 \text{ mm}
\end{cases}
\end{align*}
$$

where $u$ denotes the displacement of node at which the base support element exists, for example, $x_2^a$ or $y_2^a$. A harmonic external force having an amplitude of 100 N acts on node 2 in the $Y_2$ direction and on node 6 in the $Y_3$ and $Z_3$ direction.

6.3 Condition of stability analysis

The stability analysis is executed under the condition that the number of modes is set to $M_1 = M_II = 20$ for the mode I and mode II methods. In order to construct the reduction model with the most dominant twenty modes, modes were extracted by the following procedures:

1. A temporary reduction model involving modes up to the 45th order is constructed.
2. Some elements of the 2nd-order Fourier coefficient matrices of $K^a$
are selected in the order of decreasing absolute value [2].

3. The twenty modes corresponding to the elements selected in step (2) are extracted in the order of magnitude.

The Fourier coefficient matrices of $\hat{K}^*$ up to the 18th order were used for the stability analysis. Thus, the dimension of the eigenvalue problem to compute the characteristic exponents becomes 400. In contrast, the dimension increases to 6,840 if the techniques of the reduction model and the extraction of modes are not adopted.

6.4 Frequency response and stability analysis

Figures 5 to 8 show the results of the frequency response for the maximum amplitude of the periodic solution at node 3 of the main system. Figures 5 and 7 show the frequency response in the $Y_3$ direction, and Figs. 6 and 8 show that in the $Z_3$ direction. The stability of the solution is determined by the mode II method in Figs. 5 and 6, and by the mode I method in Figs. 7 and 8. The solid and broken lines designate the stable and the unstable solutions, respectively, and the $\bigcirc$, $\square$ and $\triangle$ symbols represent the bifurcation points of the saddle-node type, Hopf type and pitchfork type, respectively. The chain double-dashed line denotes the switching point of the spring characteristic of the piecewise linear spring (5 mm).

The primary resonances of the 1st through 4th modes are shown in each figure. The response curve at the resonant region leans to the right as a result of the nonlinear character of the hardening type. Moreover, the gradient of the response curve varies largely at the spring characteristic switching. With respect to the computational accuracy of the

![Graph](image1)

Fig. 5  Frequency response of $y_3^*$ and stability analysis by Mode II method

![Graph](image2)

Fig. 6  Frequency response of $z_3^*$ and stability analysis by Mode II method

![Graph](image3)

Fig. 7  Frequency response of $y_3^*$ and stability analysis by Mode I method

![Graph](image4)

Fig. 8  Frequency response of $z_3^*$ and stability analysis by Mode I method
approximate solution, the periodic solution by the 9th order can be regarded as sufficiently accurate because the form of the frequency response deforms only slightly from that obtained by the solution approximated up to the 11th or greater order.

With respect to the stability of the solution, the feature of this analytical model is that there exist not only the unstable regions caused by the saddle-node bifurcation but also many unstable regions caused by the Hopf and the pitchfork bifurcations. The results of the stability analysis by the mode II method shown in Figs. 5 and 6 can be regarded as sufficiently accurate because the results of the stability analysis using 21 or more modes almost never changes from these in Figs. 5 and 6.

On the other hand, by comparing the results obtained by the mode I method shown in Figs. 7 and 8 with the results shown in Figs. 5 and 6, some disagreement is observed. That is, the location of the Hopf bifurcation point and the saddle-node bifurcation point, which must appear at the point in which the tangent of the response curve becomes vertical, differ. Based on these results, it is determined that the mode II method is superior to the mode I method with respect to the accuracy of the stability analysis, although the mode II method has a slight disadvantage with respect to the required computational burden, as pointed out in the previous paper [2].

7. Conclusion

An incremental transfer stiffness coefficient method and a stability analysis method using a reduction model are applied to a three-dimensional tree structure, and an analytical algorithm for longitudinal, flexural and torsional coupled nonlinear vibration is formulated. In addition, the validity of these methods is determined from the concrete numerical computation results. Based on the obtained results, it was confirmed that the steady-state vibration and its stability for a three-dimensional tree structure with a substantial DOF supported by nonlinear element can be analyzed at a high computation speed and with high accuracy. Because the analysis methods developed and presented in the previous papers [1], [2] and the present paper are superior based upon computation speed and accuracy for the vibration analysis of large-scale nonlinear systems, it is believed that method should be adopted for these sorts of nonlinear problems.

References


