A Regularization Method for a Stepwise Primary Diagnosis Method of a Beam Structure Using a Force Identification Technique*

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Abstract
In this study, first, the optimum adoption of singular values was investigated by a numerical example when the unknown parameters were the nodal forces of a finite element. It was shown that the distribution of the singular values depended on the sensor location and that accurate estimation results could be obtained by using one singular value in this example based on the condition number. Then the new expression of the virtual additional force was proposed. That is one of the regularization methods of inverse analysis, and the magnitude of the impulsive force is set as an unknown parameter. The numerical example showed that the estimated location of the abnormality determined by the proposed method agreed with that determined by the adoption of one singular value using the previous method in which the nodal forces were set as the unknown parameters. Also, the numerical and the experimental results showed that the accurate estimated location of the abnormality could be obtained by adopting one singular value using the proposed method. As a result, it was shown that the proposed method could be used in the actual application.

Key words: Model-Based Diagnosis Approach, Inverse Analysis, Regularization, Force Identification, Modal Analysis

1. Introduction
Machine condition monitoring and diagnosis have become increasingly important, and the application of these processes to beam structures and rotating machinery has been widely investigated. At the early stage of diagnosis, abnormal data are encountered, and a primary diagnosis is required to identify the location and cause of the abnormality.

There have been many studies regarding the primary diagnosis of abnormality. Some of them use a knowledge-based approach, i.e., an expert system in which fault-symptom matrices, if-then rules, fuzzy logic or neural networks are used. Others use a model-based approach in which the abnormal response is calculated from a mathematical model having a certain cause of abnormality, and the residual between the measured and simulated response is checked; the correct cause of abnormality can then be identified as the cause leading to a minimum residual. Many studies use a model-based approach to the crack diagnosis of a beam (1,2) and to the crack or unbalance diagnosis of a rotor system (3-5). The authors previously proposed a stepwise diagnosis method (6) that was a model-based approach in which the location of the abnormality was first estimated using the force identification approach. The point of the proposed method is that the abnormality is considered to be an
additional local force in the early stage of abnormality. After that, the cause of the abnormality was identified. A numerical example showed that the location and cause of the abnormality could be identified with sufficient accuracy. Then the authors developed a new diagnosis approach to increase the robustness of the stepwise primary diagnosis method (6) where the mathematical model was modified based on the difference of the response between the measurement and the simulation (7). The validity and applicability of the proposed method were shown with respect to the experimental data of a free-free uniform beam excited at the center of the beam.

In our previous studies (6),(7), to regularize the ill-posed problem of inverse analysis, some small singular values were truncated. In the studies, we used a beam element that had four nodal forces. Measuring the dynamic response at four points, the transfer function matrix between the unknown external forces and the measured responses was a $4 \times 4$ matrix. It had four singular values, and we adopted one singular value by checking the magnitude of the identified external force. But the optimum truncation order of the singular values was not clear. And when the proposed method is applied to a complex structure, a finite element has many nodal forces. This means that many sensors are required.

In this paper, first, the relationship of sensor location and the significant singular values is found using a numerical example, and the optimum truncation order of singular values is discussed. Then a new expression of the virtual external force is proposed in order for it to be applicable to complex structures. This is one of the regularization methods of the inverse analysis; the virtual external force is assumed to be an impulsive force. The applicability of the proposed method is also checked by numerical and experimental studies.

This paper will focus on the estimation of the location of abnormality. The identification of the cause of abnormality is omitted because the procedure is the same as in previous studies (6)-(7).

2. Stepwise fault diagnosis method

In this section, the stepwise diagnosis method (6),(7) is briefly explained.

2.1. Construction of the mathematical model

The structure to be diagnosed is modeled as a structure without damping. In this diagnosis system, a vibration test will be performed at regular intervals to monitor the health of the structure. The external force for the vibration test can be measured, and the responses are measured at several points. The mass matrix $[M]$ and the stiffness matrix $[K]$ of the structure under normal conditions were constructed in advance using the FEM.

A harmonic excitation with frequency $\omega$ is considered as the external force. The equation of motion and the output equation under the normal condition are, respectively, as follows:

$$([K] - \omega^2[M])X_n(\omega) = [F(\omega)] , \quad \{X_n(\omega)\} = [C]\{X_m(\omega)\} ,$$

where $\{X_n(\omega)\}$ and $\{F(\omega)\}$ are the magnitude of the response and the external force, respectively. $\{X_m(\omega)\}$ is the magnitude of the measured response, and $[C]$ is the coefficient matrix which expresses the measurement positions.

2.2. Estimation of the location of abnormality

In the abnormal condition, the response changes because the mass and/or stiffness matrices change due to abnormality, even if the same external force acts on the structure in the vibration test. However, the change in the response is considered to be the result of a virtual additional external force, $\{F_a(\omega)\}$, exerted on the normal structure as follows:

$$([K] - \omega^2[M])X_n(\omega) = [F(\omega)] + \{F_a(\omega)\} ,$$

where $[F(\omega)]$ is the external force applied during the vibration test.
where
\[
\{X_0(\omega)\} = \{X_n(\omega)\} + \{\Delta X(\omega)\}, \quad \{\Delta X_n(\omega)\} = [C]\{\Delta X(\omega)\}.
\] (3)

Moreover, an abnormality in the early stage occurs locally in the structure, so the additional external force is considered to act only on the \(i\)-th element of the finite element model. The force elements can be identified using the relationship
\[
\{\Delta X_n(\omega)\} = [H_i(\omega)]\{F_0(\omega)\},
\] (4)

where \([H_i(\omega)]\) is a compliance matrix between the measured responses and the assumed external force. The dimension of \([F_0(\omega)]\), \(n_r\), depends on the type of the finite element. When a beam element is used, the dimension of \([F_0(\omega)]\) is 4. And the dimension of \([\Delta X_n(\omega)]\), that is the number of sensors, \(n_s\), has to be equal to or more than the one of \([F_0(\omega)]\).

Here in the model-based fault diagnosis approach, a sufficiently accurate mathematical model has to be constructed to reproduce the dynamic response under the normal condition. The model update process can be applied, but a perfectly exact mathematical model cannot be obtained, and slight differences in the response cannot be avoided. In the previous study (7), the uncertainty in the mathematical model was considered as described below.

Under the normal condition, the measured magnitude of response is indicated as \(X_n(\omega)\) at the measurement point \(S_j\) \((j = 1, \cdots, n_s)\). Similarly, the simulated magnitude of response obtained by the mathematical model is indicated as \(X_0(\omega)\). Here the response ratio is defined as follows:
\[
r_j(\omega) = X_n(\omega)/X_0(\omega) \quad (j = 1, \cdots, n_s).
\] (5)

Under the abnormal condition, the response change \(\Delta X_n(\omega)\) is measured, and to adjust it to the mathematical model, we modify it by multiplying the ratio \(r_j(\omega)\) as follows:
\[
\Delta X_n(\omega) = r_j(\omega) \cdot \Delta X_0(\omega).
\] (6)

In the ideal case, the measured and the simulated responses are quite equal, so that Eq. (4) is written as follows:
\[
\{\Delta X_i(\omega)\} = [H_i(\omega)]\{F_0(\omega)\}.
\] (7)

But by using the relationship expressed in Eq. (6), Eq. (7) is rewritten as follows:
\[
\{\Delta X_n(\omega)\} = [R(\omega)]^{-1}[H_i(\omega)]\{F_0(\omega)\} = [\overline{H}_i(\omega)]\{F_0(\omega)\},
\] (8)

where
\[
[R(\omega)] = \begin{bmatrix} r_1(\omega) & 0 \\ r_2(\omega) & \ddots \\ 0 & \cdots & r_n(\omega) \end{bmatrix}.
\] (9)

In the actual practice of solving Eq. (8), the compliance matrix \([\overline{H}_i(\omega)]\) is often ill-posed, and thus the truncation of small singular values is needed. Then \([F_0(\omega)]\) is identified, and \([\Delta X_n(\omega)]\) can be obtained.

The next objective function, \(J_i\), is calculated for every element number \(i\), as follows:
\[
J_i = \| \Delta X_n(\omega) \| - \| \Delta X_m(\omega) \|,
\] (10)

where \(\| \cdot \|\) means the Euclid norm. It is considered that the abnormality will occur at the element where \(J_i\) is significantly small.

The vibration test will be carried out for several frequencies by a random excitation test (7), so that the location of the abnormality is identified for every frequency, and the mean
value and the standard deviation are calculated. Therefore, the location of the abnormality can be estimated with reliability.

Even if the multiple abnormal elements exist, the equivalent location of the abnormality can be estimated. But the estimated location has little useful information to diagnose the structure. Other approaches have to be used to diagnose the multiple abnormal elements.

2.3. Identification of the cause of abnormality

The location of the abnormality has been estimated in Sec.2.2 as the \( I \)-th element, then we consider some causes of the abnormality. One cause of the abnormality is set as \((j)\). A mathematical model with the cause of abnormality \((j)\) in the \( I \)-th element is constructed. The mass and stiffness matrices are expressed as \([M'_{ij}]\) and \([K'_{ij}]\), respectively. When the external force for the vibration test acts on the mathematical model, the response \(\{X'_{ij}(\omega)\}\) is calculated using the equation of motion,

\[
([K'_{ij}] - \omega^2[M'_{ij}])\{X'_{ij}(\omega)\} = \{F(\omega)\},
\]

and the response changes at the measurement points are obtained as follows:

\[
\{\Delta X'_{ij}(\omega)\} = [C]\{\{X'_{ij}(\omega)\} - \{X_{ij}(\omega)\}\},
\]

where \(\{X_{ij}(\omega)\}\) is the normal response obtained by using the mathematical model.

The next objective function, \(J_2\), is calculated as

\[
J_2 = \|[R(\omega)\{\Delta X_{in}(\omega)\} - \{\Delta X'_{in}(\omega)\}]\|,
\]

and the cause of abnormality is identified when \(J_2\) is significantly small.

3. Improvement of the expression of the virtual external force

In our previous papers (6),(7), a beam structure was considered. When the mathematical model is constructed using a beam element, there are four nodal forces \((n_f = 4)\), that is, the transverse and the rotational forces at both ends. Therefore, if we measure the dynamic response at four points, that means we use four sensors \((n_s = 4)\), the compliance matrix \([H_i(\omega)\] in Eq. (8) is a \(4 \times 4\) matrix, and the inverse problem can be solved. When we consider a plate structure and use a triangle element, there are nine nodal forces, that is, the transverse force in the out-of-plane direction and the rotational forces around two axes at each node. This means that nine sensors are required.

We would like to use a few sensors for the primary diagnosis, so we propose a new expression of the virtual external force. The point is that an impulsive force is considered as the virtual external force, and the location and the magnitude of the external force are set as unknown parameters. The proposed method can be applied for various types of the finite element, and in this study, the concrete procedure when the four sensors are used for a beam element is described as follows.

The external force \(Q(x)\) is generally transformed into the nodal forces by the FEM using the shape function \(\{L(x)\}\) as follows:

\[ F_{xa}^i \cdot \delta(x_{ai}) \]

Fig.1 Impulsive force that acts on the element
\[
\{F\} = \int_0^1 \{L(x)\}Q(x)dx,
\]
where \(\{L(x)\}\) is given as
\[
\{L(x)\} = \{L_0(x) \ L_1(x) \ L_2(x) \ L_3(x)\}^T,
\]
\[
L_0(x) = 1 - 3(x/h)^2 + 2(x/h)^3, \quad L_1(x) = h[(x/h)^2 - 2(x/h)^3 + (x/h)^5]
\]
In the proposed method, the external force is assumed to be an impulsive force that acts at \(x'_a\) in the \(i\)-th element as shown in Fig. 1, i.e., \(Q(x) = F'_m \cdot \delta(x'_a)\). The unknown external force \(\{F_m(\omega)\}\) in Eq. (8) is replaced with \(\{F'_m(\omega,x'_a)\}\), which is defined as
\[
\left\{F'_m(\omega,x'_a)\right\} = \int_0^1 \{L(x)\}F'_m(\omega)\delta(x'_a)dx = \{L(x'_a)\}F'_m(\omega),
\]
where \(F'_m(\omega)\) is the magnitude of the impulsive force. When the location of the external force \(x'_a\) is discretely given, the unknown parameter is only \(F'_m(\omega)\). So Eq. (8) becomes
\[
\left\{ \Delta X_m(\omega) \right\} = [\Pi(\omega)]\{L(x'_a)\}F'_m(\omega) = [\Pi(\omega)]F'_m(\omega),
\]
where \(\{\Pi(\omega)\}\) is the 4×1 compliance vector. This procedure can be applied to a complex structure; even for the plate structure, the unknown parameter is only the magnitude of the impulsive force.

The solution of Eq. (17) can be obtained by the singular value decomposition as follows,
\[
F'_m(\omega) = V[B\Pi[U]^T \{\Delta X_m(\omega)\}]
\]
where \([B]\) is the 1×4 vector whose first element is the inverse of the singular value and the others are zero, and \([U]\) is the base matrix. \(V\) is a score because \([B]\) is the 1×4 vector, but it is generally the base matrix, too. The advantage of the proposed method is that the compliance matrix \([\Pi(\omega)]\) has four singular values so that the truncation order must be discussed; however, the new method is used, and there is one singular value so that the discussion of the truncation order can be avoided. Moreover, even if a finite element used for modeling has many nodal variables, the unknown parameter is only the magnitude of the impulsive force, thus, the inverse problem can be easily solved.

4. Experimental instrument and mathematical model

In this study, the relationship of sensor location and the significant singular values is discussed using a numerical example, and the new expression of the virtual external force is investigated using a numerical example and the experimental data. In this chapter, the experimental instrument and the mathematical model are shown.
4.1. Experimental instrument and normal displacement

The structure to be diagnosed is a free-free uniform beam whose length $l$ is 800.0 mm, width $w$ is 16.0 mm and thickness $t$ is 8.0 mm, as shown in Fig. 2. The beam is excited at the center with random noise to 100 Hz by an exciter (IMV: m030/MA1) for the regular vibration test. The beam and the attachment are bonded together with an adhesive. The external force is measured by a load cell (PCB: 208C01), and the accelerations at four points $S_i$ ($i = 1, \cdots, 4$) are measured by accelerometers (PCB:353B15), as shown in Fig. 2.

The compliances under the normal condition at four points near the first natural frequency are shown in Fig. 3, where the frequency resolution is 0.125 Hz. These data are obtained by the transformation of acceleration data and normalized by the magnitude of external force measured by the load cell.

4.2. Construction of the mathematical model

To construct the mathematical model by using the FEM, we divide the beam into 160 beam elements so the length of one element $h$ is 5.0 mm. The density $\rho_0$ and the Young's modulus $E_0$ of the beam are $7.894 \times 10^3$ kg/m$^3$ and 213.84 GPa, respectively, as determined by actual measurement. The mass of the joint part between the beam and the piezoelectric sensor $m_{jp}$ is 18.27 g as determined by measurement. The width of the attachment is 10.0 mm, which coincides with the length of two elements. In this paper, the density of the two elements of the beam with the attachment, which is called the joint element, changes as follows:

$$
\rho_s = \rho_0 + \frac{1}{2} \frac{m_{jp}}{\Delta V} \left( \Delta V = w \times t \times h \right).
$$

(19)

Fig.3 Compliance obtained by the experiment under the normal condition

Fig.4 Compliance obtained by the model updated mathematical model
As the displacement is normalized by the magnitude of the external force, the magnitude of the external force is set as 1.0 N. The external force acts on the joint elements, so the distributed external force for unit length \( q \) is obtained as follows:

\[
q = \frac{1.0}{2h} = 100.0 \text{ N/m.}
\]  

(20)

4.3. Model update and the response ratio

The mathematical model has to be modified so that the simulated response will agree with the measured one. In this study, the stiffness in the joint element is modified. The Young's modulus of the joint element is changed as follows:

\[
E_a = \gamma_E \times E_0.
\]

(21)

The parameter \( \gamma_E \) is determined so as to minimize the residual of the response in the frequency range except near the resonance frequency, i.e., from 60.000Hz to 64.750Hz and from 66.125Hz to 70.000Hz (7). The optimum parameter is obtained as \( \gamma_E = 0.937 \). The compliances using the updated mathematical model are shown in Fig. 4, and this agrees well with Fig. 3. The ratios \( r_j(\omega) \) can then be calculated.

5. Consideration of adopted singular values using a numerical example

In this chapter, the relationship of the sensor location and the significant singular values is discussed using a numerical example.

5.1. The case in which four elements of the external force are unknown parameters

5.1.1. Singular values and the condition number

The setting of the diagnosis approach is the same as in previous research (6a)(7). In this study, the behavior of the singular values is the focus, and the objective is to determine the optimum truncation order of the singular values when the structure to be diagnosed and the sensor locations are given.

As an example, the singular values \( \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) are shown in Fig. 5 in the case of \( \omega = 63.0 \text{Hz} \). The horizontal axis represents the element number of the virtual external force. The singular values show characteristic changes at the element numbers 21, 61, 111 and 151, which are the locations of the sensors. When the virtual external force is assumed between the 61st and the 110th element, the four singular values are significant values. When it is assumed between the 21st and the 60th element or between the 111th and the 150th element, three singular values are significant values. And when it is assumed between the 1st and the 20th element or between the 151st and the 161st element, two singular values are significant values. Therefore, according to the behavior of the singular values, the truncation order of the singular value is dependent on the location of the virtual

Fig.5 Distribution of singular values at the excitation frequency \( \omega = 63.0 \text{ Hz} \)
external force. The condition number, which is defined as the ratio between the maximum and the minimum singular value, is then determined as shown in Fig. 6. In the figure, for example, \( \frac{\lambda_1}{\lambda_2} \) shows the condition number when \( \lambda_1 \) and \( \lambda_4 \) are truncated. From the figure, when the virtual external force is assumed to be between the 61st and the 110th element and four singular values are adopted, the condition number is about \( 10^{7.2} \), which is too large for actual application. Even if two singular values are adopted, the condition number is about \( 10^2.0 \) in average, but this is allowable in the actual application.

As a result, for this example, the optimum truncation order of the singular value is one or two as determined from the condition number.

5.1.2 Estimation of location of abnormality

Additional mass is considered as an example of an abnormal condition (6),(7). In the experiment, the small weight shown in Fig. 7, whose properties are shown in Table 1, is attached to the beam. To discuss the truncation of the singular value by a numerical example, the lightest weight W03 is considered and attached at the 50th, 100th or 140th element on the beam, and the normal and the abnormal response are assumed to be ideal data without noise.

Figure 8 shows the objective function \( J_1 \). In the figure, for example, the blue line of \( \lambda_1 - \lambda_2 \) shows the result when \( \lambda_1 \) and \( \lambda_4 \) are truncated. Here \( J_1 \) is normalized by \( \| \Delta X_\omega \| \). In the case of Fig. 8(a), the additional mass is at the 50th element, where three singular values are significant as shown in Fig. 6. From the figure, it is recognized that the
abnormal data can be accurately reconstructed using three singular values whenever the virtual external force is set between the 20th and the 60th element, so that the location of the abnormality cannot be estimated. Therefore, two singular values should be adopted in this case.

In the case shown in Fig. 8(b), the additional mass is at the 100th element, where four singular values are significant, as shown in Fig. 6. Based on the same considerations as in Fig. 8(a), three singular values may be adopted. From the figure, however, one or two singular values have to be adopted because the condition number $\lambda_1/\lambda_2$ is large in this case.

In the case of Fig. 8(c), the additional mass is at the 140th element. Two singular values may be adopted as in the case of Fig. 8(a), but one singular value has to be adopted.

The condition numbers $\lambda_1/\lambda_2$ at the 50th, the 100th and the 140th elements are 74.1,
247.7 and 91.9, respectively, from Fig. 6. These condition numbers are almost same but in the case of the 50th element of the additional mass, the accurate location can be obtained using two singular values. Therefore, the threshold of the condition number may be considered to be about 80 in this example. In the actual case, we do not know the correct location of the abnormality, so the adoption of one singular value is feasible.

5.2. The case in which one element of the external force is an unknown parameter

The virtual external force is expressed as in Eq. (16). As an example, the singular value
is shown in Fig. 9 in the case of $\omega=63.0$Hz. The horizontal axis is the location of the impulsive force that is set at 10 points in one element. In the figure, the largest singular value in Sec. 5.1 is also shown. The singular values show almost the same behavior, so that it is considered that the method in which one element of the external force is an unknown parameter corresponds to the adoption of one singular value in the case in which four elements of the external force are unknown parameters.

The estimated location of the abnormality is shown in Fig. 10. The behavior of $J_1$ is almost the same as in the case of the adoption of one singular value when four elements of the external force are unknown parameters.

Therefore, the new expression of the virtual external force can be used in the actual application because it is applicable to a complex structure if one singular value is used in the inverse analysis. For other excitation frequencies, it is confirmed that the same results can be obtained.

6. Diagnosis using the experimental data

The new expression of the virtual external force is checked using the experimental data. The actual abnormality is the additional mass, whose properties are shown in Table 1 on the 50th, 100th and 140th elements on the beam. As an example of the estimated location of the abnormality, the results in the case of $\omega=63.0$ Hz and W03 are shown in Fig. 11. Figures 11(a), (b) and (c) are the results for four unknown parameters corresponding to Sec.5.1.2, and Fig.11 (d) is the result for one unknown parameter corresponding to Sec. 5.2. The magnitudes of the objective functions in Fig.11 are larger than the ones in Figs.8 and 10, because the numerical example were carried out using the data without noise while the experimental data has some errors. From Figs. 11(a)-(c), the accurate location of the abnormality can be obtained by using one singular value, and the adoption of two singular values leads to an inaccurate result because of the large condition number. From Fig. 11(d), the accurate results are found by using the new expression of the external force in which there is only one unknown parameter; there is one singular value. For other excitation frequencies, we can obtain the same results.
The final result for the location of the abnormality is determined as a mean value of the results of various excitation frequencies \(^{(7)}\), as described in Sec. 2.2. As an example of the estimation result, the distribution of the estimated location of the virtual additional external force is shown in Fig. 12 in the case of W03 at the 100th element on the beam. Then the location of the abnormality is determined as a mean value of the estimated location, i.e., 100.2, and the final result is 100. For all cases, the results of the location of the abnormality are shown in Table 2. From the table, when the abnormality occurs at the 100th element, the estimated results are sufficiently accurate for all weights. In that case, the maximum error of the location of the abnormality is 10.0 mm. When the weight is attached at the 50th or 140th element, the maximum error is 25.0 mm and it is considered to be allowable.

From these results, it is recognized that the proposed regularization method for inverse analysis can be used in the actual application.

### 7. Conclusions

In this study, first, the optimum adoption of singular values was investigated by a numerical example when the unknown parameters were the nodal forces of a finite element. It was shown that accurate estimation results could be obtained by using one singular value in this example based on the condition number. A new expression of the virtual additional force was then proposed. This is one of the regularization methods of inverse analysis, and the magnitude of the impulsive force is set as an unknown parameter. The numerical and experimental results showed that the accurate estimated location of the abnormality could be obtained by the adoption of one singular value using the proposed method. As a result, it was shown that the proposed method was feasible for actual application.

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