Nonlinear Parametric Vibrations of Elastic Structures Containing Two Cylindrical Liquid-Filled Tanks*

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Abstract

Nonlinear vibrations of elastic structures containing two cylindrical liquid-filled tanks subjected to vertical harmonic excitation are investigated. When a 2:1:1 ratio of internal resonance is satisfied between the natural frequencies of the structure and the sloshing in the two liquid tanks, nonlinear parametric vibrations can be observed. The equation of motion for the structure is derived considering the nonlinearity of the hydrodynamic force, and the modal equations for sloshing are derived by using Galerkin's method. Then, frequency response curves are determined by using van der Pol's method. The respective influences of the liquid levels and the deviation of the internal resonance ratio 2:1:1 on the frequency response curves are discussed. The results of the response curves differed from those obtained in the case where a structure had only one liquid-filled tank installed. It is found that two types of vibrations may occur depending on the excitation frequency and the liquid levels: a single-mode solution (sloshing occurs in either tank) and a double-mode solution (sloshing occurs in both tanks). Amplitude and phase modulated motion appears in the single-mode solution when some deviation of the internal resonance ratio exists. The validity of the theoretical results was verified by the experiments.

Key words: Nonlinear Parametric Vibration, Sloshing, Two Cylindrical Tanks, Frequency Response Curve, Vertical Excitation, Single-Mode and Double-Mode Solutions, Amplitude and Phase Modulated Motion

1. Introduction

Investigating vibrations of elastic structures with liquid-filled tanks, as seen in elevated-water tanks and liquefied gas storage tanks, is crucial in the fields of mechanical and civil engineering in terms of disaster prevention. Thousands of the technical papers were cited in the review paper by Ibrahim et al. ⁽¹⁾ and in his book ⁽²⁾. Recently, partially filled liquid tanks, called Tuned Liquid Dampers (TLDs), have been utilized as a vibration suppression device instead of a dynamic vibration absorber. The idea of using TLDs was first proposed by Modi and Welt ⁽³⁾ and it was developed by Hagiuda ⁽⁴⁾, Fujino et al. ⁽⁵⁾, and Kaneko and Ishikawa ⁽⁶⁾. It is well known that the nonlinearity of the hydrodynamic force becomes significant as the liquid sloshing approaches high amplitudes. In order to ensure accurate results, this nonlinearity must be taken into account in the theoretical analysis. Ikeda et al. ⁽⁷⁾⁽⁸⁾ showed the frequency response curves of the elastic structures, subjected to horizontal excitation, with a rectangular tank or a cylindrical tank by considering the

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nonlinear liquid force. In addition, nonlinear parametric vibrations of structures containing either a cylindrical tank or a rectangular tank have been also investigated by Ibrahim et al. ⁽⁹⁾⁻⁽¹¹⁾ and Ikeda and Murakami ⁽¹²⁾⁽¹³⁾, and Ikeda ⁽¹⁴⁾ where the structures are subjected to vertical excitation. The theoretical analyses used in the previous research have dealt with structures containing only one tank. However, in reality, multiple tanks are used, because they have some advantages in manufacturing, installing and maintaining. There appears to be only two papers by Ikeda ⁽¹⁵⁾⁽¹⁶⁾ who has theoretically and experimentally investigated the vibrations of structures containing two rectangular tanks under either vertical or horizontal excitation.

In this paper, nonlinear vibrations of elastic structures containing two cylindrical liquid-filled tanks (a dual-tank system) subjected to vertical harmonic excitation are investigated. Nonlinear parametric vibrations occur when a 2:1:1 ratio of internal resonance is satisfied between the natural frequencies of the structure and the sloshing in the two tanks. The equation of motion of the structure is derived taking into account the nonlinearity of the hydrodynamic force and the modal equations for sloshing are derived by using Galerkin's method. Then, van der Pol's method is applied to determine frequency response curves for the displacement of the structure and liquid free surfaces. The influences of the liquid levels and the deviation of the internal resonance ratio on the response curves are discussed. The purpose of this paper is to illustrate the differences between the dynamic responses of the structures when either one cylindrical tank, investigated previously by Ikeda and Murakami ⁽¹²⁾, or two cylindrical tanks are installed. The validity of the theoretical analysis was confirmed by experimental data.

2. Theoretical Analysis

2.1. Equations of Motion

Figure 1 shows the theoretical model. A real structure is modeled as a single-degree-of-freedom system with a mass m , a spring constant k , and a viscous damping coefficient c . Two cylindrical liquid-filled tanks (tanks 1 and 2) with a radius R , are installed on the structure. Hereafter, the symbols of system parameters have a subscript ' i ' ($i=1, 2$) to distinguish the two different tanks. The tanks are partially filled with liquid to depths h_i . In Fig. 1, the coordinate systems $O_i-x_iy_iz_i$ and $O_i-r_i\theta_iz_i$, are fixed to the tanks. The x_iy_i -planes are defined as the planes which coincide with the undisturbed liquid free surfaces. The vertical displacement of the structure is z_0 . The elevations of the liquid surfaces are η_i at the position (r_i, θ_i) . Here, the potential theory is applied to investigate the liquid motion without considering the fluid viscosity. Hence, the velocity potentials can exist and they are designated as ϕ_i . However, the viscous damping terms are later factored into the modal

Fig. 1 Theoretical model.

Fig. 2 Mode shapes.

equations. The other parameters of the liquids are as follows: P_i are fluid pressures, ρ is fluid density and F_i are the fluid forces acting on the structure from the tank bottoms. The structure is vertically excited by the external force $F_0 \cos \omega t$. The following dimensionless quantities are introduced:

$$\begin{aligned} z'_0 &= z_0/R, \eta'_i = \eta_i/R, r'_i = r_i/R, z'_i = z_i/R, h'_i = h_i/R, \\ v &= m/M, v_i = m_i/(\pi M h'_i), k' = k/(M p_{11}^2), c' = c/(M p_{11}), \\ \phi'_i &= \phi_i/(R^2 p_{11}), F'_0 = F_0/(M R p_{11}^2), F'_i = F_i/(M R p_{11}^2), \\ P'_i &= P_i/(\rho R^2 p_{11}^2), \xi_{mn} = \lambda_{mn} R, \omega' = \omega/p_{11}, t' = p_{11} t, \\ P'_{mn} &= p_{mn}/p_{11}, \bar{P}'_{mn} = \bar{p}_{mn}/p_{11}, \end{aligned} \quad (1)$$

where

$$M = m + m_1 + m_2, \quad p_{mn} = \sqrt{g \lambda_{mn} \tanh(\lambda_{mn} h_1)}, \quad \bar{p}_{mn} = \sqrt{g \lambda_{mn} \tanh(\lambda_{mn} h_2)}, \quad (2)$$

and g is the acceleration of gravity. In Eq. (2), where m and n are integers, p_{mn} and $\bar{p}_{mn}$ represent the natural frequencies of (m, n) sloshing mode in tanks 1 and 2, respectively. λ_{mn} in Eqs. (1) and (2) are the n -th positive roots of the following equations:

$$\frac{d}{dr_i} [J_m(\lambda r_i)] = 0 \quad (\text{at } r_i = R), \quad (3)$$

where J_m denotes the m -th order Bessel function of the first kind. Hereafter, the primes ' ' will be omitted. Using the velocity potentials ϕ_i , we obtain Laplace's equations

$$\frac{\partial^2 \phi_i}{\partial r_i^2} + \frac{1}{r_i} \frac{\partial \phi_i}{\partial \theta_i} + \frac{1}{r_i^2} \frac{\partial^2 \phi_i}{\partial \theta_i^2} + \frac{\partial^2 \phi_i}{\partial z_i^2} = 0, \quad (4)$$

and the fluid pressures from Euler's energy equations

$$P_i = -\frac{\partial \phi_i}{\partial t} - \frac{1}{2} \left\{ \left(\frac{\partial \phi_i}{\partial r_i} \right)^2 + \frac{1}{r_i^2} \left(\frac{\partial \phi_i}{\partial \theta_i} \right)^2 + \left(\frac{\partial \phi_i}{\partial z_i} \right)^2 \right\} - \frac{z_i}{\psi_{11}} - \ddot{z}_0 z_i, \quad (5)$$

where $\psi_{11} = \lambda_{11} \tanh(\lambda_{11} h_1)$. The boundary conditions at the tank walls and bottoms are given by

$$\frac{\partial \phi_i}{\partial r_i} = 0 \quad (\text{at } r_i = 1), \quad \frac{\partial \phi_i}{\partial z_i} = 0 \quad (\text{at } z_i = -h_i). \quad (6a, b)$$

In addition, because the vertical velocity of the free surface is equal to the velocity of a liquid particle on the surface, the kinematic boundary conditions are given by

$$\frac{\partial \phi_i}{\partial z_i} = \frac{\partial \eta_i}{\partial t} + \frac{\partial \phi_i}{\partial r_i} \frac{\partial \eta_i}{\partial r_i} + \frac{1}{r_i^2} \frac{\partial \phi_i}{\partial \theta_i} \frac{\partial \eta_i}{\partial \theta_i} \quad (\text{at } z_i = \eta_i). \quad (7)$$

Because $P_i = 0$ on the free surfaces, Eq. (5) is reduced to

$$\frac{\partial \phi_i}{\partial t} + \frac{1}{2} \left\{ \left(\frac{\partial \phi_i}{\partial r_i} \right)^2 + \frac{1}{r_i^2} \left(\frac{\partial \phi_i}{\partial \theta_i} \right)^2 + \left(\frac{\partial \phi_i}{\partial z_i} \right)^2 \right\} + \frac{z_i}{\psi_{11}} + \ddot{z}_0 z_i = 0 \quad (\text{at } z_i = \eta_i). \quad (8)$$

The equation of motion for the structure is given by

$$v \ddot{z}_0 + c \dot{z}_0 + k z_0 = F_1 + F_2 + F_0 \cos \omega t + \frac{v_1 \pi h_1}{\psi_{11}} + \frac{v_2 \pi h_2}{\psi_{11}}, \quad (9)$$

where F_i represent the fluid forces acting on the structure from the bottoms of the tanks and are expressed by

$$F_i = -v_i \int_0^{2\pi} \int_0^1 r_i P_i(r_i, \theta_i, z_i, t) |_{z_i=-h_i} dr_i d\theta_i \quad (i = 1, 2), \quad (10)$$

where the fluid pressures P_i are given by Eq. (5).

2.2. Modal Equations

The solutions of forced vibrations are assumed to be expressed in terms of the eigenfunctions, which are obtained from the free vibration analysis for the corresponding linear system, as follows:

$$\phi_1(r_1, \theta_1, z_1, t) = a_c(t) + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \{a_{mn}(t) \cos m\theta_1 + b_{mn}(t) \sin m\theta_1\} J_m(\xi_{mn} r_1) \times \cosh\{\xi_{mn}(z_1 + h_1)\} / \cosh(\xi_{mn} h_1), \quad (11a)$$

$$\eta_1(r_1, \theta_1, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \{c_{mn}(t) \cos m\theta_1 + d_{mn}(t) \sin m\theta_1\} J_m(\xi_{mn} r_1), \quad (11b)$$

$$\phi_2(r_2, \theta_2, z_2, t) = \bar{a}_c(t) + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \{\bar{a}_{mn}(t) \cos m\theta_2 + \bar{b}_{mn}(t) \sin m\theta_2\} J_m(\xi_{mn} r_2) \times \cosh\{\xi_{mn}(z_2 + h_2)\} / \cosh(\xi_{mn} h_2), \quad (11c)$$

$$\eta_2(r_2, \theta_2, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \{\bar{c}_{mn}(t) \cos m\theta_2 + \bar{d}_{mn}(t) \sin m\theta_2\} J_m(\xi_{mn} r_2), \quad (11d)$$

where $a_c, a_{mn}, b_{mn}, c_{mn}, d_{mn}, \bar{a}_c, \bar{a}_{mn}, \bar{b}_{mn}, \bar{c}_{mn}$ and $\bar{d}_{mn}$ are the unknown functions of time. c_{mn}, d_{mn} and $\bar{c}_{mn}, \bar{d}_{mn}$ represent the amplitudes of (m, n) sloshing modes in tanks 1 and 2, respectively. Figure 2 shows the shapes of $(0, 1)$, $(1, 1)$ and $(2, 1)$ modes. The thin lines, shown in the top views, represent the nodal lines of the sloshing modes. The small parameter ε is introduced, and the variables and parameters are assumed in the same way by Hutton⁽¹⁷⁾ as follows:

$$\begin{aligned} a_{11}, \bar{a}_{11}, b_{11}, \bar{b}_{11}, c_{11}, \bar{c}_{11}, d_{11}, \bar{d}_{11}, z_0; c, \xi_{mn}, \bar{\xi}_{mn} &\approx O(\varepsilon^{1/3}), \\ a_c, \bar{a}_c, a_{01}, \bar{a}_{01}, c_{01}, \bar{c}_{01}, a_{21}, \bar{a}_{21}, b_{21}, \bar{b}_{21}, c_{21}, \bar{c}_{21}, d_{21}, \bar{d}_{21} &\approx O(\varepsilon^{2/3}), \\ a_{31}, \bar{a}_{31}, b_{31}, \bar{b}_{31}, c_{31}, \bar{c}_{31}, d_{31}, \bar{d}_{31}; F_0 &\approx O(\varepsilon^{3/3}). \end{aligned} \quad (12)$$

In Eq. (9), the amplitude of excitation F_0 is first assumed to be $O(\varepsilon)$. It should be noted that the resultant equations, Eqs. (13a)-(13c), include cubic terms. Then, conducting the nonlinear theoretical analysis within the accuracy of $O(\varepsilon)$, we assume that the displacement of the structure z_0 and the amplitudes of $(1, 1)$ sloshing mode in the tanks have the magnitude of $O(\varepsilon^{1/3})$ because they appeared predominantly in experiments discussed later. Other significant sloshing modes, $(0, 1)$ and $(2, 1)$, are assumed to be $O(\varepsilon^{2/3})$ in order to improve the accuracy of z_0 and the amplitudes of $(1, 1)$ sloshing mode. Substituting Eq. (5) together with Eq. (11) into Eq. (10), and considering Eq. (12), we obtain the fluid forces F_i within the accuracy of $O(\varepsilon)$. Substituting the resulting fluid forces into Eq. (9), we can rewrite the equation of motion for the structure as follows:

$$Q_1 \ddot{z}_0 + c \dot{z}_0 + k z_0 + H_1(z_0, c_{11}, d_{11}, \bar{c}_{11}, \bar{d}_{11}) = F_0 \cos \omega t, \quad (13a)$$

where $Q_1 = \nu + \pi \nu_1 h_1 + \pi \nu_2 h_2$. Q_1 represents the dimensionless quantity corresponding to the sum of the masses of the structure and liquids. Therefore, $Q_1 = 1$ must be always satisfied [see Eqs. (1) and (2)].

Expanding Eqs. (7) and (8) in Taylor's series about $\eta_i = 0$, then substituting Eq. (11) into the resulting equations and equating the coefficients of $J_0(\xi_{01} r_i)$, $J_m(\xi_{m1} r_i) \cos m\theta_i$ and $J_m(\xi_{m1} r_i) \sin m\theta_i$ (where $m=1, 2$) on both sides of the resulting equations within the accuracy of $O(\varepsilon)$, results in the modal equations for sloshing in tank 1 as follows:

$$\left. \begin{aligned} \ddot{c}_{11} + 2\zeta_{11} \dot{c}_{11} + (1 + \psi_{11} \ddot{z}_0) c_{11} + H_2(c_{11}, d_{11}) &= 0, \\ \ddot{d}_{11} + 2\zeta_{11} \dot{d}_{11} + (1 + \psi_{11} \ddot{z}_0) d_{11} + H_3(c_{11}, d_{11}) &= 0, \\ \ddot{c}_{01} + 2\zeta_{01} \dot{c}_{01} + \omega_{01}^2 (1 + \psi_{11} \ddot{z}_0) c_{01} + H_4(z_0, c_{11}, d_{11}) &= 0, \\ \ddot{c}_{21} + 2\zeta_{21} \dot{c}_{21} + \omega_{21}^2 (1 + \psi_{11} \ddot{z}_0) c_{21} + H_5(z_0, c_{11}, d_{11}) &= 0, \\ \ddot{d}_{21} + 2\zeta_{21} \dot{d}_{21} + \omega_{21}^2 (1 + \psi_{11} \ddot{z}_0) d_{21} + H_6(z_0, c_{11}, d_{11}) &= 0, \end{aligned} \right\} \quad (13b)$$

and the modal equations for sloshing in tank 2 as follows:

$$\left. \begin{aligned} \ddot{\bar{c}}_{11} + 2\bar{\zeta}_{11}\bar{\omega}_{11}\dot{\bar{c}}_{11} + \bar{\omega}_{11}^2(1 + \bar{\psi}_{11}\ddot{z}_0)\bar{c}_{11} + H_7(\bar{c}_{11}, \bar{d}_{j1}) &= 0, \\ \ddot{\bar{d}}_{11} + 2\bar{\zeta}_{11}\bar{\omega}_{11}\dot{\bar{d}}_{11} + \bar{\omega}_{11}^2(1 + \bar{\psi}_{11}\ddot{z}_0)\bar{d}_{11} + H_8(\bar{c}_{11}, \bar{d}_{j1}) &= 0, \\ \ddot{\bar{c}}_{01} + 2\bar{\zeta}_{01}\bar{\omega}_{01}\dot{\bar{c}}_{01} + \bar{\omega}_{01}^2(1 + \bar{\psi}_{11}\ddot{z}_0)\bar{c}_{01} + H_9(z_0, \bar{c}_{11}, \bar{d}_{11}) &= 0, \\ \ddot{\bar{c}}_{21} + 2\bar{\zeta}_{21}\bar{\omega}_{21}\dot{\bar{c}}_{21} + \bar{\omega}_{21}^2(1 + \bar{\psi}_{11}\ddot{z}_0)\bar{c}_{21} + H_{10}(z_0, \bar{c}_{11}, \bar{d}_{11}) &= 0, \\ \ddot{\bar{d}}_{21} + 2\bar{\zeta}_{21}\bar{\omega}_{21}\dot{\bar{d}}_{21} + \bar{\omega}_{21}^2(1 + \bar{\psi}_{11}\ddot{z}_0)\bar{d}_{21} + H_{11}(z_0, \bar{c}_{11}, \bar{d}_{11}) &= 0. \end{aligned} \right\} \quad (13c)$$

where $\omega_{mn}^2 = \psi_{mn}/\psi_{11}$, $\psi_{mn} = \lambda_{mn} \tanh(\lambda_{mn}h_1)$, $\bar{\omega}_{mn}^2 = \bar{\psi}_{mn}/\psi_{11}$, and $\bar{\psi}_{mn} = \lambda_{mn} \tanh(\lambda_{mn}h_2)$. H_j ($j=1,2,\dots,11$) represent the nonlinear terms which consist of the variables shown in the parentheses as given in the Appendix. The amplitudes a_{mn} , $\bar{a}_{mn}$, b_{mn} and $\bar{b}_{mn}$ have been eliminated in Eqs. (13a)-(13c), because the focus is on the liquid surface elevations instead of ϕ_i . In order to consider the damping of the sloshing, the viscous damping terms, such as $2\zeta_{mn}\omega_{mn}\dot{c}_{mn}$, have been factored into Eqs. (13b) and (13c).

2.3. Frequency Response Curves

According to the simulation results from Eqs. (13a)-(13c), we use van del Pol's method⁽¹⁸⁾ to obtain the expressions for the frequency response curves of the structure and sloshing. Here, the positions of the nodal lines of the (1, 1) sloshing mode can be assumed to coincide with the y_i axes. This assumption never loses the universality of the theoretical analysis. Therefore, $d_{11} = 0$ is satisfied. Then, $d_{21} = 0$ can be also assumed, because d_{21} is included in the form of the product with d_{11} in Eq. (13) (see the Appendix). Therefore, the solutions of the forced vibrations in Eq. (13) with $d_{mn} = 0$ and $\bar{d}_{mn} = 0$ can be assumed as follows:

$$\left. \begin{aligned} z_0 &= u_1 \cos \omega t - v_1 \sin \omega t, \\ c_{11} &= u_2 \cos \frac{1}{2} \omega t - v_2 \sin \frac{1}{2} \omega t + e_1 \cos \frac{3}{2} \omega t - f_1 \sin \frac{3}{2} \omega t, \\ c_{01} &= e_2 \cos \omega t - f_2 \sin \omega t + r_0, \\ c_{21} &= e_3 \cos \omega t - f_3 \sin \omega t + r_2, \\ \bar{c}_{11} &= \bar{u}_2 \cos \frac{1}{2} \omega t - \bar{v}_2 \sin \frac{1}{2} \omega t + \bar{e}_1 \cos \frac{3}{2} \omega t - \bar{f}_1 \sin \frac{3}{2} \omega t, \\ \bar{c}_{01} &= \bar{e}_2 \cos \omega t - \bar{f}_2 \sin \omega t + \bar{r}_0, \\ \bar{c}_{21} &= \bar{e}_3 \cos \omega t - \bar{f}_3 \sin \omega t + \bar{r}_2, \end{aligned} \right\} \quad (14)$$

where the amplitudes, u_i , v_i , $\bar{u}_2$, $\bar{v}_2$, e_j , f_j , $\bar{e}_j$ and $\bar{f}_j$ ($i=1,2$; $j=1,2,3$), and the constant terms, r_0 , r_2 , $\bar{r}_0$ and $\bar{r}_2$ are assumed to vary slowly over time and hence their orders of magnitudes are assumed, satisfying with Eq. (12), as follows:

$$\left. \begin{aligned} u_i, v_i, \bar{u}_2, \bar{v}_2 &= O(\varepsilon^{1/3}), \quad e_j, f_j, r_0, r_2, \bar{e}_j, \bar{f}_j, \bar{r}_2 = O(\varepsilon^{2/3}), \\ \dot{u}_i, \dot{v}_i, \dot{\bar{u}}_2, \dot{\bar{v}}_2 &= O(\varepsilon^{3/3}), \quad \dot{e}_j, \dot{f}_j, \dot{r}_0, \dot{r}_2, \dot{\bar{e}}_j, \dot{\bar{f}}_j, \dot{\bar{r}}_2 = O(\varepsilon^{4/3}), \\ \ddot{u}_i, \ddot{v}_i, \ddot{\bar{u}}_2, \ddot{\bar{v}}_2 &= O(\varepsilon^{5/3}), \quad \ddot{e}_j, \ddot{f}_j, \ddot{r}_0, \ddot{r}_2, \ddot{\bar{e}}_j, \ddot{\bar{f}}_j, \ddot{\bar{r}}_2 = O(\varepsilon^2). \end{aligned} \right\} \quad (15)$$

Substituting Eq. (14) into Eq. (13), and taking harmonic balance in terms of the frequencies ω , 2ω and 3ω and the constant term under the assumption (15), we obtain the equations within the accuracy of $O(\varepsilon)$ as follows:

$$\left. \begin{aligned} \dot{u}_1 &= G_1(u_i, v_i, \bar{u}_2, \bar{v}_2, e_1, f_1, \bar{e}_1, \bar{f}_1), & \dot{v}_1 &= G_2(u_i, v_i, \bar{u}_2, \bar{v}_2, e_1, f_1, \bar{e}_1, \bar{f}_1), \\ \dot{u}_2 &= G_3(u_i, v_i, e_j, f_j, r_0, r_2), & \dot{v}_2 &= G_4(u_i, v_i, e_j, f_j, r_0, r_2), \\ \dot{\bar{u}}_2 &= G_5(u_i, v_i, \bar{u}_2, \bar{v}_2, \bar{e}_j, \bar{f}_j, \bar{r}_0, \bar{r}_2), & \dot{\bar{v}}_2 &= G_6(u_i, v_i, \bar{u}_2, \bar{v}_2, \bar{e}_j, \bar{f}_j, \bar{r}_0, \bar{r}_2), \\ G_m(u_i, v_i, e_j, f_j, r_0, r_2) &= 0, & G_n(u_i, v_i, \bar{u}_2, \bar{v}_2, \bar{e}_j, \bar{f}_j, \bar{r}_0, \bar{r}_2) &= 0, \\ (i=1,2; \quad j=1,2,3; \quad m=7,8,\dots,14; \quad n=15,16,\dots,22) \end{aligned} \right\} \quad (16)$$

where G_l ($l=1,2,\dots,22$) are linear functions in terms of e_j , f_j , $\bar{e}_j$, $\bar{f}_j$, r_0 , r_2 , $\bar{r}_0$ and $\bar{r}_2$ but linear and nonlinear functions in terms of u_i , v_i , $\bar{u}_i$ and $\bar{v}_i$. The steady state solutions of Eq. (16) can be calculated numerically, and their stability can be determined by introducing small deviations from the steady state solutions⁽¹⁴⁾.

In order to thoroughly examine the effects of the liquid motion on the structure's response, it is necessary to consider the response curve for the structure when the liquid surface is assumed to be undisturbed. We can give the amplitude of the frequency response for the structure with a solid mass as same as the liquid mass as follows:

$$A = F_0 / \sqrt{(k - Q_1 \omega^2)^2 + (c\omega)^2} \quad (17)$$

3. Results of Numerical Calculation

In the numerical calculations, the values of parameters are chosen in such a way that the natural frequency $p_0 (= \sqrt{k/Q_1})$ of the structure is approximately equal to twice the natural frequencies p_{11} and $\bar{p}_{11}$ of (1,1) modes of sloshing in tanks 1 and 2. Therefore the tuning condition $p_0 \cong 2p_{11} \cong 2\bar{p}_{11}$ is satisfied resulting in harmonic oscillations and 1/2-order subharmonic oscillations occurring in the structure and on the liquid surfaces of the two tanks, respectively.

Figure 3 shows the frequency response curves of a single-tank system, as demonstrated in the previous paper ⁽¹²⁾, in order to compare them to those of a dual-tank system. The values of the parameters are $\nu=0.87$, $c=0.013$, $k=4.0$, $h_1=1.2$, $\nu_1=0.0345$, $\zeta_n=0.008$, $h_2=0$, $\nu_2=0$, and $F_0=0.0021$. In this case, the internal resonance ratio $p_0 = 2p_{11}$ is exactly satisfied. Figures 3(a) and 3(b) show the response curves for the component of the frequency ω in z_0 (with an amplitude $\sqrt{u_1^2 + v_1^2}$) and for the component of the frequency $(1/2)\omega$ in c_{11} (with an amplitude $\sqrt{u_2^2 + v_2^2}$), respectively. The solid and broken lines represent the stable and unstable steady state solutions, respectively. In order to confirm the accuracy of the response curves, we conducted an FFT analysis of the time histories which were obtained by numerically integrating Eqs. (13a)-(13c). The results of the FFT analysis are plotted in the symbols '•'. The branch $A_0B_0C_0D_0$ in Fig. 3(a) corresponds to the amplitude given by Eq. (17). However, because the zero-amplitude branch B_1C_1 in Fig. 3(b) becomes unstable due to sloshing, the branch B_0C_0 in Fig. 3(a) also becomes unstable. Although the branch B_0C_0 is unstable, diverging vibrations never occur because of the nonlinearity of the fluid force. Instead, steady-state oscillations appear as shown by the branch E_1F_1 (drawn in blue). In other words, when the relationship $p_0 \cong 2p_{11}$ is satisfied, the vibration of the structure influences the liquid sloshing resulting in a parametric excitation. Hence the sloshing occurs at finite amplitudes, because the equations of motion for liquid elevations are nonlinear. Once sloshing occurs in the tank, it is coupled with the motion of the structure. Comparing the response curves of the structure with one tank installed to that of the hypothetical case where the liquid acts as a solid mass, we can see the significant difference in amplitude and stability.

Figure 4 shows the response curves of a dual-tank system for $h_1=h_2=1.2$, $\nu_1=\nu_2=0.0172$ and $\bar{\zeta}_n=0.008$. As observed in a single-tank system (Fig. 3), when a 2:1:1 ratio of internal resonance is satisfied ($p_0 = 2p_{11} = 2\bar{p}_{11}$), harmonic oscillations and 1/2-order subharmonic oscillations occur in the structure and on the liquid surfaces of the two tanks, respectively. The sum of the masses of the liquids in tanks 1 and 2 of Fig. 4 is equal to the mass of the liquid in the tank of Fig. 3. The liquid levels are the same as in that of Fig. 3, and the values of ν_1 and ν_2 are given half the value of ν_1 of Fig. 3. The values of the other

Fig. 3 Frequency response curves of a single-tank system for $k=4.0$, $h_1=1.2$ and $\nu_1=0.0345$.

Fig. 4 Frequency response curves of a dual-tank system for $k=4.0$, $h_1=h_2=1.2$ and $\nu_1=\nu_2=0.0172$.

Fig. 5 Frequency response curves of a dual-tank system for $k=4.0$, $h_1=1.2$, $h_2=1.1$ and $\nu_1=\nu_2=0.0180$.

parameters are the same as those of Fig. 3. Figures 4(a), 4(b) and 4(c) show the frequency response curves for the structure and sloshing in the two tanks, respectively. The shape of the curves of Figs. 4(b) and 4(c) is perfectly alike, because the dimensions and the liquid levels of both tanks are identical. However, it is interesting that the responses of sloshing in the two tanks are different. As the excitation frequency decreases from $\omega=2.1$, sloshing starts to occur near $\omega=2.04$ in both of the two tanks, as shown by the branches 'c₁' and 'c₂' (drawn in blue in Figs. 4(b) and 4(c)). Corresponding to these branches, the branch 'c₀' (drawn in blue in Fig. 4(a)) appears in the structure. We call this type of response a 'double-mode' solution. The curves corresponding to these branches 'c₁' and 'c₂' appeared at same amplitude. When the excitation frequency decreases further, the system undergoes the bifurcations at points P_j ($j=0,1,2$) and the branches E_jP_j become unstable. Two possible responses can occur at points P_1 and P_2 in tanks 1 and 2, respectively. Either of these responses can occur depending on the initial conditions. Considering tank 1 at point P_1 , the amplitude of sloshing may jump to either branch 'a₁' (drawn in red) or 'b₁' (drawn in green) as shown by the arrows $\uparrow$ and $\downarrow$, respectively. If the amplitude of sloshing in tank 1 jumps up to 'a₁', the amplitude of sloshing in tank 2 jumps down to branch 'a₂' of zero amplitude. Therefore sloshing occurs only in tank 1 and is termed a 'single-mode' solution, observed on the left hand side of Fig. 4(b). The single-mode solution can also occur in tank 2 (shown by the branch 'b₂' drawn in green). However, the amplitude of the branch Q_0R_0 in Fig. 4(a) is identical whichever single-mode solution of the liquid slosh occurs. It is also found that the structure vibrates at little bit smaller amplitude on the branch Q_0R_0 than that on the branch E_0P_0 which corresponds to the stable branch of the structure with a single-tank system. Vyas and Bajaj⁽¹⁹⁾ have reported the occurrence of single-mode and double-mode solutions in a SDOF structure with multiple pendulums installed.

Figure 5 shows the response curves when h_2 changes from 1.2 in Fig. 4 to 1.1. In this case, $\nu_1=\nu_2=0.0180$ because Q_1 must be equal to 1.0. The branches 'a₁' and 'b₂' corresponding to a single-mode solution can appear in either tank. However, the branches

' c_1 ' and ' c_2 ' corresponding to a double-mode solution can appear simultaneously in both tanks. Because the liquid level h_2 in Fig. 5 is lower than that of Fig. 4, a small deviation of the internal resonance ratio ($p_0 \cong 2\bar{p}_{11}$) exists, and this in turn causes both ' c_1 ' and ' c_2 ' to split into two solid lines. Therefore, compared to Fig. 4, the amplitudes of the single-mode and double-mode solutions (' a_1 ' and ' c_1 ') are larger in tank 1, while smaller in tank 2 (' b_2 ' and ' c_2 ').

Next, Fig. 6 shows the response curves when only the value of k changes from 4.0 in Fig. 5 to 3.78. Using the detuning parameters $\sigma_1 = p_0 - 2p_{11}$ and $\sigma_2 = p_0 - 2\bar{p}_{11}$ as measures, we can observe the deviation of the internal resonance ratio, $\sigma_1 = -0.056$ and $\sigma_2 = -0.044$ in Fig. 6. The branches ' c_1 ' and ' c_2 ' corresponding to the double-mode solution appear separately in two parts. In addition, bifurcations occur at points G_i and H_i ($i=1, 2$) on the branches ' a_1 ' and ' b_2 ' which correspond to the single-mode solutions. Unstable steady-state solutions appear at the intervals G_1H_1 and G_2H_2 . It should be noted that the interval G_1H_1 is wider than G_2H_2 .

Figure 7 shows the stationary time histories which are calculated within the interval G_1H_1 in Fig. 6 by using Eq. (13). The corresponding time histories of the liquid surface in tank 2 are not shown because their amplitudes are all zeros. Amplitude and phase modulated motions appear in the structure and liquid free surface of tank 1. The amplitudes change once during each cycle of the envelopes at $\omega=1.903$ as shown in Fig. 7(a). They change twice at $\omega=1.913$ as shown in Fig. 7(b). In Fig. 7(c), the amplitudes of the envelopes change irregularly. This type of motion seems to be a chaotic vibration. From these figures, it is now clear that points G_i and H_i ($i=1, 2$) are Hopf bifurcation points.

Fig. 6 Frequency response curves of a dual-tank system for $k=3.78$, $h_1=1.2$, $h_2=1.1$ and $\nu_1=\nu_2=0.0180$.

Fig. 7 Time histories in Fig. 6.

Fig. 8 Lyapunov exponents (a) near the interval G_1H_1 in Fig. 6(b); (b) near the interval G_2H_2 in Fig. 6(c).

The Lyapunov exponents are calculated by using the method of Wolf et al. ⁽²⁰⁾ to prove the occurrence of chaotic vibrations when the single-mode solutions appear in Eq. (13). Figures 8(a) and 8(b) show the largest three Lyapunov exponents near the intervals G_1H_1 and G_2H_2 in Figs. 6(b) and 6(c), respectively. The interval 'A' in Fig. 8(a) represents the excitation frequency range where the amplitudes jump down from modulated amplitude solutions to the zero-amplitude solutions. The positions of the excitation frequencies marked by the arrows at points 'a'-'c' in Fig. 8(a), correspond to the time histories of Figs. 7(a)-(c), respectively. It can be verified from Fig. 8(a) that the time histories of Fig. 7(c) exhibit chaotic vibrations, because the largest Lyapunov exponent takes a positive value at point 'c'.

In order to investigate the influence of the liquid level on the response curve, the values $h_1=h_2=0.4$ are used, which are smaller than those of Fig. 4. Figure 9 shows an example of the influence of the liquid level on the response curves for $\nu=0.96$ and $\nu_1=\nu_2=0.0159$. The results show that the single-mode solutions 'a₁' and 'b₂' appear on the right hand side of the response curves of Fig. 9 as opposed to the left hand side as in Fig. 4.

4. Experimental Results

Figure 10 shows a sketch of the experimental apparatus. The configuration of this apparatus was similar to the one used in the previous paper ⁽¹²⁾. A T-shaped structure was supported by a spring (see the front view). Using two 500 mm length arms with ball bearings in their joints enabled the structure to move approximately only in the vertical direction. The structure was excited sinusoidally and stepwise at each excitation frequency

Fig. 9 Frequency response curves of a dual-tank system for $\nu=0.96$, $k=4.0$, $h_1=h_2=0.4$ and $\nu_1=\nu_2=0.0159$.

Fig. 10 Experimental apparatus.

Table 1 Dimensions of apparatus A, B, C and D.

Apparatus	A	B	C	D
Structure mass: m kg	7.294	8.087	8.179	8.840
Spring constant: k N/m	9560	13150	13280	13210
Damping coefficient: c Ns/m	4.05	5.87	5.73	7.11
Tank radius : R mm	60	47.5	47.5	47.5
Water depth of Tank 1: h_1 mm	72	57	57	57
Water depth of Tank 2: h_2 mm	---	57	52	52
Water mass of Tank 1: m_1 kg	0.814	0.404	0.404	0.404
Water mass of Tank 2: m_2 kg	---	0.404	0.369	0.369
Damping ratios: $\zeta_n, \bar{\zeta}_n$	0.075	0.012	0.012	0.012
Force amplitude: F_0 N	0.402	0.659	0.672	0.725
Natural frequency: p_0 Hz	5.465	6.120	6.130	5.900
Natural frequency: p_{11} Hz	2.727	3.065	3.065	3.065
Natural frequency: $\bar{p}_{11}$ Hz	---	3.065	3.047	3.047
Detuning: $\sigma_1 = p_0 - 2p_{11}$ Hz	0.011	-0.010	0	-0.230
Detuning: $\sigma_2 = p_0 - 2\bar{p}_{11}$ Hz	---	-0.010	0.036	-0.194
Positions of S_i ($i=1, \dots, 4$): r_i mm	50	40	40	40

by an electromagnetic exciter through a thin aluminum plate. The head of the exciter was controlled by a vibration exciter controller in order to keep the amplitude of the head constant regardless of the value of the excitation frequency. Two cylindrical tanks were installed on the structure. A water solution consisting of 0.28% concentration of white paint was used as a test liquid so that the light of the laser sensor could be reflected on the liquid surface. The laser sensor S_0 (drawn in red) measured the displacement z_0 of the structure. In the experiments, the direction of the diametric nodal line of the (1, 1) sloshing mode did not necessarily coincided with the y_i -axis or the x_i -axis. Because of this, the elevations η_i of the water surfaces were calculated by the square root of the sum of amplitudes, which were measured by the laser sensors S_1 and S_2 for tank 1 and by the laser sensors S_3 and S_4 for tank 2 set just over the x_i - and y_i -axes, respectively (see the top view).

The dimensions of the experimental apparatuses A to D are given in Table 1. In each apparatus except for apparatus D, the relationship, $p_0 = 2p_{11} = 2\bar{p}_{11}$, was approximately satisfied. The value of the damping coefficient c of the structure was determined by a curve-fitting method using the experimental data and the theoretical response curve given by Eq. (17). The values of the damping ratios ζ_{mn} of the water sloshing were determined by a curve-fitting method using the experimental data and the theoretical response curves, given by Eq. (16), for the nonlinear-coupled system.

Figure 11 shows the frequency response curves for apparatus A, when only one tank (a single-tank system) was installed on the structure, in order to compare them to those of a dual-tank system. The liquid level of the tank was $h_1=72$ mm and the tank radius was $R=60$ mm ($h_1/R=1.2$). Figures 11(a) and 11(b)

Fig. 11 Comparison between the theoretical and experimental response curves for apparatus A.

show a comparison between the theoretical and experimental response curves of the structure and the water surface, respectively. The symbols $\circ$ and $\bullet$ represent the amplitudes of the time histories measured when the structure had a tank partially filled with liquid. The symbol $\bullet$ represents the amplitude of the displacement of the structure when the water was replaced by a solid mass. Vertical arrows shown in the figures represent the occurrences of the jump phenomena of the amplitudes. Theoretical response curves are also drawn using solid lines for the stable solutions and broken lines for the unstable solutions. It can be seen that the theoretical response curves are in quantitative agreement with the experimental data. This result demonstrates that one must consider the nonlinearity of the hydrodynamic force due to the sloshing in the theoretical analysis. The same results observed in Fig. 11 were also obtained in the previous paper⁽¹²⁾.

Figure 12 shows the comparison between the theoretical and experimental response curves for apparatus B, using the dual-tank system. The liquid levels and radii of the two tanks were identical, i.e., $h_1=h_2=57$ mm and $R=47.5$ mm ($h_1/R=h_2/R=1.2$). Figures 12(b) and 12(c)

Fig. 12 Comparison between the theoretical and experimental response curves for apparatus B.

Fig. 13 Photos showing the movement of the liquid surfaces at $f=6.020$ Hz in Fig. 12.

represent the response curves of sloshing in tanks 1 and 2, respectively. As predicted in the theoretical results in Fig. 4, the single-mode solutions plotted by either ● for 'a₁' or ● for 'b₂', appeared exclusively in tanks 1 and 2, respectively. In the experiments, the excitation frequency range where a single-mode solution appeared was wider than that the theoretical results. As the excitation frequency increased, the amplitudes jumped to the double-mode solutions plotted by the symbol ●. Compared to Fig. 11, the effectiveness of the dual-tank system to suppress vibrations decreased in some degrees, because the single-mode solutions appeared in a wider excitation frequency range. The experimental data also deviated from the theoretical curves at low slosh amplitudes on the right sides of Figs. 12(b) and 12(c). This kind of phenomenon, termed a 'tailing' effect, was also observed by Virnig et al. ⁽²¹⁾. They demonstrated that the deviation of the experimental data from the theoretical curve was due to the surface tension which is also a possible factor in Figs. 12(b) and 12(c). Another cause of the deviation may be due to the surface state of the side wall of the tank. The experimental data obtained by Case and Parkinson ⁽²²⁾, when using both polished and unpolished cylinders, demonstrated that the slosh damping was significantly influenced by the surface state. Furthermore contamination of the side wall with wax or oil is possible and may significantly influence the damping of liquid sloshing. Therefore, a relatively minor change in the surface state sometimes causes a significant change in the degree of the slosh damping ⁽²²⁾.

Figure 13 shows the photos of the movements of the liquid surfaces, which were taken at $f=6.020$ Hz in Fig. 12. Figures 13(a) and 13(b) instantiate a single-mode solution which occurred in either tank 1 or tank 2. Figure 13(c) instantiates a double-mode solution where the sloshing occurred in both tanks.

Figure 14 shows the response curves for apparatus C. The liquid level h_2 of tank 2 changed from 57 mm (Fig. 12) to 52 mm ($h_2/R=1.095$). Figure 14 corresponds to the

Fig. 14 Comparison between the theoretical and experimental response curves for apparatus C.

Fig. 15 Comparison between the theoretical and experimental response curves for apparatus D.

Fig. 16 Experimental time histories in Fig. 15.

theoretical results shown in Fig. 5. It is similar to Fig. 12, but the branches corresponding to the double-mode solutions, 'c₁' and 'c₂', split into two solid lines on the left hand sides.

Figure 15 shows the response curves for apparatus D, which shares the same values as apparatus C, except the mass of the structure was increased, and hence the natural frequency p_0 became lower. As shown in Table 1, the detuning parameters were $\sigma_1 = -0.230$ Hz ($\sigma_1/p_{11} = -0.075$) and $\sigma_2 = -0.194$ Hz ($\sigma_2/p_{11} = -0.063$). These values yielded Hopf bifurcations on the branches corresponding to the single-mode solutions, and amplitude modulated motion occurred as plotted by the symbols ● and ● (designated by AMM).

Figures 16(a) and 16(b) show the time histories observed at $f = 5.715$ Hz and 5.745 Hz respectively, on the branch 'a₁' in Fig. 15. η_{S1} and η_{S2} represent the elevations of sloshing at the positions of sensors S₁ and S₂. The amplitudes of the envelopes changed once each cycle regularly in Fig. 16(a). However, they changed irregularly in Fig. 16(b) and they exhibited a chaotic vibration.

5. Conclusions

Nonlinear parametric vibrations of an elastic structure containing two cylindrical liquid-filled tanks under vertical harmonic excitation have been investigated by showing the frequency response curves, the simulation results in the nonlinear theoretical analysis and the experimental data. The results can be summarized as follows:

- (1) Harmonic oscillations and 1/2-order subharmonic oscillations may occur in the structure and on the liquid surfaces in the two tanks, respectively, when a 2:1:1 ratio of internal resonance is satisfied.
- (2) A dual-tank system causes a part of the stable branch in a single-tank system to become unstable and to generate new stable branches. Their branches have the effectiveness in decreasing the amplitude of the structure.
- (3) A single-mode solution in either tank and a double-mode solution in both tanks occur depending on the excitation frequencies and liquid levels.
- (4) A single-mode solution appears on the left hand side of the response curves for high liquid levels, while it appears on the right hand side for low liquid levels.
- (5) The deviation of the internal resonance ratio yields Hopf bifurcation points on the stable branches corresponding to the single-mode solutions. Amplitude and phase modulated motions including chaotic motions appear within the interval of Hopf bifurcation points.
- (6) Experimental data confirm the validity of the theoretical analysis.

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Appendix

The nonlinear terms H_i ($i=1,2,\dots,11$) in Eq. (13) are given as follows:

$$\begin{aligned}
 H_1 &= S_1(1+\psi_{11}\ddot{z}_0)(c_{11}^2+d_{11}^2)+S_2(\dot{c}_{11}^2+\dot{d}_{11}^2)+\bar{S}_1(1+\psi_{11}\ddot{z}_0)(\bar{c}_{11}^2+\bar{d}_{11}^2)+\bar{S}_2(\dot{\bar{c}}_{11}^2+\dot{\bar{d}}_{11}^2), \\
 H_2 &= S_3\dot{c}_{01}\dot{c}_{11}+S_4(\dot{c}_{11}\dot{c}_{21}+\dot{d}_{11}\dot{d}_{21})+S_5c_{11}\dot{c}_{11}^2+S_6c_{11}\dot{d}_{11}^2+S_7\dot{c}_{11}\dot{d}_{11}d_{11} \\
 &\quad +S_8c_{01}c_{11}+S_9c_{11}c_{21}+S_{10}d_{11}d_{21}+S_{11}c_{11}^3+S_{12}c_{11}d_{11}^2,
 \end{aligned}$$

$$\begin{aligned}
 H_3 &= S_{13}\dot{c}_{01}\dot{d}_{11} + S_{14}(\dot{c}_{11}\dot{d}_{21} - \dot{c}_{21}\dot{d}_{11}) + S_{15}\dot{c}_{11}^2 d_{11} + S_{16}\dot{d}_{11}^2 d_{11} + S_{17}c_{11}\dot{c}_{11}\dot{d}_{11} \\
 &\quad + S_{18}c_{01}d_{11} + S_{19}c_{11}d_{21} + S_{20}c_{21}d_{11} + S_{21}d_{11}^3 + S_{22}c_{11}^2 d_{11}, \\
 H_4 &= S_{23}(1 + \psi_{11}\ddot{z}_0)(c_{11}^2 + d_{11}^2) + S_{24}(\dot{c}_{11}^2 + \dot{d}_{11}^2), \\
 H_5 &= S_{25}(1 + \psi_{11}\ddot{z}_0)(c_{11}^2 - d_{11}^2) + S_{26}(\dot{c}_{11}^2 - \dot{d}_{11}^2), \\
 H_6 &= S_{27}(1 + \psi_{11}\ddot{z}_0)c_{11}d_{11} + S_{28}\dot{c}_{11}\dot{d}_{11}, \\
 H_7 &= \bar{S}_3\dot{c}_{01}\dot{c}_{11} + \bar{S}_4(\dot{c}_{11}\dot{c}_{21} + \dot{d}_{11}\dot{d}_{21}) + \bar{S}_5\bar{c}_{11}\dot{c}_{11}^2 + \bar{S}_6\bar{c}_{11}\dot{d}_{11}^2 + \bar{S}_7\dot{c}_{11}\dot{d}_{11}\bar{d}_{11} \\
 &\quad + \bar{S}_8\bar{c}_{01}\bar{c}_{11} + \bar{S}_9\bar{c}_{11}\bar{c}_{21} + \bar{S}_{10}\bar{d}_{11}\bar{d}_{21} + \bar{S}_{11}\bar{c}_{11}^3 + \bar{S}_{12}\bar{c}_{11}\bar{d}_{11}^2, \\
 H_8 &= \bar{S}_{13}\dot{c}_{01}\dot{d}_{11} + \bar{S}_{14}(\dot{c}_{11}\dot{d}_{21} - \dot{c}_{21}\dot{d}_{11}) + \bar{S}_{15}\dot{c}_{11}^2 \bar{d}_{11} + \bar{S}_{16}\dot{d}_{11}^2 \bar{d}_{11} + \bar{S}_{17}\bar{c}_{11}\dot{c}_{11}\dot{d}_{11} \\
 &\quad + \bar{S}_{18}\bar{c}_{01}\bar{d}_{11} + \bar{S}_{19}\bar{c}_{11}\bar{d}_{21} + \bar{S}_{20}\bar{c}_{21}\bar{d}_{11} + \bar{S}_{21}\bar{d}_{11}^3 + \bar{S}_{22}\bar{c}_{11}^2 \bar{d}_{11}, \\
 H_9 &= \bar{S}_{23}(1 + \bar{\psi}_{11}\ddot{z}_0)(\bar{c}_{11}^2 + \bar{d}_{11}^2) + \bar{S}_{24}(\dot{\bar{c}}_{11}^2 + \dot{\bar{d}}_{11}^2), \\
 H_{10} &= \bar{S}_{25}(1 + \bar{\psi}_{11}\ddot{z}_0)(\bar{c}_{11}^2 - \bar{d}_{11}^2) + \bar{S}_{26}(\dot{\bar{c}}_{11}^2 - \dot{\bar{d}}_{11}^2), \\
 H_{11} &= \bar{S}_{27}(1 + \bar{\psi}_{11}\ddot{z}_0)\bar{c}_{11}\bar{d}_{11} + \bar{S}_{28}\dot{\bar{c}}_{11}\dot{\bar{d}}_{11}, \tag{18}
 \end{aligned}$$

where S_n and $\bar{S}_n$ ($n=1, 2, \dots, 28$) are constants to be determined by using the values of the system parameters.

Nomenclature

c	damping coefficient of the structure
F_1, F_2	vertical fluid forces from tanks 1 and 2, respectively
F_0	amplitude of excitation
g	acceleration of gravity
h_1, h_2	liquid levels in tanks 1 and 2, respectively
k	spring constant of the structure
M	summation of the masses of the structure and the liquids ($=m+m_1+m_2$)
m	mass of the structure
m_1, m_2	masses of the liquids in tanks 1 and 2, respectively
P_1, P_2	fluid pressures in tanks 1 and 2, respectively
p_0	natural frequency of the structure (or the main system)
$p_{mn}, \bar{p}_{mn}$	natural frequencies of the sloshing mode (m, n) in tanks 1 and 2 (m, n : integers), respectively
R	tank radius
(r_i, θ_i, z_i)	circular cylindrical coordinate systems (see Fig. 1)
t	time
(x_i, y_i, z_i)	rectangular coordinate systems (see Fig. 1)
z_0	vertical displacement of the structure
η_1, η_2	elevations of the liquid surfaces in tanks 1 and 2, respectively
$\nu = m/M$	
$\nu_i = m_i R / (\pi M h_i)$	
ρ	fluid density
ϕ_1, ϕ_2	velocity potentials of fluids in tanks 1 and 2, respectively
ω	excitation frequency
$\zeta_{mn}, \bar{\zeta}_{mn}$	damping ratios of the sloshing mode (m, n) in tanks 1 and 2, respectively