Modal Analysis of Railway Vehicle Carbodies Using a Linear Prediction Model

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Abstract
This paper reports on a study regarding modal property identification for railway vehicles using a linear prediction model. The relationship between input (excitation force or axlebox acceleration) and output (carbody acceleration) of an actual railway vehicle obtained by stationary or running tests is expressed by means of an ARX (Auto-Regressive eXogenous) model, and the procedure for the extraction of modal properties is described in detail. Determination of an appropriate model order (i.e., the order of the prediction coefficients in the ARX model) is specifically discussed from the viewpoint of practical use. The implementation of average estimation errors for two different parts of the analyzed data is proposed, and their effectiveness in determining the model order is evaluated. Suitability for the MIMO (multiple-input multiple-output) problem using the ARX model is also described. It is shown that detailed modal characteristics can be successfully identified using the proposed method from measured data for both stationary and running tests.

Key words : Railway, Modal Analysis, Linear Prediction Model, Signal Processing, Bending Vibration

1. Introduction
The suppression of vertical bending vibration in carbodies is important in improving the ride quality of railway vehicles. The first step to take in vibration countermeasures is to identify vibration characteristics such as the frequency and modal properties of the carbody. Running and stationary vibration tests are usually conducted for this purpose. Running tests on an actual commercial-service track are carried out to analyze frequency characteristics during running and evaluate the ride quality of vehicles. Stationary vibration testing is suitable for use in ascertaining the modal properties of a carbody because the relationship between input forces and response acceleration is clear. Since the costs and effort involved in conducting railway vehicle measurement tests are high, it is very effective to be able to evaluate both ride quality and modal properties from a single measurement test. The authors have already reported on a method of evaluating running quality from stationary testing(1)(2). This paper deals with a technique to evaluate the modal properties of carbodies from running tests.

The input/output relationships of a railway vehicle are complex and unsteady during running; the vehicle is subjected to multiple-inputs, and the excitation conditions change over short periods of time. It has therefore been difficult to ascertain the detailed frequency and modal properties of carbodies from running tests. To address this challenge, the authors examined the application of a linear prediction model (LPM) such as ARX (the Auto-Regressive eXogenous model) to analyze railway vehicle vibration(3)(7), since LPMs are advantageous in treating short-time data and multiple-input multiple-output (MIMO) problems. However, determining the model order (i.e., the order of the prediction coefficients in the ARX model) is problematic. This paper describes identification of the modal properties of railway vehicle carbodies using the ARX model, and the determination of appropriate model order is specifi-
cally discussed from the viewpoint of practical use.

2. Necessity of Introducing a Linear Prediction Model (LPM)

The excitation conditions of railway vehicles vary from moment to moment during running because the track conditions and/or running velocity are always changing. Accordingly, the magnitude of the induced bending vibration in a carbody also changes every moment. When analyzing such unsteady vibration data using FFT (Fast Fourier Transform) with sufficient frequency resolution, e.g., calculation of acceleration PSD (Power Spectral Density) with \( \Delta f = 0.1 \text{Hz} \), it is difficult to obtain reliable results because of a lack of averaging. This difficulty can be avoided by using a sufficient data length such as 60 seconds. However, a vehicle running at 300 km/h, for example, travels 5 km in 60 seconds, so the FFT method is not necessarily suitable for the analysis of such limited-length data for some specified sections where remarkable bending vibration occurs.

The authors studied the application of an LPM to analyze railway vehicle vibration characteristics, and showed that it is advantageous in ascertaining the modal properties of carbodies as well as frequency characteristics such as acceleration PSDs during running. Usually, railway vehicles are subjected to eight vertical excitation inputs during running because they run on eight wheels. The LPM can easily be expanded to accommodate multiple-input conditions, which is another advantage over FFT-based modal analysis.

3. Modal Analysis Using the LPM

This section outlines the procedure for application of the LPM to modal analysis of railway vehicle carbodies by treating a railway vehicle as a multiple-input multiple-output (MIMO) system. The analytical process here is based on the literatures previously published.

3.1. Calculation of Prediction Coefficients

Suppose that the input signals are \( u(n) \), the output signals are \( y(n) \), and that these signals form a discretized data series with arbitrary sampling time \( \Delta t \). Here, \( n \) indicates the data sample number. Now, we consider prediction of the output signal at \( n \) by using \( M \) samples of past input and output data with weighting coefficients (prediction coefficients) \( a_m^{(M)} \) and \( b_m^{(M)} \) as follows:

\[
\hat{y}(n) = \sum_{m=1}^{M} a_m^{(M)} y(n-m) + \sum_{m=1}^{M} b_m^{(M)} u(n-m),
\]  

(1)

where \( \hat{y} \) denotes the predicted values. When the difference between the actually measured output signals \( y(n) \) and the predicted values are expressed as the prediction error \( e_y^{(M)}(n) \), we obtain the following equation:

\[
y(n) = \sum_{m=1}^{M} a_m^{(M)} y(n-m) + \sum_{m=1}^{M} b_m^{(M)} u(n-m) + e_y^{(M)}(n).
\]  

(2)

This equation expresses the relationship between input and output signals as an \( M \)-th order ARX (Auto-Regressive eXogenous) model. The input and output signals are taken as vectors \( u(n) = [u_1(n), \ldots, u_P(n)]^T \) and \( y(n) = [y_1(n), \ldots, y_Q(n)]^T \), where \( P \) and \( Q \) denote the number of input and output data series, respectively (\( [.]^T \) expresses the vector transpose), and \( a_m^{(M)} \) and \( b_m^{(M)} \) are matrices with sizes \( Q \times Q \) and \( Q \times P \), respectively.

Next, we consider the reversed form of the input and output relationship in Eq. (2).

\[
u(n) = \sum_{m=1}^{M} d_m^{(M)} u(n-m) + \sum_{m=1}^{M} c_m^{(M)} y(n-m) + e_u^{(M)}(n),
\]  

(3)

where \( d_m \) and \( c_m \) denote prediction coefficient matrices with sizes \( P \times P \) and \( P \times Q \), respectively, and \( e_u^{(M)} \) denotes the prediction error for \( u(n) \). Since Eqs. (2) and (3) are independent, these two models can be combined, and the following equation is obtained:
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where \( x(n) = \sum_{m=1}^{M} A_{m}^{(M)} x(n-m) + \epsilon^{(M)}(n) \), \( (4) \)

where \( x(n) = [u^{T}(n) \ y^{T}(n)]^{T} \) is a combined vector consisting of input and output time series data, \( \epsilon^{(M)}(n) = [\epsilon_{n}^{(M)}(n) \ \epsilon_{y}^{(M)}(n)]^{T} \) expresses the combined error vector, and \( A_{m}^{(M)} \) denotes the following block matrix containing prediction coefficients:

\[
A_{m}^{(M)} = \begin{bmatrix} d_{m}^{(M)} \ c_{m}^{(M)} \\ b_{m}^{(M)} \ a_{m}^{(M)} \end{bmatrix},
\]

Equation (4) means that \( x(n) \) can be expressed as an AR (Auto-Regressive) model with \( P + Q \) independent variables, and \( A_{m}^{(M)} \) can be calculated by applying existing algorithms for an ordinary AR model. In this study, we employ the Burg-method\(^{(11)} \), which is considered advantageous in spectrum estimation from short-time data and has several efficient algorithms for numerical calculation of prediction coefficients.

In addition to the forward ARX model expressed in Eqs. (2)-(4), we define another backward ARX model in order to apply the Burg-method as follows:

\[
y(n - M) = \sum_{m=1}^{M} e_{m}^{(M)} y(n - M + m) + \sum_{m=1}^{M} f_{m}^{(M)} u(n - M + m) + \eta_{y}^{(M)}(n), \quad (5)
\]

\[
u(n - M) = \sum_{m=1}^{M} g_{m}^{(M)} u(n - M + m) + \sum_{m=1}^{M} h_{m}^{(M)} y(n - M + m) + \eta_{u}^{(M)}(n), \quad (6)
\]

\[
x(n - M) = \sum_{m=1}^{M} b_{m}^{(M)} x(n - M + m) + \eta^{(M)}(n), \quad (7)
\]

where the backward prediction error vector \( \eta^{(M)}(n) \) and the backward prediction coefficient matrix \( B_{m}^{(M)}(n) \) can be expressed as

\[
\eta^{(M)}(n) = \begin{bmatrix} \eta_{u}^{(M)}(n) \\ \eta_{y}^{(M)}(n) \end{bmatrix}, \quad B_{m}^{(M)} = \begin{bmatrix} h_{m}^{(M)} & g_{m}^{(M)} \\ f_{m}^{(M)} & e_{m}^{(M)} \end{bmatrix}.
\]

The basic concept of the Burg-method is that prediction coefficients are calculated in order to minimize the sum of prediction error variances in Eqs. (4) and (7) by solving the following recurring formulas obtained by such a condition:

\[
A_{m}^{(M)} = R^{(M)} \left[ \sum_{n=M}^{N} \eta^{(M-1)}(n-1) \eta^{(M-1)}(n-1) \right]^{-1}, \quad (8)
\]

\[
B_{m}^{(M)} = R^{(M)} T \left[ \sum_{n=M}^{N} \epsilon^{(M-1)}(n) \epsilon^{(M-1)}(n) \right]^{-1}, \quad (9)
\]

\[
\epsilon^{(M)}(n) = \epsilon^{(M-1)}(n) - A_{m}^{(M)} \eta^{(M-1)}(n-1), \quad (10)
\]

\[
\eta^{(M)}(n) = \eta^{(M-1)}(n-1) - B_{m}^{(M)} \epsilon^{(M-1)}(n), \quad (10)
\]

where \( N \) denotes the data length, and \( R^{(M)} \) is expressed as

\[
R^{(M)} = \sum_{n=M}^{N} \epsilon^{(M-1)}(n) \eta^{(M-1)}(n-1) T\]
3.2. Extraction of Modal Properties

Suppose the prediction coefficients in Eq. (4) of the $M$-th order are obtained by the procedure outlined above. In this subsection, we calculate modal properties using these prediction coefficients. Note that only $a^{(M)}_m$ and $b^{(M)}_m$ are used in the coefficient matrix $A^{(M)}_m$.

By introducing following state vectors
\[ y^n(n) = [y^T(n) \ y^T(n-1) \ \ldots \ y^T(n-M+1)]^T, \]
\[ u^n(n) = [u^T(n) \ u^T(n-1) \ \ldots \ u^T(n-M+1)]^T, \]
Eq. (2) can be expressed as follows:
\[
\begin{align*}
\{ \quad \begin{aligned}
y^{n+1} &= A^n y^{n} + B^n u^{n}, \\
y^{n} &= C^n y^{n},
\end{aligned}
\end{align*}
\] (11)
where superscript $s$ denotes values in state space, and $A^s, B^s$ and $C^s$ are block matrices expressed as follows:
\[
\begin{align*}
A^s &= \begin{bmatrix}
a^{(M)}_1 & a^{(M)}_2 & a^{(M)}_3 & \ldots & a^{(M)}_M \\
I & 0 & 0 & \ldots & 0 \\
0 & I & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & I & 0
\end{bmatrix}, \\
B^s &= \begin{bmatrix}
b^{(M)}_1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}.
\end{align*}
\]

\[ C^s = \left[ \begin{array}{ccc}
I & 0 & \cdots & 0
\end{array} \right], \]

where $I$ and $0$ are the unit matrix and the null matrix, respectively. By applying the $z$ transformation to Eq. (11), the following equations can be obtained:
\[
\begin{align*}
z Y^s(z) &= \tilde{A}^s Y^s(z) + \tilde{B}^s U(z), \\
Y(z) &= \tilde{C}^s Y^s(z),
\end{align*}
\] (12)
where, $Y(z), Y^s(z)$ and $U(z)$ are the $z$ transformation of $y(n), y^s(n)$ and $u(n)$, respectively. Here, a relationship whereby the $z$ transformation of $u(n-m)$ can be written as $z^{-m}U(z)$ is used, and
\[
\tilde{A}^s = A^s, \quad \tilde{B}^s = \left[ \sum_{m=1}^{M} z^{1-m}b^{(M)}_m \ 0 \ \cdots \ 0 \right]^T, \quad \tilde{C}^s = C^s. 
\] (13)

From the first formula in Eq. (12), we obtain
\[
Y^s(z) = \left[ z I - \tilde{A}^s \right]^{-1} \tilde{B}^s U(z). 
\] (14)
Substituting Eq. (14) into the second formula of Eq. (12) yields the following relationship.
\[
Y(z) = G(z) U(z),
\] (15)
where $G(z)$ is the pulse transfer matrix between the input and output signals written as follows:
\[
G(z) = \tilde{C}^s \Psi [ z I - \Lambda ]^{-1} \Psi^{-1} \tilde{B}^s. 
\] (16)
In this equation, $\Lambda(z)$ denotes the diagonal matrix containing $\lambda_i$ as the $i$-th diagonal element, where $\lambda_i$ is the $i$-th eigenvalue of the block coefficient matrix $A^s$, and $\Psi(z)$ is the matrix whose $i$-th column consists of the eigenvector corresponding to $\lambda_i$. The size of the matrices $\Lambda(z)$ and $\Psi(z)$ is $QM \times QM$. Note that $A^s$ is supposed to have no duplicated eigenvalues.
Partial fraction decomposition of this pulse transfer matrix yields
\[
G(z) = \sum_{i=1}^{QM} \frac{R_i}{z - \lambda_i}, 
\] (17)
where $R_i$ is a complex residue of $G(z)$, and can be written using the $QM \times QM$ matrix $A_i$, whose $i$-th diagonal element is 1 and the others are all 0, as follows:

$$R_i = \tilde{C} \Psi \Lambda_i \Psi^{-1} \tilde{B}^{*} |_{z = \lambda_i} .$$  \hfill (18)

By transferring $G(z)$ into the $s$-domain under the condition that the impulse response is invariant, and classifying the roots to $r$ couples of complex conjugate roots and the others, we obtain Eq. (19). The $r$ couples of the complex conjugate roots denote the vibration mode of the system.

$$G(s) = \sum_{i=1}^{r} \left( \frac{R_i}{s - s_i} + \frac{R_i^*}{s - s_i^*} \right) + \sum_{i=r+1}^{Q} \left( \frac{R_i}{s - s_i} \right) + A_0 .$$  \hfill (19)

The first term on the right side, $R_i$ (which is the $Q \times P$ matrix containing the response to the $p$-th input in the $p$-th column) corresponds to the modal shape of the system. Natural frequency $f_i$ and corresponding modal damping ratio $\zeta_i$ for the $i$-th mode can be expressed as

$$f_i = \frac{|s_i|}{2\pi} = \frac{|\log \lambda_i|}{2\pi \Delta t}, \quad \zeta_i = \cos \left\{ \pi \frac{(\log \lambda_i)}{2\pi \Delta t} \right\} .$$  \hfill (20)

### 4. Identification of Modal Properties from Stationary Vibration Testing

#### 4.1. Outline of a Stationary Vibration Test for an Actual Railway Vehicle

Measurement tests for actual railway vehicles were conducted to evaluate the validity of the LPM-based modal property identification method. A stationary vibration test using an exciter (a suitable technique to ascertain modal properties clearly since the input/output relationship is definite and the results are easily comparable to those of other methods) was carried out first. In this test, a railway vehicle was situated on the rail and excited using an exciter to measure the response vibration of the carbody.

Figure 1 shows the railway vehicle tested here. This is a test vehicle belonging to the Railway Technical Research Institute (RTRI), and has almost identical carbody structures to those of current commuter-type vehicles used in commercial service. The carbody shell is made of stainless steel, which is the dominant material for commuter-type vehicles in Japan. The test vehicle has no service equipment such as passenger seats or lighting, and the air-conditioner on the roof is a dummy unit with mass and inertia moments equivalent to those of an actual model. The length, width and height of the carbody are 19.5 m, 2.95 m and 2.67 m, respectively, and its mass (without bogies) is approximately 10.7 tons.

The acceleration measurement points on the carbody and the excitation point are shown in Fig. 2. In this vibration test, a total of 43 acceleration pickups were attached to the carbody (17 points on the floor in the vertical direction, 14 points on the roof in the vertical direction, 6 points on the side panel in the lateral direction, and 3 points on each end panel in the longitudinal direction), and an electro-dynamic exciter (maximum excitation capacity 1 kN) was fixed in the center of the floor from underneath with a driving rod. A load cell was installed between the driving rod and the carbody to measure the excitation force. Thus, the numbers of input and output signals were $P = 1$ and $Q = 43$, respectively, in this case.

A band-limited random signal with a uniform frequency component in the range of 5-30 Hz was used to excite the vehicle, and the excitation test duration was 120 s. The measured data were recorded in digital format with a sampling time of $\Delta t = 0.005$ s (200 Hz), and the cut-off frequency of the anti-aliasing filter was set to 80 Hz.

#### 4.2. Determination of Model Order

To perform modal property identification using the proposed method, the model order $M$ has to be determined in advance. For this purpose, the AIC(12) is widely used in general; that is, the model order is determined so that AIC is minimized. In published literatures(8)(9), the
The model order was determined referring to the decrease in the trace of covariance matrices of forward and backward prediction error $V^{(M)} = \mathbf{e}^{(M)}(n)\mathbf{b}_M^T(n)$ and $W^{(M)} = \mathbf{y}^{(M)}(n)\mathbf{y}^{(M)T}(n)$.

We first checked these index values using measured data obtained from the above-mentioned stationary vibration test; that is, considering the excitation force as $u(n)$ and the acceleration data as $y(n)$, the trace of covariance matrix trace $V^{(M)}$ of forward prediction error $\mathbf{e}^{(M)}(n)$ and the multiple variable AIC (MAIC) defined as follows\(^{(12)(13)}\) were calculated.

$$\text{MAIC}(M) = N \log |V^{(M)}| + 2M(P + Q)^2. \quad (21)$$

Figure 3 shows the calculation results. Here, the trace of error covariance matrix trace $V^{(M)}$ (left side axis) and MAIC (right side axis) are plotted against the model order. Note that both values are indicated in normalized form. It can be seen that both indexes have similar tendencies; both of them decrease continuously as the model order increases, and they have no minimum value. The trace $V^{(M)}$ values show a radical drop at around $M = 2$, but this is not an adequate model order to identify modal properties, as described later. Thus, it is found that determination of a correct model order using these traditional indexes is difficult for vibration data measured from a railway vehicle carbody.

The authors therefore determined the model order heuristically by considering information such as the features of mode shapes in their former work\(^{(3)-(7)}\). If a more objective and realistic method can be established in the determination process of model order, it is expected that the reasonability of analysis will increase and the number of trials needed will decrease. Referring to Fig. 3, it seems that the change in each index is reduced when the model order
becomes 10 or larger, especially in the case of MAIC. If a certain threshold can be specified, we can determine a reasonable model order. However, the value of MAIC or the trace of covariance matrices of prediction errors may vary depending on the analyzed data, which makes it difficult to specify the correct threshold.

In consideration of the features of the problem to be analyzed (i.e., identification of vibration properties from measured data rather than prediction of time series representations in the future), we propose a realistic determination procedure as follows:

(1) Divide the data measured with length \( N \) into two parts of length \( N_1 \) and \( N_2 \) (\( N = N_1 + N_2 \)), and use one of these parts to calculate the prediction coefficients of the ARX model. Hereafter, the \( N_1 \) part will be used for calculation of the prediction coefficients in this study.

(2) Estimate time series representations for each of the \( N_1 \) and \( N_2 \) parts using the prediction coefficients obtained, and evaluate the error between the measured and estimated data \( \epsilon_{1y}(n) = \epsilon_{y}^{(M)}(n_1) \) and \( \epsilon_{2y}(n) = \epsilon_{y}^{(M)}(n_2) \), where, \( n_1 \) and \( n_2 \) denote data in the \( N_1 \) and \( N_2 \) parts, respectively.

(3) Iterate this procedure by increasing \( M \), and employ \( M \) as the model order when \( \epsilon_{1y}^{(M)}(n) \) and \( \epsilon_{2y}^{(M)}(n) \) become sufficiently small.

In this method, the measured data are divided into two parts; one is used for prediction coefficient calculation, and the other is used for evaluation. Note that this method supposes that the measured data do not vary greatly in terms of their properties, e.g., the variation of running velocity and/or track conditions is small. The method also assumes that estimation for the \( N_2 \) part is sufficiently accurate if the model order and the prediction coefficients using the \( N_1 \) part are suitable. The following can be employed for evaluation of estimation error \( \epsilon_{y}^{(M)} \) of the time series:

\[
S_{err}(M) = \frac{1}{Q} \sum_{q=1}^{Q} \frac{\epsilon_{q}^{(M)}}{\bar{y}_q} \times 100
\]  

(22)

where \( \epsilon_{q}^{(M)} \) and \( \bar{y}_q \) respectively denote the RMS (root mean square) of the estimation error (the difference between the estimated and measured values) and the RMS of the measured data itself for the \( q \)-th output. Note that \( Q \) denotes the number of output signals. The \( S_{err}(M) \) value expresses the average estimation error ratio (%).

Figure 4 shows the average estimation error ratio \( S_{err}(M) \) for the same measured data as Fig. 3. In this case, the data measured with a total length of \( N = 24,000 \) (120s) are divided into two parts with length \( N_1 = N_2 = 12,000 \) (60s) and the former part \( N_1 \) is used for calculation of the prediction coefficients. The green and black lines respectively denote \( S_{err}(M) \) for \( N_1 \) and \( N_2 \). It can be seen that the \( S_{err}(M) \) values also decrease gradually as \( M \) increases and have no minimum value in the same way as with MAIC. However, the \( S_{err}(M) \) values have a physical meaning, representing the estimation error ratio versus the actual measured value.

We can therefore set a certain threshold for the values preliminarily, which is advantageous for application of the ARX model to practical problems. Below, we express the average estimation error ratios for the \( N_1 \) and \( N_2 \) parts as \( S_{err1} \) and \( S_{err2} \), respectively.
The estimation error for the $N_1$ part used for calculation of the prediction coefficients will decrease monotonically as the model order increases; on the other hand, the estimation error for the $N_2$ part may increase if the prediction coefficients fit excessively for the $N_1$ part. For the problems dealt with in this paper, appropriate prediction coefficients are those that identify the modal properties of railway vehicles correctly; if the coefficients are appropriate, they should be usable for accurate estimation of other time series representations for carbody response. Accordingly, such prediction coefficients cannot be suitable for our purposes here unless they give a small estimation error for the $N_1$ part but a large error for the other part.

To check this, we introduce the difference in the estimation error for the $N_1$ and $N_2$ parts $\Delta S_{err1} = S_{err2} - S_{err1}$. Figure 5 shows the change in $\Delta S_{err1}$ versus the model order $M$. $\Delta S_{err1}$ reaches its minimum when $M$ is slightly larger than 20, and increases thereafter. As shown in Fig. 4, both $S_{err1}$ and $S_{err2}$ decrease monotonically as $M$ increases. In this situation, $\Delta S_{err1}$ increases when the rate of decrease for $S_{err2}$ is smaller than that for $S_{err1}$. If we consider the increase in the value of $\Delta S_{err1}$ as a sign of over-fitting to the $N_1$ part, we can use the $M$ value at which $\Delta S_{err1}$ reaches its minimum as the upper limit of the model order.

Next, we examine the applicability of the proposed method to the analysis of short-time data. One major aspect of the Burg-method applied to calculation of the prediction coefficients may be exerted on treating short-time data. To demonstrate this feature, we calculate FRFs (Frequency Response Functions) using different-length data. Figure 6 shows the FRFs between input force and response acceleration measured on the carbody floor just above the bogie center. The bold green line shows the FRF obtained using FFT for 60 s data, and the thin black, blue dotted and red dashed lines show FRFs calculated using the ARX model with 60 s, 15 s and 5 s data, respectively. The model order for this ARX model is $M = 12$. The FRFs obtained by the ARX model using 60 s and 15 s agree well with the FFT results. However, the FRF obtained with the 5 s data shows relatively large fluctuation, indicating a lower level of FRF reliability.

The $\Delta S_{err1}$ values for model order $M$ are plotted in Fig. 7. In the case of the data length is 5 s, $\Delta S_{err1}$ increases for $M > 8$, which suggests that the model order used in Fig. 6 ($M = 12$) is too large. We therefore modified the model order as $M = 8$ and re-calculated the FRF using 5 s data. The results are shown in Fig. 8. The fluctuation of the FRF decreased, and the level
of reliability increased. This example shows the effectiveness of upper-limit determination for the model order using the $\Delta S_{err1}$ value.

4.3. Modal Analysis for Stationary Vibration Tests

Modal analysis was then performed with the identified prediction coefficients of the ARX model for the stationary vibration tests. The order of the ARX model (Eq. (1)) was set as $M = 12$. The reason for this choice is described in the following section.

Figure 9 shows the modal properties identified (vibration shapes and corresponding natural frequencies). In this figure, the thin black lines show the undeformed shapes linking the original positions of the acceleration pickups, and the thick red lines express the vibration shapes corresponding to the natural modes. Note that the deformation of the end panels is not displayed in the figure. Notations such as Z-10, S-11, etc. in Figs. 9, 11 and 12 represent features related to the vibration shapes. The prefixes S and A mean that the roof and floor deformed in the same and opposite directions in the longitudinal center of the carbody, respectively. Z is used when the directions of the roof and/or floor are not obvious, or when the amplitudes of the roof and floor are extremely different. J represents mode shapes with shear deformation in the cross section of the carbody shell. The numbers following the above
letters are composed of two parts. The former and latter represent the number of vibration loop(s) observed in the roof and floor structures, respectively. Note that the second number is omitted when the vibration shapes are related to J, since the roof and floor always have the same number of vibration loops in these vibration tests. Numerical values with units in Hz show natural frequencies related to the corresponding vibration modes.

The natural frequencies identified and the modal damping ratios based on the ARX model are summarized in Table 1, and are compared with those identified based on the conventional FFT method. The natural frequencies and damping ratios obtained using both methods are very close. Note that it has been confirmed that the mode shapes with both methods (mode shapes obtained by FFT are omitted due to the space limitations of this paper) are also very similar, so we can conclude that the proposed method is appropriate.

Table 2 shows the natural frequencies identified with different-order ARX models. Figures in parentheses show that the natural mode is insufficiently separated from that of another mode. In this study, the dependency or correlation between modes was checked with respect to MAC (Modal Assurance Criterion)\(^{(14)(15)}\), and parentheses are used for frequencies with MAC values larger than 0.3 in Table 2.

Since the natural frequencies of modes Z-10 and J-1 are close for this test vehicle, these two modes are not separated clearly in the case where the order \(M\) is small. The order should therefore be \(M \geq 12\) in this case to enable identification of the modal parameters corresponding to these two modes separately in accordance with Table 2. In general, although a small-order \(M\) value is desirable from the viewpoint of calculation cost, it is expected that a relatively large-order value will enable more accurate identification of the modal parameters; the upper limit of the value of \(M\) can be determined using \(\Delta S_{err21}\). Additionally, the number of eigenvalues increases according to the value of \(M\), and fake modes tend to appear when the value is large. It is confirmed that the focused modal properties can be identified even in such cases by observing the damping ratios and/or \(\text{RPF}^{(16)}\).

5. Analysis of Multiple-input Data under Running Conditions

This section describes modal analysis performed with the data measured in the running tests. The tested car, which is similar but different to the one shown in Fig. 1, has a stainless steel carbody, interior elements and underfloor equipment. In Japan, ordinary railway passenger vehicle carbodies are supported by two bogies, each of which has two wheelsets (a wheelset consists of a two-wheel pair and an axle), so a railway vehicle runs on eight
It is known that the vertical vibration of a running railway vehicle is strongly related to vertical track irregularities\(^{(17)}\). It is therefore expected that the vertical accelerations of the wheels have much higher rigidity than rubber tires for automobiles. Accordingly, in this study, the vertical accelerations of the axleboxes were considered as carbody input signals. Additionally, because railway vehicles run on a pair of rails (right and left), the axlebox acceleration values on the same rail can be regarded as the same signals with certain time delays. A set of time series data, with a total length of \(N = 10,000\) (50 s) measured under running conditions at constant velocity of 83 km/h, was used for the analysis. The average estimation errors for two different parts, \(S_{\text{err}1}\) for first \(N_1 = 6,000\) samples (30 s) and \(S_{\text{err}2}\) for the following \(N_2 = 4,000\) samples (20 s) of the data are calculated to determine the order of the prediction coefficients for the ARX model. Figure 10 shows the calculated average estimation errors and their difference \(\Delta S_{\text{err}21} = S_{\text{err}2} - S_{\text{err}1}\). Note that the prediction coefficients were identified with the first \(N_1\) samples of the data. Both \(S_{\text{err}1}\) and \(S_{\text{err}2}\) decrease as the order \(M\) of the ARX model increases.

Table 1  Comparison of natural frequencies and modal damping obtained using the proposed and conventional methods

<table>
<thead>
<tr>
<th>Mode</th>
<th>Proposed (ARX)</th>
<th>Conventional (FFT)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Natural frequency (f_i), Hz</td>
<td>(\zeta_i), %</td>
</tr>
<tr>
<td>Z-10</td>
<td>8.27</td>
<td>2.93</td>
</tr>
<tr>
<td>J-1</td>
<td>8.28</td>
<td>1.13</td>
</tr>
<tr>
<td>Z-20</td>
<td>9.28</td>
<td>3.04</td>
</tr>
<tr>
<td>Z-30</td>
<td>10.64</td>
<td>2.32</td>
</tr>
<tr>
<td>Z-40</td>
<td>14.47</td>
<td>1.33</td>
</tr>
<tr>
<td>A-31</td>
<td>14.90</td>
<td>1.20</td>
</tr>
</tbody>
</table>

Table 2  Natural frequencies versus the order of the ARX model

<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural frequency Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Order of ARX model (M)</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td>Z-10</td>
<td>(8.88)</td>
</tr>
<tr>
<td>J-1</td>
<td>(8.38)</td>
</tr>
<tr>
<td>Z-20</td>
<td>(8.38)</td>
</tr>
<tr>
<td>Z-30</td>
<td>10.86</td>
</tr>
</tbody>
</table>

Fig. 10  Average estimation errors for the two parts of the data measured during running at 83km/h and their difference.
increases. We also observe a tendency whereby $\Delta S_{err21}$ increases in the range where $M$ is larger than a certain value (around 10 in this case). Thus, the order is determined as $M = 10$ in the modal analysis for the running conditions.

Figure 11 shows the modal properties identified with the accelerations in the first $N_1 = 6,000$ samples (30 s) of the data measured under the conditions of running at 83 km/h. Note that only the shapes of the carbody are shown in Fig. 11, although the accelerations on the bogies were also measured in the running tests and used in the modal analysis.

Figure 12 shows the modal properties identified with the data (60 s) acquired in the stationary vibration tests for the same car in Fig. 11. The order of the prediction coefficients is determined as $M = 12$ according to the similar investigation above with $S_{err1}$, $S_{err2}$ and $\Delta S_{err21}$ in Fig. 10.

The mode shapes and natural frequencies in Figs. 11 and 12 are almost identical except for the fact that mode J-1 is not identified in the stationary vibration tests (Fig. 12). Note that the differences in the natural frequencies of the corresponding modes between both conditions are less than 3%. The absence of mode J-1 in the stationary vibration tests may be caused by the choice of the excitation point; that is, the exciter was set at the very center of the carbody.

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Fig. 11 Mode shapes and natural frequencies of a commuter vehicle as obtained from a running test. ($V = 83$ km/h)

Fig. 12 Mode shapes and natural frequencies of a commuter vehicle as obtained from stationary vibration testing.
(a node of the mode J-1) in the stationary test.

Note that it has been confirmed that the acceleration PSD on the floor (not shown in this paper) under running conditions implies that mode J-1 influences ride comfort. It should therefore be understood that some important modes may be dropped from the results obtained according to the excitation position. On the other hand, it is thought that such problems can be avoided by performing modal analysis with the data acquired in the running test because these data contain all modes that affect ride comfort under running conditions.

Since the excitation input during running is determined by the track conditions and cannot be set artificially, it is not always possible to acquire acceleration data including bending vibration with sufficient length. The proposed method discussed above enables identification of modal properties using measurement data with a relatively short length under running conditions.

6. Conclusions

This paper reports on a study of modal property identification for railway vehicles using a linear prediction model (LPM) with multiple-input and multiple-output (MIMO). Determination of an appropriate model order (i.e., the order of the prediction coefficients for the LPM model) is specifically discussed from the viewpoint of practical use. The authors proposed an average estimation error ratio calculated using the ratio of the root-mean-square value of the estimated error to that of the measured output to determine the model order. A remarkable feature of the proposed determination process is the employment of average estimation error ratios for two different parts of the analyzed data – one for identification of the prediction coefficients and the other to check the validity of the results obtained. It was confirmed that the difference between the average estimation error ratios for the two parts was effective in estimating the upper limit of the model order.

Modal properties can be identified successfully using the proposed method from data measured in both stationary and running tests. The characteristics identified in stationary tests (single-input condition) were compared with the results based on conventional FFT base procedures, and then with the results of the running tests (multiple-input condition). The model-order determination process has previously been dependent on trial and error, but this paper outlines a reasonable and practical method that can be used to perform the task directly. The proposed determination process improves the value of the analysis method based on the LPM as a tool to evaluate vibration in railway vehicles. We will accumulate vibration data for various sort of railway vehicle carboies and discuss the thresholds of the average estimation error ratios as the future work.

Acknowledgements

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