Analysis of Steady Collision Vibration in Cantilever Beam Having an Attached Mass (The Case Where Structural Damping and Viscous Damping is Considered in System)*

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Abstract

This paper deals with response analysis of collision vibration in continuous system excited by periodic displacement with arbitrary functions. A system of steady vibration in a cantilever beam having an attached mass at free-end is considered. The cantilever beam has structural damping in the beam, and it is being put in the viscous fluid with hydrodynamic drag. The attached mass collides elastically with the coil spring clamped on asymmetrical faces when the amplitude of mass exceeds the clearance between the coil spring and the attached mass. Then, the restoring force is assumed to be an asymmetric piecewise-linear system. For such a system, the beam undergoes a nonlinear vibration when the attached mass collides with the coil spring. In order to analyze harmonic, superharmonic and subharmonic resonances for the system, Fourier series method is applied to obtain an exact solution for resulting vibration. Next, the numerical calculation is performed to obtain the resonance curves. The numerical results show effects of the mass ratio, the amplitude ratio of excitation, the spring constant ratio, the structural damping ratio and the external damping ratio on the resonance curves. The numerical experiments are also carried out to verify the numerical results.

Key words: Theory of Vibration, Method of Vibration Analysis, Vibration of Continuous System, Cantilever Beam, Collision Vibration, Nonlinear Vibration, Resonance Curve, Viscous Damping, Structural Damping

1. Introduction

Since piping systems in chemical plants are frequently subjected to high temperatures some leeway is left in the connections to flanges and other components attached to piping to allow for thermal expansion. This creates a gap between the supports and the supported elements, which is well known to generate collision vibrations⁽¹⁾. Collision vibrations are highly nonlinear and are not generally predictable using standard analysis methods. A new model must be constructed for each system in order to analyze its vibrations. Watanabe⁽²⁾ and Aoki *et al.*⁽³⁾⁽⁴⁾ have conducted analyses of vibrations arising from single harmonic excitation of continuous systems subjected to collision vibration by Fourier series method. We have also examined collision vibration of continuous systems subjected to periodic excitation with an arbitrary function⁽⁵⁾. All of these analyses were limited to main resonance

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regions; they neglected external and structural damping and they also assumed that the waveform of the resulting vibration is symmetric. However, components of structures such as chemical systems are not usually subject to single harmonic excitation. Rather, they are subject to a wide variety of simultaneous excitations, resulting in asymmetrical vibration waveforms. In spite of this, no reports have been published that analyze asymmetrical resulting vibration waveforms. In recent year, composite materials have been extensively used as filling materials in piping systems and other systems. In addition to damping of vibrations by these laminated structures, there is also damping due to bolts, adhesives, welds and other connecting structures and components, and structural damping due to the dispersion of energy from the piping systems to exterior components. Some systems also have viscous damping and other types of damping, produced by their environments, fluid resistance, and other factors. There do not seem to be any published analyses of these nonlinear collision vibration systems with structural and viscous damping. A model of a continuous system was created for this study. It consisted of a cantilever beam with an attached mass and structural and external damping, as described by the Euler-Bernoulli theory. The mass was attached to the tip of the beam, which vibrated in lateral bending mode. The mass collided elastically with a spring which was attached to a wall displacing through clearance of e_0 . The steady collision vibration of this system was analyzed under a periodic excitation with an arbitrary function by a displacement oscillation. The restoring force accompanying the compression of the spring by the mass was approximated by an asymmetric piecewise linear force and expanded in a Fourier series, as described in previous paper⁽²⁾⁻⁽⁵⁾. An analytical procedure was proposed that did not require imposing prior limitations on the resulting vibration waveform and mode, and an exact solution was derived. In addition, a numerical calculation was also conducted using the analysis results to obtain a periodic solution for three ranges of the resulting vibrations: the main resonance region (i.e., harmonic resonance); the superharmonic vibration region; and the subharmonic vibration region. Resonance curves were created using several parameters: the ratio of the tip mass to the beam mass; the ratio of the spring constant of the spring to the spring constant of the beam; the ratio of the excitation amplitude; the structural damping ratio; and the external damping ratio. The effect of each parameter on resonance was examined numerically. The mode equation corresponding to this model was directly integrated numerically in an experiment, and the numerical calculation results were found to agree well with the numerical experiment results.

2. Theoretical Analysis

2.1 System characteristics and equation of motion

The analytical model consists of a cantilever beam incorporating a mass m_a at its tip (referred to below as “the beam”), as shown in Fig. 1. The beam is displaced by a vibration $q(t)$ with an arbitrary period, causing m_a to elastically collide with a spring with spring constant K that is attached to a wall, and the steady collision vibration is examined.

Figure 2 shows the restoring force in the spring during collision with the mass, which occurs after the mass is displaced through a clearance of distance $z=e_0$. This clearance is called the dead zone or zone I. The restoring force of spring can then be expressed by an asymmetric piecewise linear system. If region I is taken to be the standard, region II is considered to be nonlinear region.

Consider a beam with cross-sectional area A , second moment of area I , and density ρ , Young's modulus includes the effect of structural damping $\bar{E}$ and the external damping coefficient c_e . If the relative displacement of the beam with respect to the base is z_r , the absolute displacement is z_a , and t is time, then according to the Euler-Bernoulli theory the equation of motion for this beam is given by

Fig.1 Dynamical model for collision vibration in cantilever beam

Fig.2 Characteristics of restoring force

$$\rho A \frac{\partial^2 z_a}{\partial t^2} + c_e \frac{\partial z_a}{\partial t} + \bar{E} I \frac{\partial^4 z_r}{\partial x^4} = 0 \quad (1)$$

If the exciting displacement vibration of arbitrary period is $q(t)$, then the relationship between the absolute and relative displacements of the beam is given by:

$$z_a(x, t) = z_r(x, t) + q(t) \quad (2)$$

When the Young's modulus of the beam is E and the structural damping coefficient is c_s , the modulus that includes the effect of structural damping is given by

$$\bar{E} = E + c_s \frac{\partial}{\partial t} \quad (3)$$

Substituting Eqs. (2) and (3) into Eq. (1), we obtain the following equation of motion for the relative beam displacement z_r :

$$\rho A \frac{\partial^2 z_r}{\partial t^2} + c_e \frac{\partial z_r}{\partial t} + c_s I \frac{\partial^5 z_r}{\partial x^4 \partial t} + \bar{E} I \frac{\partial^4 z_r}{\partial x^4} = -\rho A \frac{\partial^2 q}{\partial t^2} - c_e \frac{\partial q}{\partial t} \quad (4)$$

2.2 Fourier series method

The displacement $q(t)$ acting on the system is assumed to be described by an arbitrary periodic function, and can therefore be expanded into the following complex Fourier series and a real Fourier series.

$$q(t) = \sum_{n=-\infty}^{\infty} h_n e^{jn\omega t} = \frac{f_0}{2} + \sum_{n=1}^{\infty} (f_n \cos n\omega t + g_n \sin n\omega t) \quad (5)$$

where ω is the circular frequency and h_n, f_0, f_n and g_n are Fourier coefficients.

The absolute and relative displacements are assumed to be separable. Subharmonic resonance occurs when the external excitation vibration is far from the main resonance frequency of the system but close to an integer multiple of any of the natural frequencies of the system. This consists of dominant vibration components having frequencies that are fractional multiples of the natural frequency. Thus, we assume a periodic solution with the $(1/s)$ th subharmonic. The absolute and relative displacements are expressed in terms of the complex Fourier series given in Eq. (6). The periodic solutions for superharmonic and main resonance at $s=1$ are assumed to be:

$$\left. \begin{aligned} z_a(x, t) &= \sum_{n=-\infty}^{\infty} X_{an}(x) e^{\frac{jn\omega t}{s}} \\ z_r(x, t) &= \sum_{n=-\infty}^{\infty} X_{rn}(x) e^{\frac{jn\omega t}{s}} \\ (s &= 1, 2, \dots) \end{aligned} \right\} \quad (6)$$

The periodic displacement excitation in Eq. (5) transforms the frequency as follows:

$$q(t) = \sum_{n=-\infty}^{\infty} h_n e^{\frac{jn\omega t}{s}} \quad (7)$$

Substituting Eqs. (6) and (7) into Eq. (4) provides the following ordinary differential equation for the relative displacement z_r :

$$\frac{d^4 X_{rn}}{dx^4} = \frac{\rho A \frac{n^2 \omega^2}{s^2} - c_e \frac{jn\omega}{s}}{EI + c_s I \frac{jn\omega}{s}} \left(X_{rn} + h_n \frac{1}{s} \right) \quad (8)$$

If the coefficients in Eq. (8) are

$$\lambda_n^4 = \frac{\rho A \frac{n^2 \omega^2}{s^2} - c_e \frac{jn\omega}{s}}{EI + c_s I \frac{jn\omega}{s}} \quad (9)$$

then λ_n are the eigenvalues and X_m can be formulated as follows:

$$\left. \begin{aligned} X_{r0} &= a_r x^3 + b_r x^2 + c_r x + d_r \\ X_{rn} &= A_{rn} \cosh \lambda_n x + B_{rn} \sinh \lambda_n x + C_{rn} \cos \lambda_n x + D_{rn} \sin \lambda_n x - h_n \frac{1}{s} \end{aligned} \right\} \quad (10)$$

Where, $a_r, b_r, c_r, d_r, A_{rn}, B_{rn}, C_{rn}, D_{rn}$ are presently unknown constants that will be

determined by the boundary conditions. The eigenfrequencies ω_i of the beam including the mass are given by

$$\omega_i = \frac{(\beta_i l)^2}{l^2} \sqrt{\frac{EI}{\rho A}} \quad (i = 1, 2, 3, \dots) \quad (11)$$

Where, β_{il} are determined by the ratio between m_a and the mass of the beam itself, $m_b (= \rho A l)$; β_{il} are calculated using the following frequency equation:

$$\frac{m_a}{m_b} \beta l = \frac{1 + \cosh \beta l \cos \beta l}{\cosh \beta l \sin \beta l - \sinh \beta l \cos \beta l} \quad (12)$$

where

$$\left. \begin{aligned} \Omega &= \frac{\omega}{\omega_1} \quad (\text{frequency ratio}) \\ \Omega_i &= \frac{\omega_i}{\omega_1} = \left(\frac{\beta_i l}{\beta_1 l} \right)^2 \quad (\text{eigenfrequency ratio}) \\ \zeta_{ei} &= \frac{c_e}{2\rho A \omega_i} = \frac{\omega_1}{\omega_i} \zeta_{e1} = \frac{\zeta_{e1}}{\Omega_i} \quad (\text{external damping ratio}) \\ \zeta_{si} &= \frac{c_s \omega_i}{2E} = \frac{\omega_i}{\omega_1} \zeta_{s1} = \Omega_i \zeta_{s1} \quad (\text{structural damping ratio}) \end{aligned} \right\} \quad (13)$$

When the identities in Eq. (13) are substituted into Eq. (9), we obtain the following:

$$(\lambda_n l)^4 = (\beta_1 l)^4 \frac{\left(\frac{n}{s} \right)^2 \Omega^2 - 2j\zeta_{e1} \frac{n}{s} \Omega}{1 + 2j\zeta_{s1} \frac{n}{s} \Omega} \quad (14)$$

The phase angle θ is found from

$$\theta = \frac{\omega t}{s} \quad (15)$$

The resulting displacement normalized to clearance e_0 is introduced into Eq. (15), providing the following expression:

$$\frac{z_r(x/l, \theta)}{e_0} = \sum_{n=-\infty}^{\infty} \frac{X_{rn}(x/l)}{e_0} e^{jn\theta} \quad (16)$$

The boundary conditions in the analytical model in Fig.1 can be rewritten by incorporating the motion of the beam tip mass m_a :

$$\left. \begin{aligned} z_r|_{x=0} &= 0 \\ \frac{\partial z_r}{\partial x}|_{x=0} &= 0 \\ \bar{EI} \frac{\partial^2 z_r}{\partial x^2}|_{x=l} &= 0 \\ \bar{EI} \frac{\partial^3 z_r}{\partial x^3}|_{x=l} &= g(z_r|_{x=l}) + m_a \left(\frac{\partial^2 z_r}{\partial t^2}|_{x=l} + \frac{\partial^2 q(t)}{\partial t^2} \right) \end{aligned} \right\} \quad (17)$$

Where, $g(z_r|_{x=l})$ is the restoring force exerted by the spring on m_a during collision; this has a restoring force characteristic described by the graph in Fig. 2, so it is expressed in the two regions as follows:

$$g(z_r|_{x=l}) = \begin{cases} 0 & ; z_r|_{x=l} < e_0 \quad \text{Region I} \\ K(z_r|_{x=l} - e_0) & ; z_r|_{x=l} \geq e_0 \quad \text{Region II} \end{cases} \quad (18)$$

The period of the restoring force $g(z_r|_{x=l})$ is assumed to be equal to that of the resulting vibration. It is expanded into the following Fourier series:

$$g(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{jn\theta} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \quad (19)$$

where c_n , a_0 , a_n , and b_n are Fourier coefficients.

2.3 Derivation of analytical equation

In this section, the boundary conditions given in the previous section are used to derive the analytical equation for the resulting vibration. If the boundary conditions in Eq. (17) are substituted into the resulting waveform expression in Eq. (16), we can derive the dimensionless resulting displacement ratio with this equation:

$$\begin{aligned} \frac{z_r}{e_0} &= \frac{N_0 f_0^*}{2} + \frac{M_0 x_0}{2} \\ &+ \sum_{n=1}^{\infty} \left\{ \left(\text{Re}[N_{rn}] f_{\frac{n}{s}}^* + \text{Im}[N_{rn}] g_{\frac{n}{s}}^* \right) \cos n\theta + \left(\text{Re}[N_{rn}] g_{\frac{n}{s}}^* - \text{Im}[N_{rn}] f_{\frac{n}{s}}^* \right) \sin n\theta \right\} \\ &+ \sum_{n=1}^{\infty} \{ (\text{Re}[M_n] x_n + \text{Im}[M_n] y_n) \cos n\theta + (\text{Re}[M_n] y_n - \text{Im}[M_n] x_n) \sin n\theta \} \end{aligned} \quad (20)$$

In order to simplify the format of Eq. (20), the following dimensionless quantities and coefficients were introduced. “Re” and “Im” in Eq. (20) designate the real and imaginary parts, respectively.

$$\left. \begin{aligned}
 \frac{f_n}{e_0} &= \frac{f_n^*}{s}, \quad \frac{g_n}{e_0} = \frac{g_n^*}{s}, \quad \frac{a_n}{ke_0} = x_n, \quad \frac{b_n}{ke_0} = y_n, \\
 \Delta_{rn} &= 2 \left\{ - (1 + \cosh \lambda_n l \cos \lambda_n l) + \frac{m_a \frac{n^2 \omega^2}{s^2}}{\lambda_n^3 \left(EI + c_s I \frac{j n \omega}{s} \right)} (\cosh \lambda_n l \sin \lambda_n l - \sinh \lambda_n l \cos \lambda_n l) \right\}, \\
 \overline{XHA_n} &= \frac{1}{\Delta_{rn}} \left\{ - (1 + \cosh \lambda_n l \cos \lambda_n l + \sinh \lambda_n l \sin \lambda_n l) - 2 \frac{m \frac{n^2 \omega^2}{s^2}}{\lambda_n^3 \left(EI + c_s I \frac{j n \omega}{s} \right)} \sinh \lambda_n l \cos \lambda_n l \right\}, \\
 \overline{XHB_n} &= \frac{1}{\Delta_{rn}} \left\{ (\cosh \lambda_n l \sin \lambda_n l + \sinh \lambda_n l \cos \lambda_n l) + 2 \frac{m \frac{n^2 \omega^2}{s^2}}{\lambda_n^3 \left(EI + c_s I \frac{j n \omega}{s} \right)} \cosh \lambda_n l \cos \lambda_n l \right\}, \\
 \overline{XHC_n} &= \frac{1}{\Delta_{rn}} \left\{ - (1 + \cosh \lambda_n l \cos \lambda_n l - \sinh \lambda_n l \sin \lambda_n l) + 2 \frac{m \frac{n^2 \omega^2}{s^2}}{\lambda_n^3 \left(EI + c_s I \frac{j n \omega}{s} \right)} \cosh \lambda_n l \sin \lambda_n l \right\}, \\
 \overline{XHD_n} &= \frac{1}{\Delta_{rn}} \left\{ - (\cosh \lambda_n l \sin \lambda_n l + \sinh \lambda_n l \cos \lambda_n l) - 2 \frac{m \frac{n^2 \omega^2}{s^2}}{\lambda_n^3 \left(EI + c_s I \frac{j n \omega}{s} \right)} \cosh \lambda_n l \cos \lambda_n l \right\}, \\
 \overline{XCA_n} &= \frac{k}{\lambda_n^3 \left(EI + c_s I \frac{j n \omega}{s} \right)} \frac{(\sinh \lambda_n l + \sin \lambda_n l)}{\Delta_{rn}}, \\
 \overline{XCB_n} &= - \frac{k}{\lambda_n^3 \left(EI + c_s I \frac{j n \omega}{s} \right)} \frac{(\cosh \lambda_n l + \cos \lambda_n l)}{\Delta_{rn}}, \\
 \overline{XCC_n} &= - \frac{k}{\lambda_n^3 \left(EI + c_s I \frac{j n \omega}{s} \right)} \frac{(\sinh \lambda_n l + \sin \lambda_n l)}{\Delta_{rn}}, \\
 \overline{XCD_n} &= \frac{k}{\lambda_n^3 \left(EI + c_s I \frac{j n \omega}{s} \right)} \frac{(\cosh \lambda_n l + \cos \lambda_n l)}{\Delta_{rn}}, \\
 N_{rn}(x) &= (\overline{XHA_n} \cosh \lambda_n x + \overline{XHB_n} \sinh \lambda_n x + \overline{XHC_n} \cos \lambda_n x + \overline{XHD_n} \sin \lambda_n x - 1), \\
 M_n(x) &= (\overline{XCA_n} \cosh \lambda_n x + \overline{XCB_n} \sinh \lambda_n x + \overline{XCC_n} \cos \lambda_n x + \overline{XCD_n} \sin \lambda_n x)
 \end{aligned} \right\} \quad (21)$$

The following identities are also defined for simplification:

$$\left. \begin{aligned}
 p_0 &= \operatorname{Re}[N_{r0}|_{x/l=1}] f_0^* + \operatorname{Re}[M_0|_{x/l=1}] x_0 \\
 p_n &= \operatorname{Re}[N_{rn}|_{x/l=1}] \frac{f_n^*}{s} + \operatorname{Im}[N_{rn}|_{x/l=1}] \frac{g_n^*}{s} + \operatorname{Re}[M_n|_{x/l=1}] x_n + \operatorname{Im}[M_n|_{x/l=1}] y_n \\
 q_n &= -\operatorname{Im}[N_{rn}|_{x/l=1}] \frac{f_n^*}{s} + \operatorname{Re}[N_{rn}|_{x/l=1}] \frac{g_n^*}{s} - \operatorname{Im}[M_n|_{x/l=1}] x_n + \operatorname{Re}[M_n|_{x/l=1}] y_n
 \end{aligned} \right\} \quad (22)$$

When Eq. (22) is substituted into Eq. (20), the resulting displacement ratio is transformed by Fourier series expansion into the following expression:

$$\frac{z(\theta)}{e_0} = \frac{z_r(x/l, \theta)}{e_0} \Big|_{x/l=1} = \frac{p_0}{2} + \sum_{n=1}^{\infty} (p_n \cos n\theta + q_n \sin n\theta) \quad (23)$$

The dimensionless Fourier coefficients x_n and y_n in Eq. (19) are substituted into the restoring force in Eq. (19), and dimensionless restoring force is

$$\frac{g(\theta)}{ke_0} = \sum_{n=-\infty}^{\infty} \frac{c_n}{ke_0} e^{jn\theta} = \frac{x_0}{2} + \sum_{n=1}^{\infty} (x_n \cos n\theta + y_n \sin n\theta) \quad (24)$$

2.4 Introduction of the switching-over condition

Considerations of the resulting waveform at steady state must account for the possibility that the beam will enter displacement zones I and II multiple times during a single cycle. Here, we call the condition at the boundary between displacement regions in the phase region (region in the time domain) the “switching-over condition”, and the phase angle at that point, the “switching-over phase angle”. The vibration mode is defined by the number of times the resulting displacement enters region II from region I: if this transition occurs once, it is called a type I vibration; if it occurs twice, it is called a type II vibration. If the solution is periodic, the number of passages through the border between regions is double the number of entries into region II, i.e., two passages in type I and four passages in type II per cycle, so that there are always an even number of passages in a full cycle. Displacement regions I and II are designated with superscripts ι . The phase regions are designated with $(\iota-\tau)$ from the lowest region up, and the type under consideration is designated $\tau_{\max}$. The transition from region I to region II with respect to time is marked (I II), and the reverse is (II I). The switching-over conditions are expressed as follows:

Fig.3 Resulting waveforms for superharmonic vibration

$$\left. \begin{aligned} \theta = \theta_{\tau}^{\text{III}}, \theta_{\tau}^{\text{III}} : \frac{z}{e_0} = 1 \\ \tau = \text{I}, \text{II}, \dots, \tau_{\max} \end{aligned} \right\} \quad (25)$$

The switching-over phase angle is expressed as follows:

$$\left. \begin{aligned} 0 < \theta_{\tau}^{\text{III}} - \theta_{\tau}^{\text{III}} < 2\pi \\ \theta_1^{\text{III}} < \theta_1^{\text{III}} < \theta_{\text{II}}^{\text{III}} < \theta_{\text{II}}^{\text{III}} < \dots < \theta_{\tau_{\max}}^{\text{III}} < \theta_{\tau_{\max}}^{\text{III}} \end{aligned} \right\} \quad (26)$$

Figure 3 was drawn by making no assumptions about these resulting vibrations, i.e., not assuming any relationship between the switching-over condition in Eq. (25) and the phase region in Eq. (26), or the resonance solution expressing the relationships among the excitation, response, and the spring restoring force. Figure 3 depicts a typical type II 2nd superharmonic resonance. Figure 3(a) shows an excitation of arbitrary frequency (the dashed line represents the fundamental wave). Figure 3(b) shows the resulting vibration (hatched areas mark angular regions spent in zone II), and Fig. 3(c) shows the restoring force in the spring during collision. Here, the only consideration was for regions in which the switching-over phase angle is defined; only regions where Eq. (26) is not satisfied were neglected. Both positive and negative values were accepted.

If the switching-over conditions in Eq. (25) are substituted into Eq. (23), we obtain

$$\left. \begin{aligned} \frac{p_0}{2} + \sum_{n=1}^{\infty} (p_n \cos n\theta_{\tau}^{\text{I II}} + q_n \sin n\theta_{\tau}^{\text{I II}}) = 1 \\ \frac{p_0}{2} + \sum_{n=1}^{\infty} (p_n \cos n\theta_{\tau}^{\text{II I}} + q_n \sin n\theta_{\tau}^{\text{II I}}) = 1 \end{aligned} \right\} \quad (27)$$

If we sum from $\tau = \text{I}$ to $\tau = \tau_{\max}$ in Eq. (27) and substitute it into Eq. (23), we can express z_r/e_0 in dimensionless form as:

$$\frac{z_r}{e_0} = 1 + \sum_{n=1}^{\infty} \left[\left\{ \cos n\theta - \frac{1}{2\tau_{\max}} \sum_{\tau=1}^{\tau_{\max}} (\cos n\theta_{\tau}^{\text{I II}} + \cos n\theta_{\tau}^{\text{II I}}) \right\} p_n \right. \\ \left. + \left\{ \sin n\theta - \frac{1}{2\tau_{\max}} \sum_{\tau=1}^{\tau_{\max}} (\sin n\theta_{\tau}^{\text{I II}} + \sin n\theta_{\tau}^{\text{II I}}) \right\} q_n \right] \quad (28)$$

2.5 Determining dimensionless Fourier coefficients

The real dimensionless Fourier coefficients x_n and y_n ($n=1, 2, 3, \dots$) are defined in expressions 3 and 4 of Eq. (21). These undetermined coefficients were introduced on expansion of the equation describing the restoring force in the spring after collision, so they must be set to satisfy the piecewise linear curve of the restoring force. From this condition, we obtain the following linear simultaneous equation for x_0 , x_n and y_n :

$$\left. \begin{aligned} x_0 &= \sum_{n=1}^{\infty} \sum_{\tau=1}^{\tau_{\max}} \overline{X0_{n\tau}} x_n + \sum_{n=1}^{\infty} \sum_{\tau=1}^{\tau_{\max}} \overline{Y0_{n\tau}} y_n + \sum_{n=1}^{\infty} \sum_{\tau=1}^{\tau_{\max}} \overline{C0_{n\tau}} \\ x_m &= \sum_{n=1}^{\infty} \sum_{\tau=1}^{\tau_{\max}} \overline{XC_{mn\tau}} x_n + \sum_{n=1}^{\infty} \sum_{\tau=1}^{\tau_{\max}} \overline{YC_{mn\tau}} y_n + \sum_{n=1}^{\infty} \sum_{\tau=1}^{\tau_{\max}} \overline{CC_{mn\tau}} \\ y_m &= \sum_{n=1}^{\infty} \sum_{\tau=1}^{\tau_{\max}} \overline{XS_{mn\tau}} x_n + \sum_{n=1}^{\infty} \sum_{\tau=1}^{\tau_{\max}} \overline{YS_{mn\tau}} y_n + \sum_{n=1}^{\infty} \sum_{\tau=1}^{\tau_{\max}} \overline{CS_{mn\tau}} \\ (m &= 1, 2, 3, \dots) \end{aligned} \right\} \quad (29)$$

where the following identities apply:

$$\left. \begin{aligned} \overline{X0_{n\tau}} &= (\overline{P0_{n\tau}} \overline{PX_n} + \overline{Q0_{n\tau}} \overline{QX_n}), \quad \overline{Y0_{n\tau}} = (\overline{P0_{n\tau}} \overline{PY_n} + \overline{Q0_{n\tau}} \overline{QY_n}), \\ \overline{C0_{n\tau}} &= \overline{P0_{n\tau}} \left(\overline{PF_n} f_n^* + \overline{PG_n} g_n^* \right) + \overline{Q0_{n\tau}} \left(\overline{QF_n} f_n^* + \overline{QG_n} g_n^* \right), \\ \overline{XC_{mn\tau}} &= (\overline{PC_{mn\tau}} \overline{PX_n} + \overline{QC_{mn\tau}} \overline{QX_n}), \quad \overline{YC_{mn\tau}} = (\overline{PC_{mn\tau}} \overline{PY_n} + \overline{QC_{mn\tau}} \overline{QY_n}), \\ \overline{CC_{mn\tau}} &= \overline{PC_{mn\tau}} \left(\overline{PF_n} f_n^* + \overline{PG_n} g_n^* \right) + \overline{QC_{mn\tau}} \left(\overline{QF_n} f_n^* + \overline{QG_n} g_n^* \right), \\ \overline{XS_{mn\tau}} &= (\overline{PS_{mn\tau}} \overline{PX_n} + \overline{QS_{mn\tau}} \overline{QX_n}), \quad \overline{YS_{mn\tau}} = (\overline{PS_{mn\tau}} \overline{PY_n} + \overline{QS_{mn\tau}} \overline{QY_n}), \\ \overline{CS_{mn\tau}} &= \overline{PS_{mn\tau}} \left(\overline{PF_n} f_n^* + \overline{PG_n} g_n^* \right) + \overline{QS_{mn\tau}} \left(\overline{QF_n} f_n^* + \overline{QG_n} g_n^* \right), \\ \overline{P0_{n\tau}} &= \frac{K}{\pi k} (\overline{IC_{n\tau}} + \overline{P_n} \overline{I0_{n\tau}}), \quad \overline{Q0_{n\tau}} = \frac{K}{\pi k} (\overline{IS_{n\tau}} + \overline{Q_n} \overline{I0_{n\tau}}), \\ \overline{PC_{mn\tau}} &= \frac{K}{\pi k} (\overline{ICC_{mn\tau}} + \overline{P_n} \overline{IC_{m\tau}}), \quad \overline{QC_{mn\tau}} = \frac{K}{\pi k} (\overline{ISC_{mn\tau}} + \overline{Q_n} \overline{IC_{m\tau}}), \\ \overline{PS_{mn\tau}} &= \frac{K}{\pi k} (\overline{ICS_{mn\tau}} + \overline{P_n} \overline{IS_{m\tau}}), \quad \overline{QS_{mn\tau}} = \frac{K}{\pi k} (\overline{ISS_{mn\tau}} + \overline{Q_n} \overline{IS_{m\tau}}), \\ \overline{PX_n} &= \operatorname{Re}[M_n|_{x/l=1}], \quad \overline{PY_n} = \operatorname{Im}[M_n|_{x/l=1}], \quad \overline{QX_n} = -\operatorname{Im}[M_n|_{x/l=1}], \\ \overline{QY_n} &= \operatorname{Re}[M_n|_{x/l=1}], \quad \overline{PF_n} = \operatorname{Re}[N_{rn}|_{x/l=1}], \quad \overline{PG_n} = \operatorname{Im}[N_{rn}|_{x/l=1}], \\ \overline{QF_n} &= -\operatorname{Im}[N_{rn}|_{x/l=1}], \quad \overline{QG_n} = \operatorname{Re}[N_{rn}|_{x/l=1}], \\ \overline{P_n} &= -\frac{1}{2\tau_{\max}} \sum_{\tau=1}^{\tau_{\max}} (\cos n\theta_{\tau}^{\text{I II}} + \cos n\theta_{\tau}^{\text{II I}}), \quad \overline{Q_n} = -\frac{1}{2\tau_{\max}} \sum_{\tau=1}^{\tau_{\max}} (\sin n\theta_{\tau}^{\text{I II}} + \sin n\theta_{\tau}^{\text{II I}}), \\ \overline{IC_{n\tau}} &= \int_{\theta_{\tau}^{\text{I II}}}^{\theta_{\tau}^{\text{II I}}} \cos n\theta \cdot d\theta, \quad \overline{IS_{n\tau}} = \int_{\theta_{\tau}^{\text{I II}}}^{\theta_{\tau}^{\text{II I}}} \sin n\theta \cdot d\theta, \quad \overline{I0_{n\tau}} = \int_{\theta_{\tau}^{\text{I II}}}^{\theta_{\tau}^{\text{II I}}} d\theta, \\ \overline{ICC_{mn\tau}} &= \int_{\theta_{\tau}^{\text{I II}}}^{\theta_{\tau}^{\text{II I}}} \cos n\theta \cos m\theta \cdot d\theta, \quad \overline{ISS_{mn\tau}} = \int_{\theta_{\tau}^{\text{I II}}}^{\theta_{\tau}^{\text{II I}}} \sin n\theta \sin m\theta \cdot d\theta, \\ \overline{ISC_{mn\tau}} &= \int_{\theta_{\tau}^{\text{I II}}}^{\theta_{\tau}^{\text{II I}}} \sin n\theta \cos m\theta \cdot d\theta, \quad \overline{IC_{m\tau}} = \int_{\theta_{\tau}^{\text{I II}}}^{\theta_{\tau}^{\text{II I}}} \cos m\theta \cdot d\theta, \\ \overline{ICS_{mn\tau}} &= \int_{\theta_{\tau}^{\text{I II}}}^{\theta_{\tau}^{\text{II I}}} \cos n\theta \sin m\theta \cdot d\theta, \quad \overline{IS_{m\tau}} = \int_{\theta_{\tau}^{\text{I II}}}^{\theta_{\tau}^{\text{II I}}} \sin m\theta \cdot d\theta, \end{aligned} \right\} \quad (30)$$

The equations were rewritten to set Eq. (27) to zero and new functions ϕ_r^{I} and ϕ_r^{II} were defined thus:

$$\left. \begin{aligned} \phi_{\tau}^I &\equiv \frac{p_0}{2} + \sum_{n=1}^{\infty} (p_n \cos n\theta_{\tau}^{I\ II} + q_n \sin n\theta_{\tau}^{I\ II}) - 1 = 0 \\ \phi_{\tau}^{II} &\equiv \frac{p_0}{2} + \sum_{n=1}^{\infty} (p_n \cos n\theta_{\tau}^{II\ I} + q_n \sin n\theta_{\tau}^{II\ I}) - 1 = 0 \end{aligned} \right\} \quad (31)$$

Accordingly, solving Eqs. (29) and (31) simultaneously allows us to determine x_0 , x_n and y_n in the dimensionless Fourier series and the t switching-over phase angles $\theta_{\tau}^{I\ II}$ and $\theta_{\tau}^{II\ I}$ ($\tau=I, II, \dots, \tau_{\max}$). In turn, these allow us to solve Eqs. (28) and (24) for the resulting displacement ratio and the restoring force, respectively.

3. Analytical and Experimental Results

3.1 Comparison of calculation results and numerical experiment results

In order to verify the analytical results developed above, a numerical experiment was conducted to calculate the resulting displacement. The mode equation obtained from the above equation of motion was directly integrated numerically. In the numerical calculation based on the analytical results a value of 2 was selected for s in the $(1/s)$ th subharmonic resonance incorporated in Eq. (6), and the integration was performed including $1/2$ nd subharmonic resonance. Infinite simultaneous Eqs. (29) and (31) were calculated using the dimensionless Fourier coefficients x_n and y_n ($n=1, 2, 3, \dots$) and were performed to the term with $n, m=40$ to approximate the response. Convergence was slow when $s=2$ and $n \leq 20$, $x_n = -0.23 \sim 0.29$ and $y_n = -0.46 \sim 0.061$; whereas, convergence was acceptable with $n \geq 30$, $x_n = -0.0064 \sim 0.0054$ and $y_n = -0.0083 \sim 0.0074$.

Figure 4 presents the resonance curves for comparing the numerical calculations with the numerical experiment results. The experimental results ($\circ$ symbols) and numerical calculations (solid line) agree reasonably well. The $\bullet$ symbols in the figure indicate branching points between different types revealed by the numerical calculations. At $\Omega=2.06 \sim 2.12$ and $\Omega=2.44 \sim 2.48$, a type II resonance occurred; the beam contacted the spring twice in each cycle of the beam, which occurred during two cycles of the exciting vibration (subharmonic type II). The experimentally observed branching of the main resonance classification type I at $\Omega=2.06 \sim 2.48$ was not numerically predicted, so it may be an unstable response. Stability analysis⁽⁶⁾ is required to arrive at a rigorous conclusion as to

Fig.4 Resonance curve for comparing numerical calculation results with numerical experimental ones

whether there really is an unstable portion of the resonance curve; we intend to investigate this in a later study.

3.2 Example of numerical calculation

A numerical calculation was also conducted to seek a steady vibration solution on the basis of the above analysis. The (odd) composite function in the following equation was selected as an example of a periodic excitation with an arbitrary function:

$$q(t) = \cos \omega t + \sin 2\omega t \quad (32)$$

Figure 5 shows the resonance curve obtained, using the parameter of the mass ratio between the beam and the added mass m_a/m_b . The symbol ● in the figure is the boundary point between with and without collision. The resonance curve bends to the right when the mass at the tip of the beam collides elastically with the spring and enters the nonlinear region. When $m_a/m_b = 0$, there is no added mass on the beam. When a mass is added to the beam, the resonance region becomes slightly shorter, and the maximum amplitude increases.

Fig.5 Resonance curves in the case where mass ratio m_a/m_b is parameter

Fig.6 Resonance curves in the case where amplitude ratio f_1/e_0 of excitation is parameter

$$\{q(t) = \cos \omega t + \sin 2\omega t, m_a/m_b = 1.0, f_1/e_1 = 0.1, \zeta_{e1} = 0.01, \zeta_{s1} = 0.01\}$$

Fig.7 Resonance curves in the case where spring constant ratio K/k is parameter

$$\{q(t) = \cos \omega t + \sin 2\omega t, m_a/m_b = 1.0, f_1/e_1 = 0.1, K/k = 1.0, \zeta_{e1} = 0.01\}$$

Fig.8 Resonance curves in the case where structural damping ratio ζ_{s1} is parameter

$$\{q(t) = \cos \omega t + \sin 2\omega t, m_a/m_b = 1.0, f_1/e_1 = 0.1, K/k = 1.0, \zeta_{s1} = 0.01\}$$

Fig.9 Resonance curves in the case where external damping ratio ζ_{e1} is parameter

Figure 6 shows resonance curves in which the excitation amplitude ratio f_1/e_0 has been parameterized. As f_1/e_0 increases, the resonance region becomes wider and the maximum amplitude increases. The $\circ$ symbols in the figure represent the boundary between the types indicated by the numerical calculations. When $f_1/e_0=1.0$, a type II 2nd superharmonic resonance occurs at frequency ratios below $\Omega=0.947$ ($\circ$ symbol).

Figure 7 shows resonance curves obtained by using the spring constant ratio K/k as a parameter. As K/k increases, the rightward tilt of the resonance curve and the maximum amplitude both increase.

Figure 8 shows resonance curves with the structural damping ratio ζ_{s1} as a parameter. As ζ_{s1} increases, the maximum amplitude decreases.

Figure 9 shows resonance curves with the external damping ratio ζ_{e1} as a parameter. As ζ_{e1} increases, the maximum amplitude of the resonance curve decreases. Comparing the value of $\zeta_{s1}=0.03$ in Fig. 8 with the value of $\zeta_{e1}=0.03$ in Fig. 9 reveals that the given value of external damping allows a greater maximum amplitude. Thus, structural damping is more effective for suppressing vibrations.

Figure 10 shows curves in the region of Ω where both 2nd superharmonic resonance and 1/2nd subharmonic resonance occur. In contrast to the behavior in the vicinity of the main resonance, subharmonic resonances occurred in the vicinity of the two times frequency ratio, and superharmonic resonances occurred near the half times frequency ratio.

Fig.10 Resonance curves for region from 2nd superharmonic to 1/2nd subharmonic vibration

Fig.11 Detailed resonance curve for ultra-subharmonic vibration in Fig.10

Fig.12 Each waveform for 1/2nd subharmonic vibration of Point (A) in Fig.11

Fig.13 Detailed resonance curve for 1/2nd subharmonic vibration in Fig.10

In addition, ultra-subharmonic resonances were also identified between the main resonance and the superharmonic.

Figure 11 shows an expansion of a portion of Fig.10, showing where ultra-subharmonic resonances occurred at $\Omega=0.74\sim0.77$. This region contained two-cycle solutions. They branch away from the type II superharmonic curve and exhibit behavior classified as type IV, meaning that the mass initiates nonlinear behavior four times per cycle. These superharmonics diminish in the vicinity of resonance point and transition to type III behavior (in which the mass initiates nonlinear behavior three times per cycle). The

resonance curve has a loop in the figure.

Figure 12 shows the excitation waveforms, the resulting vibration, and the restoring force in the spring at point A in Fig. 11. At point A, the mass collides the spring four times during two excitation cycles, for a type IV 1/2nd subharmonic resonance.

Figure 13 shows the details of the resonance curve indicated in Fig. 10, in the range $\Omega = 2.0 \sim 2.4$, where a 1/2nd subharmonic vibration occurs. In the low-frequency portion of this graph, there is a relatively small type II region that rapidly splits into a type III resonance. Subsequently, around $\Omega = 2.2$, this reverts back to type II, again for a brief range of Ω , and then transitions to a type I resonance as the frequency increases and approaches the main resonance level. Closer to this level, it transitions again to a type II, then back to a type I harmonic resonance.

4. Conclusions

This paper is discussed analytical method on the steady collision vibration in a continuous system having an attached mass. The paper can be summarized as follows:

- (1) An analytical model consisted of a cantilever beam with a attached mass. The beam was subject to structural and external damping and excited by a periodic displacement with an arbitrary function that caused it to collide a spring attached to wall in an elastic collision, and the resulting steady forced vibration was analyzed. The analysis approximated the restoring force occurring during collision with the spring as an asymmetric piecewise linear system and used Fourier series method to solve it. A new analytical procedure that imposed no limits on the resulting vibration waveforms was proposed and an exact solution was derived for the resulting vibration that allows for harmonic, superharmonic and subharmonic resonances.
- (2) Numerical calculations were performed using the analytical results obtained in (1) and resonance curves were constructed to examine the effects on resonance of several factors: mass ratio m_a/m_b , excitation amplitude ratio f_1/e_0 , nonlinearity of resulting vibration K/k , structural damping ratio ζ_{s1} and external damping ratio ζ_{e1} . Several resonance curves were also constructed for closer examination of the details of the main, superharmonic and subharmonic resonance regions. The waveform of the periodic resulting vibration and the restoring force in the spring were found.
- (3) A numerical experiment was carried out to confirm the analytical results obtained in (1) and (2). The numerical calculations agreed reasonably well with the experimental results, confirming the effectiveness of this method.
- (4) Using this analytical method, it is possible to treat a periodic excitation with an arbitrary function in a general way. It was shown that, if this method can be firmly established, it is capable of providing solutions for resulting vibrations using the same equations as mentioned above.

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