Analysis of Steady State Forced Vibration in Simply-Supported-Beam Connected with a Clamped Nonlinear Spring

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Abstract
This paper deals with response analysis of nonlinear vibration in continuous system excited by periodic displacement with arbitrary functions. A system of steady state forced vibration in a simply-supported-beam connected with a nonlinear spring at midpoint of span is considered. The restoring force is assumed to be a piecewise-linear system. For such a system, the beam undergoes a nonlinear vibration when the amplitude is large. In order to analyze the main resonance for the system, the Fourier series method is applied to obtain an exact solution for response vibration. The numerical calculation is performed to obtain the resonance curves. The numerical results show effects of the stiffness of attached spring and the amplitude of excitation on the resonance curves. The experiments are also carried out to verify the numerical results.

Key words: Theory of Vibration, Nonlinear Vibration, Method of Vibration Analysis, Continuous System, Beam Structure, Resonance Curve, Main Resonance

1. Introduction

If we look at a network of long and massive structural elements, such as the piping at a chemical plant, we will observe many simple elements consisting of springs for damping vibrations at the support locations for the piping. Due to their importance in real-world applications, considerable research has been conducted on nonlinear vibrations of such spring supports, which can be considered continuous systems. There has been insufficient systematic study that incorporates the magnitude of restoring forces in the analysis of nonlinearity and spring supports of structural components. Some researchers have analyzed continuous system incorporating spring supports using the finite element method, but this is not necessarily the best method for such calculations, due to high calculation time and cost involved. If an exact solution can be obtained by analytical methods, the influence of various factors on resonance will be clear and the solution itself will be completely transparent. Kobayashi et al., Tamura et al., Pakdemirli et al., and Nayfeh et al. have published analytical studies of systems incorporating spring supports, but these reports do not sufficiently discuss the effects of various factors on resonance.

In this study, we consider a simply supported, bending beam model with an adjacent nonlinear spring at mid-span as an example of a continuous system. The restoring force of the spring is approximately symmetric piecewise-linear system. A steady-state vibration occurring under the action of a displacement excitation function with an arbitrary period is
analyzed. The analysis assumes that the restoring force from the supporting spring is finite at the extremities of the vibration. A Fourier series is expanded about this function and an exact solution is derived. A numerical calculation is also performed based on the results of this analysis. The resulting vibration in the main resonance region is predicted, and resonance curves and phase lag curve are constructed using parameters for the nonlinearity of the supporting spring and the amplitude ratio of the excitation. Finally, an experiment is performed with a real model that corresponds to the analytical model and the numerical results are found to fit the experimental results well.

2. Theoretical Analysis

2.1 Characteristics of the system and equation of motion

As shown in Fig. 1, the vibration considered in this subject is the steady-state response of a simply-supported-beam in lateral bending, with an additional nonlinear supporting spring (on pin mounts) acting at mid-span. A periodic displacement excitation was applied to this spring. The restoring force acting on the beam is the cubic curve indicated by the dotted line in Fig. 2, and was assumed to be a characteristic of the nonlinear supporting spring (spring constant \( K \)). This dotted line was treated by approximating it as a symmetric piecewise-linear spring (solid line), combining the spring constant of \( K_1 \) between the boundary lines \( z_{l/2} = \pm e_0 \) with a spring constant of \( K_1 + K_2 \) outside the boundary line. The straight lines were selected to give equal areas between the dotted line and straight lines below the curve and between the dotted line and straight lines above the curve. Thus, this analysis assumes that the restoring force can be represented by a symmetric piecewise-linear function. If Region II in Fig. 2 is the standard, then from the viewpoint of the entire system, Regions I and II are the nonlinear regions. The restoring force is represented by the symmetrical piecewise-linear sections on either side of the linear portion. The spring constant in the linear region is \( K_1 \), and in the nonlinear region it is \( K_1 + K_2 \).

The material of the beam is assumed to be homogenous. If the cross-sectional area of the beam is \( A \), the second moment of area is \( I \), the Young’s modulus is \( E \) and the mass per unit volume is \( \rho \), then the equation of motion of this system is given by:

\[
\frac{\partial^2 y}{\partial t^2} + \frac{EI}{\rho A} \frac{\partial^4 y}{\partial x^4} = 0
\]

(1)

If the excitation acting to displace this system is \( q(t) \) and the relative displacement of the beam is \( z \), then the absolute displacement \( y \) is expressed by

\[
y = z + q(t)
\]

(2)
Equation (2) can be used to express the equation of motion in terms of $z$:

$$\frac{\partial^2 z}{\partial t^2} + \frac{EI}{\rho A} \frac{\partial^4 z}{\partial x^4} = \frac{\partial^2 q(t)}{\partial t^2}$$  \hspace{1cm} (3)

2.2 Fourier series analysis

Let the excitation $q(t)$ acting on the beam have an arbitrary period; then, the real and complex Fourier series can be expanded about $q(t)$:

$$q(t) = \sum_{n=-\infty}^{\infty} s_n e^{j\omega t} = \frac{f_0}{2} + \sum_{n=1}^{\infty} \left( f_n \cos n\omega t + g_n \sin n\omega t \right)$$  \hspace{1cm} (4)

where $\omega$ is the angular frequency. If Eq. (4) is substituted into Eq. (3), we obtain the following partial differential equation:

$$\frac{\partial^2 z}{\partial t^2} + \frac{EI}{\rho A} \frac{\partial^4 z}{\partial x^4} = \sum_{n=-\infty}^{\infty} n^2 \omega^2 s_n e^{j\omega t}$$  \hspace{1cm} (5)

The goal of this analysis is to describe the steady-state vibrations in the main resonance region, so the waveform of the resulting vibration at the midpoint of the beam span is a periodic function with the same frequency as the excitation. It is assumed to be described by the following:

$$z = \sum_{n=-\infty}^{\infty} W_n(x) e^{j\omega t}, \quad y = \sum_{n=-\infty}^{\infty} X_n(x) e^{j\omega t}$$  \hspace{1cm} (6)

If Eq. (4) and (6) are substituted into Eq. (2), the following relationship is obtained:

$$W_n(x) = X_n(x) - s_n$$  \hspace{1cm} (7)

and if Eqs. (6) and (7) are substituted into Eq. (5), it is converted into the following ordinary differential equation:

$$\frac{d^4}{dx^4} X_n(x) = n^2 \omega^2 \frac{\rho A}{EI} X_n(x)$$  \hspace{1cm} (8)

If the coefficients in Eq. (8) satisfy

$$\lambda_n^4 = n^2 \omega^2 \frac{\rho A}{EI}$$  \hspace{1cm} (9)

then $\lambda_n$ are eigenvalues, and $X_n(x)$ ($n=0,1,2,...$) can be written as

$$X_0(x) = ax^3 + bx^2 + cx + d$$
$$X_n(x) = A_n \cosh \lambda_n x + B_n \sinh \lambda_n x + C_n \cos \lambda_n x + D_n \sin \lambda_n x$$  \hspace{1cm} (10)

Here, $a$, $b$, $c$, $d$, $A_n$, $B_n$, $C_n$, and $D_n$ are unknown constants that can be determined by the boundary conditions. The angular frequency $\omega_0$ is expressed by
\[ \omega_0 = \frac{Z_1^2}{l^2} \sqrt{\frac{EI}{\rho A}}, \quad Z_1 = \pi \] (11)

If the vibration frequency ratio \( \Omega (=\omega/\omega_0) \) is substituted into Eq. (9), the following relation holds:

\[ \lambda_n^4 l^4 = Z_1^2 n^2 \Omega^2 \] (12)

This analysis is limited to waveforms containing only a single entrance into the upper and lower nonlinear portions of the restoring force per cycle in the main resonance region. Figure 3 shows the assumed shape of the harmonic resonance vibration caused by the excitation acting on the beam shown in Fig. 1, by the resulting vibration \( z_{l/2}(\theta) \), and by the nonlinear restoring force \( g(\theta) \) due to the spring constant \( K_2 \) approximated by the piecewise-linear system in Fig. 2. The resulting relative displacement \( z_{l/2} \) at the beam midpoint is considered to be periodic; the response is divided into regions I, II, and III, corresponding to the regions shown in Fig. 2. The origin for measuring the phase angle \( \theta \) is defined from the peak of the fundamental waveform (dotted line). The independent variable of time \( t \) was transformed to phase angle \( \theta \), which is defined as:

\[ \theta = \omega t - \alpha \] (13)

where \( \alpha \) is the phase lag angle, which is currently undefined but will be determined later. The displacement can then be expressed as:

\[ z(x, \theta) = \sum_{n=-\infty}^{\infty} \left( X_n(x) - s_n \right) e^{jn(\theta + \alpha)} \] (14)
Expressing the interval of Region I, III by the phase angle $\theta_0$, interval I, III are regarded as the nonlinear region. The relation between the relative displacement and the phase angle (i.e., the switching-over condition) is given by the following expressions:

$$
\begin{align*}
\theta = \pm \frac{\theta_0}{2} : & z_{l/2} = e_0 \\
\theta = \pi \pm \frac{\theta_0}{2} : & z_{l/2} = -e_0
\end{align*}
$$

(15)

Meanwhile, the boundary conditions for the analytical model in Fig. 1 are expressed by:

$$
\begin{align*}
x = 0 : & z = 0, \\
x = 0 : & \frac{\partial^2 z}{\partial x^2} = 0 \\
x = \frac{1}{2} : & \frac{\partial z}{\partial x} = 0, \\
x = \frac{1}{2} : & 2EJ\frac{\partial^3 z}{\partial x^3} = f(z_{l/2})
\end{align*}
$$

(16)

where $f(z_{l/2})$ is the restoring force applied by the nonlinear supporting spring; the restoring force has the characteristics indicated in Fig. 2, so these can be expressed thus in the three regions:

$$
\begin{align*}
f(z_{l/2}) = K_1z_{l/2} + K_2(z_{l/2} - e_0) : & \frac{\theta_0}{2} \leq \theta \leq \frac{\theta_0}{2} \\
f(z_{l/2}) = K_1z_{l/2} : & \frac{\theta_0}{2} \leq \theta \leq \pi - \frac{\theta_0}{2} \\
f(z_{l/2}) = K_1z_{l/2} + K_2(z_{l/2} + e_0) : & \pi - \frac{\theta_0}{2} \leq \theta \leq \pi + \frac{\theta_0}{2}
\end{align*}
$$

(17)

In addition, once the system has reached steady state, the period of the nonlinear portion of the restoring force $g(\theta)$ shown in Fig. 3(c) can be considered to be equal to the period of the resulting vibration waveform. Then, $g(\theta)$ is expanded by a complex Fourier series.

$$
g(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)
$$

(18)

where $c_n$, $a_0$, $a_n$, and $b_n$ are Fourier coefficients.

### 2.3 Derivation of the solution equations

In this section, the solution equation for the response is derived using the switching-over conditions and the boundary conditions given in Section 2.2. This analysis addresses the steady-state solution, so the nonlinear portion $g(\theta)$ of the restoring force is considered to be the restitution force. It is treated as an external force and is incorporated as such in the equation of motion in Eq. (1). The switching-over conditions (Eq. (15)) and boundary conditions (Eq. (16)) are incorporated in Eq. (14), the expression for the resulting waveform, and the equation of motion is linearized using Eq. (18). The dimensionless response displacement ratio $z_{l/2}/e_0$ for the displacement of the beam midpoint is then derived from:
In addition, when the resulting displacement is given by Eq. (19), the excitation amplitude ratio is derived from:

\[
\frac{f_1}{\epsilon_0} = \frac{1}{\cos \alpha} \left[ \cos \theta_0 + \frac{1}{2} \sum_{n=3,5,7,\ldots} \left( N_n u_n - \frac{1-M_1 x_1}{N_1} M_n y_n \right) \cos \frac{n\theta_0}{2} \right] \]

(20)

The resulting vibration also has a phase lag with respect to the excitation of angle \( \alpha \), calculated as follows:

\[
\alpha = \tan^{-1} \left[ \frac{M_1 y_1 + \sum_{n=3,5,7,\ldots} \left( N_n u_n - \frac{1-M_1 x_1}{N_1} M_n y_n \right) \frac{\sin n\theta_0}{\sin \frac{\theta_0}{2}}}{\sqrt{1-M_1 x_1}} \right] \]

(21)

Here, in order to simplify Eqs. (19)-(21), we introduce the following normalizing equations to obtain dimensionless quantities:

\[
x_n = \alpha_n f(k \Gamma), \quad y_n = b_n f(k \Gamma), \quad k = 48EI/k^2, \quad \Omega = \omega_0^2, \quad \kappa = K_i/k
\]

\[
N_n = N_n = \frac{1}{\Delta_n} \left( \cosh \frac{\lambda_n l}{2} \sinh \frac{\lambda_n l}{2} - \cosh \frac{\lambda_n l}{2} \sin \frac{\lambda_n l}{2} \right) \frac{24}{\lambda_n^3} + \cosh \frac{\lambda_n l}{2} \sin \frac{\lambda_n l}{2} - 1
\]

\[
M_n = \frac{24}{\lambda_n^3} \Delta_n \left( \sin \frac{\lambda_n l}{2} \cos \frac{\lambda_n l}{2} - \cosh \frac{\lambda_n l}{2} \sin \frac{\lambda_n l}{2} \right)
\]

\[
\Delta_n = \left( \cosh \frac{\lambda_n l}{2} \sin \frac{\lambda_n l}{2} - \cos \frac{\lambda_n l}{2} \sinh \frac{\lambda_n l}{2} \right) \frac{24}{\lambda_n^3} + \cosh \frac{\lambda_n l}{2} \sin \frac{\lambda_n l}{2} - 2 \cosh \frac{\lambda_n l}{2} \sin \frac{\lambda_n l}{2}
\]

(22)

The amplitude \( \Gamma \) for the cosine components in the basic resulting vibration introduced for the dimensionless Fourier coefficients \( x_n \) and \( y_n \) in Eq. (22) is calculated by:

\[
\Gamma = \epsilon_0 \left[ \cos \frac{\theta_0}{2} + \sum_{n=3,5,7,\ldots} \left( N_n u_n - \frac{1-M_1 x_1}{N_1} M_n y_n \right) \cos \frac{n\theta_0}{2} \right]^{-1}
\]

(23)

The dimensionless Fourier coefficients \( x_n \) and \( y_n \) from the nonlinear portion in Eq. (18) for the restoring force are introduced:
\[ g(\theta) = \sum_{n=1,3,5,\ldots}^{\infty} (x_n \cos n\theta + y_n \sin n\theta) \]  

Equations (19)-(21), (23), and (24) obtained from the above analysis are all written in terms of the vibration frequency ratio \( \Omega \), the phase lag angle \( \alpha \), and the nonlinear phase range \( \theta_0 \) in regions I and III, and also, the independent dimensionless Fourier coefficients \( x_n \) and \( y_n \) \((n=1,2,3,\ldots)\). If \( x_n \) and \( y_n \) in these equations can be obtained, the resulting vibration can be identified. In other words, the process is opposite to the actual order of physical causation, but the resulting displacement can be found from Eq. (19); in turn, the excitation corresponding to this response can be calculated from Eq. (20), the nonlinear component of the restoring force can be calculated from Eq. (24), and the phase lag angle \( \alpha \), from Eq. (21).

2.4 Determination of dimensionless Fourier coefficients

The dimensionless Fourier coefficients \( x_n \) and \( y_n \) are defined by the first and second expressions in Eq. (22). When the Fourier series is expanded about the equation for the nonlinear component of the restoring force, these undetermined coefficients must satisfy the piecewise-linear conditions of the restoring force. The following dimensionless linear simultaneous equations are obtained from these conditions:

\[ x_m = \sum_{n=3,5,7,\ldots}^{\infty} (A_{mn} x_n - B_{mn} x_1) = I_m + \sum_{n=3,5,7,\ldots}^{\infty} I_{mn}, \quad (m = 0,1,2,\ldots) \]  

\[ y_m = \sum_{n=3,5,7,\ldots}^{\infty} (C_{mn} y_n - D_{mn} y_1) = J_m + \sum_{n=3,5,7,\ldots}^{\infty} J_{mn}, \quad (m = 0,1,2,\ldots) \]

Solving these equations together gives \( x_n \) and \( y_n \). The coefficients \( A_{mn}, B_{mn}; \cdots; J_{mn} \) in the above expressions are constant terms that are determined as parameters for \( \Omega, \alpha \) and \( \theta_0 \); they are given by the following:

\[ A_{mn} = \frac{K}{k} \frac{\theta_0}{\pi} M_n \left( \frac{\sin \frac{m+n}{2} \theta_0}{m+n} - \frac{\sin \frac{m-n}{2} \theta_0}{m-n} - \frac{2\cos n\theta_0}{2} - \frac{\sin \frac{m}{2} \theta_0}{2} \right) \]

\[ I_m = \frac{K}{k} \frac{\theta_0}{\pi} M_n \left( \frac{\sin \frac{m+1}{2} \theta_0}{m+1} - \frac{\sin \frac{m-1}{2} \theta_0}{m-1} - \frac{2\cos \theta_0}{2} - \frac{\sin \frac{m}{2} \theta_0}{2} \right) \]

\[ C_{mn} = \frac{K}{k} \frac{\theta_0}{\pi} M_n \left( \frac{\sin \frac{m-n}{2} \theta_0}{m-n} - \frac{\sin \frac{m+n}{2} \theta_0}{m+n} - \frac{\sin n\theta_0}{2} - \frac{\sin \frac{m-1}{2} \theta_0}{2} \right) \]

\[ I_{mn} = \frac{N_n}{N_1} \frac{u_n}{M_n} A_{mn}, \quad B_{mn} = M_1 I_{mn}, \quad J_{mn} = \frac{N_n}{N_1} \frac{v_n}{M_n} C_{mn}, \quad D_{mn} = M_1 J_{mn} \]

3. Experimental Apparatus and Method

An experimental apparatus was built to embody the analytical model in Fig. 1 and an experiment was carried out to verify the analysis results. Figure 4 is a diagram of the apparatus for observing steady-state vibrations in a beam satisfying the equation of motion in Eq. (3), Fig. 5 is a photograph of the apparatus\(^5\), and Table 1 provides the dimensions for
the experimental beam and spring. The test beam (plate thickness: 0.5 mm, width: 12.8 mm) was made of spring steel. The beam was assumed to be a piecewise-linear system; to model this, the linear spring ④ was connected to the beam in parallel with the supporting spring ⑤ with a gap $e_0$, to create a system with a spring constant that satisfies the property indicated by the solid line.

Beam ③ was connected to a linear spring ④ fixed to frame ⑦ was simply supported at both ends with adjacent supporting springs ⑤ (coil springs) whose near ends were separated from the linear spring ④ by symmetric gaps $e_0$ and whose far ends were fixed to frame ⑦. A response vibration occurred in the beam under the excitation vibration. Once the displacement of the beam midpoint exceeded distance $e_0$, the beam collided with the supporting springs ⑤. At this time, the displacement input vibration caused a reciprocating movement of frame ⑦ on the foundation ⑥ via the linear ball bearing, acting to cause a periodic excitation displacement vibration of the beam ③. A continuous bar ② fixed to the foundation ⑥ was connected to the shaker ①. The displacement vibration caused by the shaker ① was set by the vibration controller ⑫ connected to the function generator ⑧. Friction with the floor had little effect on the frame ⑦ from during vibration, so that this factor could be neglected. The excitation displacement was measured with a laser displacement sensor ⑨ and the resulting vibration was measured with another laser displacement sensor ⑩ mounted on the frame. The analog outputs from the sensors were stored on an analyzing recorder ⑪.

4. Numerical and Experimental Results

The values given in Table 1 were used for the dimensions of the model used for the numerical simulation of the experimental model in order to compare between the two
models. Fourier coefficients $x_n$ and $y_n$ were calculated up to $m, n = 40$ in the dimensionless Eqs. (25) and (26) in the approximation. The convergence obtained was satisfactory.

**4.1 Comparison of numerical and experimental results**

Figure 6 shows the resonance curves for comparing the numerical and experimental models. These results show quite a good agreement. An elbow in the resonance curve is apparent when the maximum vibration amplitude at the beam midpoint entered the nonlinear region (marked with ●), beyond which the curve bent sharply to the right, indicating prominent nonlinear behavior. The results in the physical model (open circles ○) showed a jump to the lower line of linear vibrations when the vibration frequency ratio exceeded a value of $\Omega \approx 1.69$ as $\Omega$ was being increased. No results were obtained in the unstable region (marked with ●).
nonlinear region in the physical model at higher values of $\Omega$. This is because damping was not considered in the analytical model, while in the physical model there was damping due to air resistance, internal friction in the material and friction at the beam supports.

Conversely, when the frequency was being reduced, there was a corresponding jump in the results in the physical model to the upper curve representing resonance near the value of $\Omega \approx 1.58$. This pattern of discontinuous results is well known in the field of nonlinear vibrations; the backbone curve of the resonance predicted by the numerical model is tilted and splits to a lower unstable branch (dotted line), while the physical model shows no such results. A stability analysis is necessary to make any rigorous conclusions about the unstable region of the resonance curve. We have already conducted a partial stability analysis\(^{(1)}\) using Hill’s equation; this deserves treatment in a follow-up report.

Figures 7 and 8 present comparisons of the experimental and numerical results for the waveform of the resulting displacement ratio $z_{1/2}(\theta)/e_0$ and the waveform of the restoring force $q(\theta)/k_f$ under the excitation $q(\theta)$ at points A and B on the resonance curve in Fig. 6. The amplitudes of these predicted values show good agreement between the experimental and numerical models, but the phase lag of the experimental model waveform was about 60° behind the phase lag of the numerical model $z_{1/2}(\theta)/e_0$, with respect to the excitation.
This discrepancy was attributed to the effects of damping terms and delay in the action of the restoring force in the supporting spring, which were not accounted for in the theoretical analysis used to develop Eq. (1).

4.2 Example of numerical calculation

A numerical analysis was performed based on these analytical results and a steady-state periodic solution was sought in the main resonance region. A numerical calculation was carried out for the combined wave (an odd function) given in Eq. (28), as an example of excitation by a function of arbitrary period.

\[ q(t) = \cos \omega t + 0.3 \sin 3\omega t \]

Figure 9(a) shows the resonance curve when the spring constant ratio is \( K_2/k \), and Fig. 9(b) shows the phase lag angle. The symbols (●) in the figure indicate the boundary between before and after the amplitude of the beam center was sufficiently great for the beam to collide with the support spring whose constant was \( K_2 \) (the nonlinear region). \( K_2/k \) expresses the magnitude of the nonlinearity of the supporting spring; when the resonance curve enters the nonlinear region, it bends, and the greater \( K_2/k \) is, the more the resonance curve bends to the right.

Figure 10 shows the resonance curve with respect to another parameter \( f_1/e_0 \), the ratio of excitation amplitude. The greater \( f_1/e_0 \) is, the wider the extent of the resonance region, and the lower \( e_0 \), the relative width of the linear region (Region I) is, the wider the resonance region.

5. Conclusions

(1) This study assumed that a model of a simply-supported-beam connected to a nonlinear spring for control of vibration could be replaced with a model of a symmetric piecewise-linear system. An analytical model was constructed of a simply-supported-beam connected to a nonlinear spring at the midpoint of the beam span. This restoring force was approximated as a symmetrical piecewise-linear vibrational system in lateral bending. A Fourier series expansion was applied to steady-state forced vibration under an excitation having an arbitrary period and an exact solution was derived for the resulting vibration.
A numerical calculation was performed based on the results of the analysis in (1). A resonance curve and a phase lag angle curve were created using the parameters $K_2/k$ (spring constant ratio) and $f_1/e_0$ (excitation amplitude ratio) and the effects of several factors on the output characteristics were revealed. The waveform of the resulting vibration in the periodic solution and the nonlinear portion of the restoring force were found and compared with the results for an experimental model.

An experiment was performed to check the analytical results for the resonance curve obtained in (1) and (2). The calculated results were shown to agree well with the experimental results. This analytical method was confirmed to be effective.

This is a general method for handling excitations with arbitrary periods. It was also shown that, once this analytical method has been established, it is possible to find a solution for response vibrations, regardless of the magnitude of the parameters described in (2).

References


