Fractal Dependence on Initial Conditions of Coupled Inverted Pendula Model of Competition and Cooperation

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Abstract
A coupled inverted pendula model of competition and cooperation is proposed to obtain a purely mechanical implementation of dynamics comparable with the Lotka-Volterra competition dynamics. It is shown numerically that the proposed model can produce the four equilibria that can be compared to ecological coexistence, dominance, and scramble. It is also shown that the proposed model exhibits fractal dependence on initial conditions. The result implies that selection of the equilibria will be uncertain under finite accuracy of knowledge on initial conditions.

Key words: Multi-agent System, Inverted Pendulum, Penalty Method, Fractal Dynamics, Uncertainty

1. Introduction

Competitive and cooperative dynamics can arise when multiple agents (autonomous entities) share common resources and environments. A large volume of research has been conducted on competitive and cooperative interactions. In the field of mathematical ecology, mechanical structures of each agent are eliminated to develop low degree-of-freedom models describing the nonlinear dynamics of group behavior (1). On the contrary, in the field of distributed autonomous robot systems (DARS), it is relatively difficult to develop the low degree-of-freedom models because the mechanical structures of each agent can not be eliminated because practical implementations of robots must be discussed (2), (3). In this manner, since the scopes of both considerably differ from each other, it seems that little attention has been given to standpoints between the two.

To provide a new insight on such problems, we have already proposed a coupled inverted pendula model (4) to obtain a low degree of freedom model in which competitive and cooperative behavior is directly caused by mechanical structures of individual agents. The proposed model consists of two inverted pendulums whose tips are jointed by the rigid link (4). Based on this model, it has previously been shown that the mechanically coupled agents can produce complex dependency on initial conditions and impulse responses, which can be interpreted as attack and counter-attack strategies of the agents (4).

In this paper, we investigate detailed structures of the dependency on initial conditions and impulse responses. Based on standard fractal analysis, we numerically demonstrate presence of complex basin of attraction having fractal self-similarity.

2. Mechanical System Representation of Competition and Cooperation

In order to obtain a mechanical competitive model in analogy with the ecological model (1), we consider the independent mechanical agents being coupled by kinematic chains (4), (5), as shown in Fig. 1.
2.1. Coupled inverted pendula model

For this purpose, we have previously proposed the coupled inverted pendula (CIP) model \(^{(4),(5)}\) as shown in Fig. 2. The CIP model consists of four points of mass, \(A_i, B_i\) \((i = 1, 2)\), where \(A_i\) represents the cart and \(B_i\) represent the tips of pendulum. It is assumed that the tips are connected by the rigid rod \(B_1B_2\). The PD control force \(u_i\) is applied to each cart \(A_i\) in the following form:

\[
u_i = u_{i}^{pd} := a(K \sin \theta_i + L \dot{\theta}_i \cos \theta_i) \quad (i = 1, 2)
\]

where \(K\) and \(L\) is a proportional and a differential gain respectively. The controller (1) acts as a PD controller with respect to the horizontal relative displacement between \(A_i\) and \(B_i\). Since this controller produces symmetric force with respect to horizontal line, by selecting \(K, L\) under \(l_0 > 2r\), all combinations of equilibria \(\theta_i = 0, \pi\) forms stable equilibria. In other words, the model in Fig. 2 exhibits quadra-stability consisting of the four steady states:

\[
(\theta_1, \theta_2) = \xi_1 := (0, 0), \quad \xi_2 := (0, \pi), \quad \xi_3 := (\pi, 0), \quad \xi_4 := (\pi, \pi).
\]

As in the previous paper\(^{(5)}\), we employ an analogy to explain the configuration \((\theta_1, \theta_2)\) as follows.

- \(\theta_i = 0 \cdots\) standing, \(\theta_i = \pi \cdots\) falling,
- \((\theta_1, \theta_2) = (0, 0) \cdots\) draw,
- \((\theta_1, \theta_2) = (0, \pi) \cdots\) left-win,
- \((\theta_1, \theta_2) = (\pi, 0) \cdots\) right-win,
- \((\theta_1, \theta_2) = (\pi, \pi) \cdots\) scramble.

Based on this analogy, we have previously proposed competitive strategies performed by each agent in the CIP model in order to generate a nonlinear dynamics comparable with attack and counter-attack games played by the agents\(^{(5)}\).

2.2. Penalty method

The motion of the model in Fig. 2 is determined by solving the differential algebraic equation (DAE). However, numerical methods for solving the DAE is still developing. For instance, standard Lagrange multiplier methods\(^{(6)}\) sometimes cause constraint violation problems. In our case, it happens that the joints \(B_i\) significantly separate during numerical integrations when applying impulsive forcing to the model as in §4. To solve this problem, in the previous paper\(^{(5)}\), we have derived the ordinary differential equations (ODE) equivalent to the original DAE by means of a penalty method as follows.

Replacing the rigid link \(B_1B_2\) in Fig. 2 with a flexible link of spring-damper with the natural length \(l_0\) as shown in Fig. 3, we obtain the ordinary differential equations that can approximate the original DAE when the stiffness and damping of the flexible link are sufficiently large. This kind of method of approximation is called a penalty method.
Let us derive the equation of motion for the model in Fig. 3. Introducing the generalized coordinates:

\[ \mathbf{q} = (x_1, \theta_1, x_2, \theta_2)^T \in \mathbb{R}^4 \]  

the control inputs \( u_1 \) and \( u_2 \) are expressed as the generalized force:

\[ \mathbf{F} = (u_1, 0, u_2, 0)^T \in \mathbb{R}^4. \]  

Evaluating the total energy of motion:

\[ T = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) + ma(\dot{x}_1 \dot{\theta}_1 \cos \theta_1 + \dot{x}_2 \dot{\theta}_2 \cos \theta_2) + \frac{ma^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2)}{2}, \]

and the total potential energy:

\[ U = 0.5 K_1 (l - l_0)^2 + mga(\cos \theta_1 + \cos \theta_2), \quad \ell^2 = (\Delta X)^2 + (\Delta Y)^2, \]

\[ \Delta X = (x_2 - x_1) + a(\sin \theta_2 - \sin \theta_1), \quad \Delta Y = a(\cos \theta_2 - \cos \theta_1), \]

we obtain the Lagrange function \( L = T - U \). The natural damping in the \( x_i \) and \( \theta_i \) directions and the damping of the flexible link \( B_1B_2 \) are expressed by the dissipation function:

\[ D = \frac{C_x (\dot{x}_1^2 + \dot{x}_2^2)}{2} + \frac{C_\theta (\dot{\theta}_1^2 + \dot{\theta}_2^2)}{2} + \frac{C_l}{2} \left( \frac{d(l - l_0)}{dt} \right)^2. \]

Substituting \( L, D, F \) into the Lagrange equations:

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = F_j, \quad (j = 1, 2, 3, 4) \]
we obtain the equations of motion of the model in Fig. 3 as follows.

\[
\begin{align*}
\dot{x}_i + C_i \ddot{x}_i &= -\frac{2}{mQ(\theta)} u_i + G_x(\theta) + C_i \frac{H_x(\theta)(l - l_0)}{(-1)^{i} \cdot l} + K_i \frac{2H_x(\theta)(l - l_0)}{(-1)^{i} \cdot l}, \\
\dot{\theta}_i + C_{\theta i} \ddot{\theta}_i &= -\frac{2 \cos \theta}{maQ(\theta)} u_i + G_{\theta}(\theta) + C_{\theta i} \frac{H_{\theta}(\theta)(l - l_0)}{(-1)^{i} \cdot l} + K_{\theta} \frac{2H_{\theta}(\theta)(l - l_0)}{(-1)^{i} \cdot l} & (i = 1, 2)
\end{align*}
\]

where

\[
\begin{align*}
I' := dl/dt, & \quad Q(\theta) := \cos 2\theta - 3, \\
G_x(\theta) := \frac{2 \sin (g(\cos \theta - a\dot{\theta}^2))}{Q(\theta)}, & \quad G_{\theta}(\theta) := \frac{2 \sin (-2g + a\dot{\theta}^2 \cos \theta)}{aQ(\theta)}, \\
H_x(\theta) := \frac{\sin (\Delta Y \cos \theta + \Delta X \sin \theta)}{mQ(\theta)}, & \quad H_{\theta}(\theta) := \frac{\Delta X \cos \theta - 2\Delta Y \sin \theta}{maQ(\theta)}.
\end{align*}
\]

It is clear that the flexible link in Fig. 3, B1B2, is almost rigid if \(K_1\) and \(C_1\) are sufficiently large. To achieve this rigid approximation, we empirically select \(K_1 = 5 \times 10^4\) and \(C_1 = 3\) for the penalty method. The other physical parameters are assumed to be \(m = l = 1, \ a = 0.3, \) and \(C_x = C_{\theta} = 0.1\). For numerical integrations, a fourth-order Runge-Kutta-Gill method is employed with the time step \(2 \times 10^{-3}\).

3. Transient Response of Coupled Inverted Pendula Model

3.1. Complex basin of attraction

Since the coupled inverted pendula model in Fig. 2 have a \((4 - 1) \times 2 = 6\) dimensional phase space, the phase trajectories would be too complex to visualize on a phase plane. In place of observing the trajectories themselves, Fig. 4 shows the color-coded area of the initial positions (\(\theta_1(0), \theta_2(0)\)) of the CIP model (9) converging to each of the equilibriums \(\xi_k (k = 1, 2, 3, 4)\). This kind of area of initial points belonging to each equilibrium is generally referred to as a basin of attraction. The initial positions are arranged on a uniform \(500 \times 500\) orthogonal grid within the plot area. In Fig. 4 (c”) with a plot range about \(0 \pi\), it is observed that the basin of attraction exhibits a multilayered structure. Furthermore, in Fig. 4 (c’), self-similarity of arcuate shapes is found in the white area placed at the end of
curved structures expanded from the center area where reduced-size copies of arcuate shapes repeat in this area.

If this complex basin of attraction is fractal, uncertainty of final position of the model is caused by finite accuracy of initial position of the model. Because, the plot range of Fig. 4 (c’’), about 0.1 rad, contains all types of basin so that it is impossible to determine which type of basin of attraction contains a given initial position if setting initial position is performed with an accuracy about 0.1 rad.
3.2. Correlation dimension

A dimension $D$ can be defined so that it takes $D = 1$ for a curve, $D = 2$ for a plane filled-figure, and $1 < D_2 < 2$ for a plane figure with self-similarity. Such a dimension is called a fractal dimension\(^7\). The fractal dimension enables us to evaluate complexity of fractal figures that cannot be smoothed by magnification. In order to calculate a correlation dimension, one of the most standard fractal dimensions, we define a correlation integral in the following form:

$$C_r = \frac{1}{N^2} \sum_{i,j=1}^{N} H(r - |x_i - x_j|)$$  \hspace{1cm} (11)

where $H$ is the Heaviside unit function:

$$H(x) = \begin{cases} 
1 & (x \geq 0) \\
0 & (x < 0) 
\end{cases}$$  \hspace{1cm} (12)

where $|x - y|$ is any norm. In order to save computational efforts, we use $1$-norm $\sum_{k=1}^{n} |x_k - y_k|$ where $n$ is a dimension of the vectors $x, y$ and $x_k, y_k$ represents $k$’th component of these vectors. Then, the exponent $D_2$ of the scaling law:

$$C_r \propto r^{D_2}$$  \hspace{1cm} (13)

is called a correlation dimension\(^7\). In practice, $D_2$ can be obtained numerically as a slope of double logarithmic plot of the cumulative frequency $C_r$ of $|x_i - x_j|$.

3.3. Fractal basin of attraction

Figure 5 shows the correlation integral $C_r$ for each color of basin in Fig. 4 (c’) where (a) is the result for white ($\xi_1$, draw), (b) for red ($\xi_2$, left-win), and (c) for black ($\xi_4$, scramble), respectively. The result is omitted for the blue basin that has the same shape as the red basin because of left-right symmetry of the model. The range of the scale $r$ is chosen to cover the linear part of $C_r$.

It is known that the linear part represents the range of scale in which self-similarity exists and that the slope of the linear part gives $D_2$. The obtained values of $D_2$ are listed in Table 1. The second low of this table represents an area ratio of each basin in Fig. 4 (c’).


Table 1  Correlation dimensions from Fig. 5

<table>
<thead>
<tr>
<th></th>
<th>Draw</th>
<th>Left-win</th>
<th>Scramble</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_2$</td>
<td>1.91</td>
<td>1.83</td>
<td>1.90</td>
</tr>
<tr>
<td>Area</td>
<td>20.7%</td>
<td>10.0%</td>
<td>59.3%</td>
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</table>

It is quantitatively identified that the multilayered structure found in Fig. 4 (c') is fractal because $D_2$ takes a fractional value about 1.8 to 1.9. Then, it is clear from Fig. 4 (c') and Table 1 that the area ratio is ranked as the scramble (black), draw (white), and each of win (red or blue) in descending order and that difference of the ratio ranges about 2 to 6.

It is also clear that there are areas filled by unicolor in a scale larger than the linear part of $C_r$ while self-similarity appears in a scale smaller than the linear part. For example, focusing on the white area, it is shown that the maximal point of linear part in Fig. 5 (a) is placed at the scale $10^{-0.36} \approx 0.44$, nearly a quarter of the horizontal range of Fig. 4(c'). This scale is smaller than that of the white area at the upper left of Fig. 4(c'). On the other hand, it also appears that arcuate white areas spread from the center area repeat their reduced-size copy in a self-similar manner. It can be said in average that this kind of multilayered structure exists in a quarter scale of the horizontal axis, which is in good agreement with placement of the linear part in Fig. 5 (a). In this manner, the maximal point of linear part represents maximal scale of presence of self-similarity, which is ranked as scramble (black), draw (white), and each of win (red or blue) in descending order. This ranking coincides with that of area ratio, and thus it seems that the maximal scale of presence of self-similarity tends to be determined relatively by the area ratio.

In addition, the linear part reaches $10^{-2}$ rad $= 0.57$ deg at least. This implies that final positions of the mechanism are hardly predictable if initial positions are given with that scale of accuracy.

4. Sensitivity to Strength of Impulse

Let us consider an open-loop control technique to switch the equilibriums generated by the PD controller $u_{pd}$. To achieve this, independent impulses are added to $u_1$ and $u_2$ (see Ref. (5) for details).

$$u_i = u_{pd}^i + P_i \delta(t) \quad (i = 1, 2)$$

where $\delta(t)$ is a Dirac’s delta function. As will be shown below, the fractal sensitivity discussed in previous section is also caused by selecting impulse strength ($P_1, P_2$).

Figure 6 shows the color-coded area of the impulse strength ($P_1, P_2$) where each color represents the final position $\xi_k$ ($k = 1, 2, 3, 4$) produced by the combination ($P_1, P_2$), which is arranged on a uniform $500 \times 500$ orthogonal grid. The rule of color-coding is the same as that in Fig. 4. Figure 6(a) shows the results for the PD gain $K = 100$ and $L = 10$. It is quite similar to the result in Fig. 4 (a) that the four equilibriums, the draw, scramble, left-win, and right-win, can be achieved by selecting the strength ($P_1, P_2$), as well as that the color-coded areas are distributed symmetrically about the line $\theta_1(0) + \theta_2(0) = 0$ ($\theta_1(0), \theta_2(0)$). It is also observed from Fig. 6 (b) and (b') that the smaller differential gain $L = 2$ produces multilayered structures with self-similarity.

Figure 7 shows the correlation integral $C_r$ for each color of basin in Fig. 6 (b') and Table 2 lists the correlation dimension $D_2$ and the area ratio for each area. All correlation dimensions take fractional values $1 < D_2 < 2$ so that each area is identified as fractal. Moreover, the area ratio is positively related to the maximal point of linear part (thick solid line) in Fig. 7. Note that the linear part represented as thin solid line in Fig. 7 (b) does not characterize fractal properties because its slope is around unity. It is also clear that the minimal point of linear part in Fig. 7 reaches a scale less than $10^{-2}$ Ns. This implies that the uncertainty of final positions may arise if an accuracy of applying the strength ($P_1, P_2$) is less than $10^{-2}$ Ns.
Fig. 6  Impulse strength \((P_1, P_2)\) belonging to the equilibria \(\xi_k\) \((k = 1, 2, 3, 4)\) for \(x_1(0) = \dot{x}_1(0) = \theta_i(0) = 0\) \((i = 1, 2)\), and \(x_2(0) > x_1(0)\). White, dark-gray, light-gray, and black regions represent the combination \((P_1, P_2)\) belonging to the draw, left-win, right-win, and scramble equilibria respectively.

Fig. 7  Correlation integrals \(C_r\) and their linear parts of the combination \((P_1, P_2)\) in Fig. 6 \((b')\), belonging to the draw as white \((a)\), left-win as dark-gray \((b)\), and scramble as black \((c)\) respectively.
Table 2 Correlation dimensions for impulse strengths from Fig. 7

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<tr>
<th></th>
<th>Draw</th>
<th>Left-win</th>
<th>Scramble</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_2$</td>
<td>1.55</td>
<td>1.71</td>
<td>1.86</td>
</tr>
<tr>
<td>Area</td>
<td>26.0%</td>
<td>8.9%</td>
<td>65.1%</td>
</tr>
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</table>

5. Conclusion

We proposed a low degree of freedom model in which competitive dynamics comparable to ecological dynamics is directly generated by the nonlinear mechanism of individual agents. We then performed a numerical experiment to obtain the following result.

- Sensitivity to the initial position and the impulse strength exhibits fractal properties with the correlation dimension satisfying $1 < D_2 < 2$.
- Consequently, uncertainty of final positions of the model may arise if the initial position and impulse strength is given with finite accuracy.

These results lead to the conclusion that correspondence from initial to final positions becomes uncertain in practical mechanical systems similar to the proposed model because in physical systems, it is hardly possible to provide initial conditions with infinite accuracy.

We plan to perform experiments to reproduce the competitive dynamics of the coupled inverted pendula model. We also plan to merge the differential game theory into our mechanically competitive framework. We also hope that our approach could provide new insights to the welfare engineering and related fields in the future.

References