Control Law Design Based on the Polynomial Method for Active Damping of Oscillatory Modes
–The Application of the Delta Operator to the Polynomial Method–

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Abstract
The polynomial or algebraic method based on the transfer function can achieve simultaneously pole and zero placement. Since the method has clear physical interpretation, it is easy to introduce many different constraints such as vibration modes and the operation limit of the actuator. However, when the controlled plant is high-order, numerical problems arise in the solution of a Diophantine equation or a Sylvester matrix. In order to make the conventional polynomial method practicable, this paper presents an improved polynomial method based on the modified delta operator δ for the microprocessor control. The modified delta form can significantly improve numerical conditioning of a Sylvester matrix on pole and zero assignment in controller design. As an example of a controlled plant, a 10th-order plant is considered. Then, the numerical sensitivity of a Sylvester matrix reduces at one millionth of the conventional one obtained on the z-plane. The simulated results show that the proposed method can be applied to a high-order plant and a robust controller can be developed. Therefore, the proposed method would be useful for the computer controlled plants in industry.

Key words: Vibration Control, Diophantine Equation, Sylvester Matrix, Delta Operator, Pole Zero Placement, Polynomial Method, Algebraic Method, Digital Control, Microprocessor Control

1. Introduction

The main advantage of the sate-space design is arbitrary pole placement with state feedback. When all the states cannot be sensed and fed back, observers can be used. The pole placement method is to arrange control law so that all poles of the closed-loop system are at desired positions. However, system response depends not only on closed-loop poles but also on closed-loop zeros. It is very difficult for usual state space method to allocate zeros to any desired position. Though zero assignment is possible by modifying usual state-space method, the procedures become involved and their physical interpretation is not clear.

On the other hand, the polynomial or algebraic method based on the transfer function can achieve pole and zero placement simultaneously and easily(1)–(6). Since the method has clear physical interpretation, it is easy to introduce many different constraints such as vibration modes ω, and the operation limit of the actuator in design procedure. Thus, the polynomial or algebraic method is suitable for vibration control. However, the pole and zero placement procedure based on the polynomial method includes a determination of two polynomials from a Diophantine equation or a Sylvester matrix. If the controlled plant is forth- or higher-order, numerical problems arise in the solution of a Sylvester matrix. Thus, it is very difficult for
controller design to apply the polynomial method to high-order plants due to the numerical problem\(^6\). Though numerical properties of the state-space design is widely studied, those of the polynomial method have not enough been studied\(^7\)–\(^10\).

By the way, as for the mechatronic systems by using microprocessors, control algorithms are in general expressed by the deference equations based on the shift operator \(z\) or the \(z\) transform. In order to damp high frequency oscillatory modes, the sampling period \(T\) must be shortened. In such situations the control algorithm based on the conventional shift form becomes numerically unstable. Word length becomes too short to express differences between polynomial coefficients at high-speed sampling. In order to avoid such numerical ill-conditions, it is well known that the polynomial representation based on the delta operator \(\delta\) is useful\(^11\)–\(^14\).

In this paper, in order to apply the polynomial method to high-order plants, an improved polynomial method is proposed. The contribution of this paper is to make the conventional polynomial method practicable so that the polynomial method can be widely used in the industry. This paper is organized as follows. Second-order plant is considered to describe problems that arise on a conventional polynomial method in § 2 and 3. In order to solve these problems, an improved polynomial method is proposed in § 4. This proposed method unifies the delta operator method with the polynomial method. The conventional delta operator is defined as \(\delta = (z - 1)/T\), where \(T\) is the sampling period. The modified delta operator \(\delta = (z - 1)/T_\delta\), where \(T_\delta\) is arbitrary positive constant called design parameter. The key idea is that the numerical sensitivity of a Sylvester matrix is reduced by adjusting design parameter \(T_\delta\) on the \(\delta\)-plain. As a result, the proposed method can be applied to controller design on high-order plants. The effectiveness of the proposed method is verified on second- and higher-order plants from the point of view of numerical sensitivity in § 5. The robustness of the controller designed based on the proposed method is verified on a 10th-order plant in § 6.

2. The problems on the conventional polynomial method by using continuous-time system design

2.1. Model matching

The two DOF controller is shown in Fig.1.

\[ H_p(s) = \frac{N_p(s)}{D_p(s)} \]  \hspace{1cm} (1)

It is a general configuration capable of arbitrary pole and zero assignment, where \(r\), \(y\), and \(d\) are the reference, the plant output, and the disturbance, respectively. The first approach is to design a continuous-time controller and transform it into a discrete-time one for digital control. Consider a proper plant \(H_p(s)\) and a stable desired closed-loop transfer function \(H_m(s)\).

Plant

\[ H_p(s) = \frac{N_p(s)}{D_p(s)} \]  \hspace{1cm} (1)
Model

\[ H_m(s) = \frac{N_m(s)}{D_m(s)} \]  

(2)

The closed loop transfer function \( H_c(s) \) is obtained from Fig.1.

\[ H_c(s) = \frac{L(s)N_p(s)}{A(s)D_p(s) + M(s)N_p(s)} \]  

(3)

Equating numerator and denominator terms determines the design equations.

\[ A(s)D_p(s) + M(s)N_p(s) = D_m(s)\bar{N}_p(s)A_o(s) =: \alpha(s) \]  

(4)

\[ L(s) = N_m(s)A_o(s)/N_l \]  

(5)

where

\[ N_p(s) = N_l s^j + \cdots + N_0 \]

\[ \bar{N}_p(s) = N^-(s) = \bar{N}_0 + \cdots + N_0/N_l \]

\( A_o(s) \) is an arbitrary polynomial called observer polynomial and the coefficient \( N_l \) of the highest power in \( N_p(s) \) is fixed to unity so that \( N_p(s) \) becomes a monic polynomial like \( \bar{N}_p(s) \). If the polynomial \( \bar{N}_p(s) \) is decomposed in two polynomials \( N^-(s) \) and \( N^+(s) \), where \( N^-(s) \) has its all stable plant zeros and \( N^+(s) \) has its all unstable plant zeros.

\[ \bar{N}_p(s) = N^-(s)N^+(s) \]

\[ = (s^j + \cdots + \bar{N}_0)N^+(s) \]

factor \( N^+(s) \) should not be included in \( \alpha(s) \), since closed loop poles are located on the right half complex plane and a closed-loop system becomes unstable. Since unstable plant zeros cannot be changed, the plant output \( y \) contains unstable zero responses. Thus, these unstable zeros should be included in the model \( H_m(s) \) and \( L(s) \) is divided by \( N^+(0) \) so that DC gain of the closed-loop system may not change.

\[ \bar{N}^- = s^j + \cdots + \bar{N}_0 \]  

(6)

\[ A(s)D_p(s) + M(s)N_p(s) = D_m(s)\bar{N}^-(s)A_o(s) =: \alpha(s) \]  

(7)

\[ L(s) = N_m(s)A_o(s)/(N_l N^+(0)) \]  

(8)

2.2. Controller output constraint

Actuators are generally nonlinear due to the saturation. Thus, the constraint on the controller output \( u \) or the actuating signal should be included in a controller design specification. This design procedure is very easy in a model matching. The following relationships are obtained from Fig.1.

\[ Y(s) = H_m(s) R(s) \]  

(9)

\[ Y(s) = H_p(s) U(s) \]  

(10)

From these equations the constraint is obtained.

\[ U(s) = \frac{H_m(s)}{H_p(s)} R(s) \]  

(11)
2.3. Design procedure

Consider following second-order systems.

Plant

\[ H_p(s) = \frac{N_0}{D_2 s^2 + D_1 s + D_0} \]  

(12)

Model

\[ H_m(s) = \frac{N_{m0}}{s^2 + D_{m1} s + D_{m0}} \]  

(13)

The transfer function on the disturbance \( d \) is obtained from Fig. 1.

\[ H_d(s) = \frac{A(s)N_p(s)}{A(s)D_p(s) + M(s)N_p(s)} \]  

(14)

A(s) appears in the numerator. Thus, in order to reject the step disturbance, \( A(s) \) must have the term \( s \) from the final value theorem in the Laplace transform. If the degree of \( A(s) \) is one, the degree of \( M(s) \) is one so that \( M(s)/A(s) \) becomes proper. In this case the term \( s^2 \) cannot be adjusted. Next, supposing that the degree of \( A(s) \) is two, continue the same procedure. As a result, the degree of both polynomials are determined as two. Then, the degree of \( L(s) \) becomes at most two so that \( L(s)/A(s) \) becomes proper. According above, \( A(s) \) and \( M(s) \) are represented as follows.

\[ A(s) = A_2 s^2 + A_1 s \]  

(15)

\[ M(s) = M_2 s^2 + M_1 s + M_0 \]  

(16)

Since the degree of the left side of Eq. (4) becomes four, the degree of the desired closed-loop characteristic equation \( \alpha(s) \) is four. Considering the degree of \( D_m(s) \) and \( N_p(s) \), the degree of the \( A_o(s) \) is determined as two. In order to find five unknowns in Eqs. (15) and (16), the coefficients of the powers of \( s \) in Eq. (4) are equated. A Sylvester matrix \( S_c \) is shown as follows.

\[ S_c X_{DN} = F \]  

(17)

where

\[ S_c = \begin{bmatrix} D_2 & 0 & 0 & 0 & 0 \\ D_1 & D_2 & 0 & 0 & 0 \\ D_0 & D_1 & N_0 & 0 & 0 \\ 0 & D_0 & 0 & N_0 & 0 \\ 0 & 0 & 0 & 0 & N_0 \end{bmatrix} \]

\[ \alpha(s) = s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0 \]

\[ X_{DN} = \begin{bmatrix} A_2 & A_1 & M_2 & M_1 & M_0 \end{bmatrix}^T \]

\[ F = \begin{bmatrix} 1 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix}^T \]

A second-order plant in which the input is an actuator current and the output is a position of controlled plant is considered, as shown in Fig. 2(15).
Plant

\[ H_p(s) = \frac{k_l/m}{s^2 + k_l/m s} \quad (18) \]

\( m = \) mass of controlled plant 0.100 kg
\( k_l = \) force constant 10.0 N/A
\( k_f = \) viscous friction coefficient 2.0 Ns/m

Model

\[ H_m(s) = \frac{\omega^2_m}{s^2 + 2\xi_m\omega_m s + \omega^2_m} \quad (19) \]

\( \xi_m = \) model damping ratio 1.0
\( \omega_m = \) model natural frequency 3000 rad/s

Since the degree of \( A_o(s) \) is two, there are two poles. Both poles are chosen as \(-40000\) from simulations.

\[ A_o(s) = (s + 40000)^2 \quad (20) \]

Also, \( L(s) \) is obtained from Eq. (5). In order to implement \( A(s) \), \( M(s) \), and \( L(s) \) as a continuous-time controller with two inputs \( r, y \) and one output \( u \), the following structure is adopted, as shown in Fig. 3. The both outputs of transfer functions \( M(s)/A(s) \) and \( L(s)/A(s) \) go to infinity in the case of the step input, since \( A(s) \) contains the pole located at \( 0 + j0 \).

Step responses \( y \) is shown in Fig.4. The disturbance \( d = -50N \) is also applied to the plant at 4ms.

Next, in order to implement \( A(s) \), \( M(s) \), and \( L(s) \) as a discrete-time controller with two inputs \( r, y \) and one output \( u \). First, \( M(s)/A(s) \) and \( L(s)/A(s) \) are digitized by using zero-order hold. Next, these two transfer functions are combined to make one controller by adopting structure as shown in Fig. 5.
Step response $y$ is shown in Fig. 6 (a) and (b). The disturbance $d = -50\text{N}$ is also applied to the plant at 4ms.

When the sampling period $T$ is 50μs, the control system is unstable. The plant output $y$ corresponds to the model one, when the sampling period $T$ must be set as 5μs or less. This instability is due to ignoring sampling process in the controller design and the approximation to the continuous-time controller. Thus, a digital controller based on the continuous-time design achieves good performance when the sampling period $T$ is very short. Since a digital controller needs a high-speed microprocessor, this design method is not suitable for an industrial application, considering power consumption and cost.

2.4. The evaluation of the numerical sensitivity based on the condition number

The condition number $C_n$ of a Sylvester matrix $S$ is defined by the ratio of the largest singular value of $S$ to the smallest one.

$$C_n = \frac{||S||}{||S^{-1}||} = \frac{\sigma_1(S)}{\sigma_{\text{min}}(S)} \quad (21)$$

A large condition number indicates a nearly singular matrix. In this paper, this condition number is adopted as a measure of the numerical sensitivity of a Sylvester matrix. The condition numbers $C_n$ are shown in Table 1.
3. The problems on the conventional polynomial method by using discrete-time system design

3.1. Design procedure based on the $z$-transform

The second approach is to transform a continuous-time plant to an equivalent step invariant discrete-time plant, as shown in Fig. 7. Then, a discrete-time controller is designed by using a polynomial method in the $z$-domain. The plant and the model Eqs. (12) and (13) are discretized by the zero-order hold.

![Fig. 7](Image)

Then, a discrete-time controller is designed by using a polynomial method in the $z$-domain.

Plant

$$H_p(z) = \frac{N_1z + N_0}{D_2z^2 + D_1z + D_0} \quad (22)$$

Model

$$H_m(z) = \frac{N_{m1}z + N_{m0}}{z^2 + D_{m1}z + D_{m0}} \quad (23)$$

As like the continuous-time design, the degrees of the $A(z)$ and $M(z)$ become two, respectively. Considering that the derivative term $s$ is expressed as $z - 1$ in the discrete-time systems, $A(z)$ and $M(z)$ are obtained.

$$A(z) = (z - 1)(\bar{A}_1z + \bar{A}_0) \quad (24)$$

$$M(z) = M_2z^2 + M_1z + M_0 \quad (25)$$

As a result, the degree of the characteristic equation $\alpha(s)$ is four. Since there appears one stable sampling zero in Eq. (22), this zero should be included in $\alpha(z)$. The degree of the observer polynomial $A_o(z)$ becomes one, when this sampling zero is considered. In the case of a very fast continuous-time pole $p_c$, the corresponding discrete-time pole $e^{p_cT}$ becomes nearly zero. Thus, $A_o(z)$ is set as follows.

$$A_o(z) = z - 0 \quad (26)$$

Thus, the characteristic equation $\alpha(z)$ is obtained.

$$\alpha(z) = \left( z^2 + D_{m1}z + D_{m0} \right) \left( z + \frac{N_0}{N_1} \right) A_o(z) \quad (27)$$

### Table 1  The comparison of the condition number

<table>
<thead>
<tr>
<th>$C_p(s)$</th>
<th>$C_p(z)$</th>
<th>$C_p(\delta)_{apl}$</th>
<th>$T_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.0851 \times 10^3$</td>
<td>$1.7160 \times 10^4$</td>
<td>$1.1666 \times 10^1$</td>
<td>$7.8778 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
In order to find five unknowns in Eqs. (24) and (25), the coefficients of the powers of $z$ are equated. The Sylvester matrix $S_z$ is shown as follows.

$$S_z X_{DN} = F$$  \hspace{1cm} (28)$$

where

$$S_z = \begin{bmatrix}
    D_2 & 0 & 0 & 0 & 0 \\
    D_1 - D_2 & D_2 & N_1 & 0 & 0 \\
    D_0 - D_1 & D_1 - D_2 & N_0 & N_1 & 0 \\
    0 - D_0 & D_0 - D_1 & 0 & N_0 & N_1 \\
    0 & 0 & 0 - D_0 & 0 & 0 & N_0
\end{bmatrix}$$

$L(z)$ is obtained from following equation.

$$L(z) = \frac{N_m(z)A_0(z)}{N_1}$$  \hspace{1cm} (29)$$

Step response $y$ at sampling period 50μs is shown in Fig. 8. The disturbance $d = -50N$ is also applied to the plant at 4ms.

![Step response](image)

Even when the sampling period $T$ is 50μs, the control system is stable. Thus, the discrete-time design method is suitable for an industrial application.

3.2. The evaluation of the numerical sensitivity based on the condition number

The condition number of $S_z$ in making the sampling period $T$ be a parameter is shown in Fig. 9.

![Condition number](image)

As the sampling period $T$ becomes shorter, the condition number $C_n(z)$ increases exponentially. In addition, the solution must be calculated within the stability region or the unit circle on the $z$-plane. Therefore, the discrete-time design method is not suitable for high-speed sampling control systems.
4. The Proposal of the improved polynomial method based on the modified delta operator

In order to solve problems on both the continuous- and discrete-time system design, an improved design methodology is proposed. Since the delta form reduces the numerical instability drastically at high-speed sampling, the delta operator method is applied to the polynomial method. The conventional delta operator $\delta$ is defined, as follows:

$$\delta = \frac{z - 1}{T}$$ (30)

where $T$ is the sampling period. The modified delta operator $\bar{\delta}$ is defined as

$$\bar{\delta} = \frac{z - 1}{T_\delta}$$ (31)

where $T_\delta$ is a positive arbitrary constant called design parameter. This modified delta operator can also reduce the numerical instability like a usual delta operator. From Eq. (31) the inverse delta operation $\bar{\delta}^{-1}$ is obtained.

$$\bar{\delta}^{-1} = \frac{T_\delta}{z - 1}$$

$$= T_\delta \frac{z^{-1}}{1 - z^{-1}}$$ (32)

The block diagram $\bar{\delta}^{-1}$ is shown in Fig. 10.

![Fig. 10 The inverse delta operation](image)

Consider the following $p$th-order systems.

$$H(z) = \frac{\sum_{i=0}^{p} b_i z^i}{\sum_{i=0}^{p-1} a_i z^i + z^p}$$ (33)

From the definition Eq. (31), the following relationship is obtained.

$$z = T_\delta \bar{\delta} + 1$$ (34)

By substituting Eq. (34) into Eq. (33), the shift form $H(z)$ is converted to the modified delta form $H(\bar{\delta})$ like the usual delta transformation, as shown in Fig. 11.

$$H(\bar{\delta}) = \frac{\sum_{i=0}^{p} \bar{b}_i \bar{\delta}_i}{\sum_{i=0}^{p-1} \bar{a}_i \bar{\delta}_i + \bar{\delta}_p}$$ (35)

![Fig. 11 Pth-order modified delta form](image)
Equation (33) is easily converted to Eq. (35) by using a state space form. A transfer function is converted to a state space form by using MATLAB function \textit{ssdat}().

\[
\begin{align*}
    x(k+1) &= A_s x(k) + B_s u(k) \\
    y(k) &= C_s x(k)
\end{align*}
\] (36)

Equation (36) can be expressed by the shift operator \( z \).

\[
\begin{align*}
    z x(k) &= A_s x(k) + B_s u(k) \\
    y(k) &= C_s x(k)
\end{align*}
\] (37)

By substituting Eq. (34) into Eq. (37),

\[
\begin{align*}
    \bar{\delta} x(k) &= A_{\bar{\delta}} x(k) + B_{\bar{\delta}} u(k) \\
    y(k) &= C_{\bar{\delta}} x(k)
\end{align*}
\] (38)

is obtained. Equation (38) is a state space form on the modified delta plane.

\[
\begin{align*}
    \bar{\delta} x(k) &= A_{\bar{\delta}} \bar{\delta} x(k) + B_{\bar{\delta}} \bar{\delta} u(k) \\
    y(k) &= C_{\bar{\delta}} \bar{\delta} x(k)
\end{align*}
\] (39)

where

\[
\begin{align*}
    A_{\bar{\delta}} &= A_s - \frac{I}{T_{\bar{\delta}}} \\
    B_{\bar{\delta}} &= \frac{B_s}{T_{\bar{\delta}}} \\
    C_{\bar{\delta}} &= C_s
\end{align*}
\]

Equation (39) is converted to the transfer function \( H(\bar{\delta}) \) by using following relation or MATLAB function \textit{tfdata}().

\[
H(\bar{\delta}) = C_{\bar{\delta}}(\bar{\delta} I - A_{\bar{\delta}})^{-1}B_{\bar{\delta}}
\] (40)

The coefficients of \( H(\bar{\delta}) \) is defined as \( \bar{a}_i \) and \( \bar{b}_i \), when \( T_{\bar{\delta}} \) is unity. Then, the following relationships are obtained.

\[
\begin{align*}
    \bar{a}_i &= \frac{\bar{a}_i}{T^i_{\bar{\delta}}} \\
    \bar{b}_i &= \frac{\bar{b}_i}{T^i_{\bar{\delta}}}
\end{align*}
\] (41)

As \( T_{\bar{\delta}} \) approaches zero, the coefficients \( \bar{a}_i \) and \( \bar{b}_i \) approach continuous-time ones. On the other hand, as \( T_{\bar{\delta}} \) approaches \( \infty \), the coefficients \( \bar{a}_i \) and \( \bar{b}_i \) approach 0. Thus, the coefficients \( \bar{a}_i \) and \( \bar{b}_i \) can be adjusted so that the condition number \( C_n \) of a Sylvester matrix is minimized. Though \( C_n \) is not generally convex on a variable \( T_{\bar{\delta}} \), finding the optimal value \( T_{\bar{\delta}} \) is a very easy task by using MATLAB due to only one variable \( T_{\bar{\delta}} \).

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**Fig. 12** The two DOF controller based on \( \bar{\delta} \)
First, the plant and the model Eqs. (22) and (23) are converted into the modified delta form, as shown in Fig. 12.

**Plant**

\[ H_p(\delta) = \frac{N_1 \delta + N_0}{D_2 \delta^2 + D_1 \delta + D_0} \]  

(43)

**Model**

\[ H_m(\delta) = \frac{N_{m1} \delta + N_{m0}}{\delta^2 + D_{m1} \delta + D_{m0}} \]  

(44)

Following relationships are obtained like the continuous- and discrete-time system design.

\[ A(\delta) = A_2 \delta^2 + A_1 \delta \]  

(45)

\[ M(\delta) = M_2 \delta^2 + M_1 \delta + M_0 \]  

(46)

\[ \alpha(\delta) = \left( \delta^2 + D_{m1} \delta + D_{m0} \right) \left( \delta + \frac{N_0}{N_1} \right) A_o(\delta) \]  

(47)

In the case of a very fast continuous-time pole \( p_c \), the corresponding modified delta pole \( \left( e^{p_c T} - 1 \right)/T \delta \) becomes \((0 - 1)/T \delta\). Thus, \( A_o(\delta) \) is set as follows.

\[ A_o(\delta) = \left( \delta + \frac{1}{T \delta} \right) \]  

(48)

\[ L(\delta) = \frac{N_m(\delta) A_o(\delta)}{N_1} \]  

(49)

\[ S_{\delta} X_{DN} = F \]  

(50)

where

\[
S_{\delta} = \begin{bmatrix}
D_2 & 0 & 0 & 0 & 0 \\
D_1 & D_2 & N_1 & 0 & 0 \\
D_0 & D_1 & N_0 & N_1 & 0 \\
0 & D_0 & 0 & N_0 & N_1 \\
0 & 0 & 0 & 0 & N_0 \\
\end{bmatrix}
\]

A discrete-time controller \( C(\delta) \) is shown in Fig. 13.

Step response is shown in Fig. 14. The disturbance \( d = -50N \) is also applied to the plant at 4ms. The same response based on the discrete-time design is obtained.
5. The evaluation of the numerical sensitivity based on the condition number

5.1. Second-order plant

The condition number $C_n$ of $S_\delta$ is shown in Fig. 15.

![Fig. 15 The condition number $C_n(\delta)$ of a second-order system](image)

The optimal $T_\delta$ is obtained from Fig. 15. The condition numbers $C_n$ based on discrete-time design methods at sampling period 50μs is shown in Table 1. The condition number becomes about one millionth or less of that based on the conventional discrete-time system design.

5.2. High-order plant

The effectiveness of the proposed method is verified in the design of the high-order plants. The following plants that are from 4th- to 10th-order are considered(15).

(a) 4th-order plant

Plant

$$H_p(s) = \frac{k_f}{s^2 + k_m} H_{f2}(s)$$

where

$$H_{f2}(s) = \frac{b_2\omega_1 s + b_1\omega_1^2}{s^2 + 2\xi_1\omega_1 s + \omega_1^2} \cdot \frac{1}{b_1}$$

Model

$$H_m(s) = \frac{\omega_m^2}{s^2 + 2\xi_m\omega_m s + \omega_m^2} \cdot \frac{\omega_m^2}{s^2 + 2\xi_m\omega_m s + \omega_m^2}$$
(b) 6th-order plant

Plant
\[ H_p(s) = \frac{k_m}{s^2 + \frac{k_m}{m} s} H_f(s) \]  
where
\[ H_f(s) = \frac{\sum_{j=1}^{2} b_j \omega_j s + b_{2j-1} \omega_j^2}{s^2 + 2 \xi_j \omega_j s + \omega_j^2} \frac{1}{2b_{2j-1}} \]

Model
\[ H_{m6}(s) = \frac{\omega_m^2}{s^2 + 2 \xi_m \omega_m s + \omega_m^2} \frac{\sum_{j=1}^{2} \omega_m^2}{2} \]

(c) 8th-order plant

Plant
\[ H_p(s) = \frac{k_m}{s^2 + \frac{k_m}{m} s} H_f(s) \]  
where
\[ H_f(s) = \frac{\sum_{j=1}^{3} b_j \omega_j s + b_{2j-1} \omega_j^2}{s^2 + 2 \xi_j \omega_j s + \omega_j^2} \frac{1}{3b_{2j-1}} \]

Model
\[ H_{m8}(s) = \frac{\omega_m^2}{s^2 + 2 \xi_m \omega_m s + \omega_m^2} \frac{\sum_{j=1}^{3} \omega_m^2}{3} \]

(d) 10th-order plant

Plant
\[ H_p(s) = \frac{k_m}{s^2 + \frac{k_m}{m} s} H_f(s) \]  
where
\[ H_f(s) = \frac{\sum_{j=1}^{4} b_j \omega_j s + b_{2j-1} \omega_j^2}{s^2 + 2 \xi_j \omega_j s + \omega_j^2} \frac{1}{4b_{2j-1}} \]

Model
\[ H_{m10}(s) = \frac{\omega_m^2}{s^2 + 2 \xi_m \omega_m s + \omega_m^2} \frac{\sum_{j=1}^{4} \omega_m^2}{4} \]

DC gain of \( H_f(s) \) is unity so that each flexible mode is not affect the plant DC gain. Simulation parameters are shown as follows.

\[
\omega_1, \omega_2, \omega_3, \omega_4 = \{2\pi50 \text{ rad/s}, 2\pi2000 \text{ rad/s}, 2\pi2000 \text{ rad/s}, 2\pi8000 \text{ rad/s}\}
\]

\[
\{b_1, b_2, b_3, b_4\} = \{-5.00 \times 10^{-3}, 1.0 \times 10^{-5}, 2.0 \times 10^{-2}, 0.0\}
\]

\[
\{b_5, b_6, b_7, b_8\} = \{0.5, 0.0, 0.1, 0.01\}
\]

\[
\{\xi_1, \xi_2, \xi_3, \xi_4\} = \{0.1, 0.01, 0.1, 0.01\}
\]

\[
\{\omega_m, \omega_m, T\} = \{1.0, 3000 \text{ rad/s}, 50 \times 10^{-6} \text{ s}\}
\]

The condition numbers of \( n \)-th order systems are shown at Table 2. The modified delta form can significantly improve numerical conditioning of a Sylvester matrix.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( C_p(z) )</th>
<th>( C_p(\delta)_{opt} )</th>
<th>( T_q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5.1238 \times 10^{13}</td>
<td>5.402 \times 10^{11}</td>
<td>1.4333 \times 10^{-11}</td>
</tr>
<tr>
<td>6</td>
<td>9.0356 \times 10^{32}</td>
<td>6.816 \times 10^{28}</td>
<td>9.0000 \times 10^{-28}</td>
</tr>
<tr>
<td>8</td>
<td>9.2342 \times 10^{32}</td>
<td>1.7497 \times 10^{21}</td>
<td>1.9667 \times 10^{-21}</td>
</tr>
<tr>
<td>10</td>
<td>1.4668 \times 10^{36}</td>
<td>2.4896 \times 10^{29}</td>
<td>3.7450 \times 10^{-29}</td>
</tr>
</tbody>
</table>
6. Verification of the robustness of the controller designed based on the proposed method

In order to verify the robustness of the proposed method, a 10th-order plant is considered, as shown in Eqs. (57) and (58). There are eight stable plant zeros and one unstable plant zero at sampling period 50μs.

6.1. Response of the 10th-order plant based on the conventional method

A Sylvester matrix $S_z$ becomes a $21 \times 21$ matrix. The step response at sampling period 50μs in which there is no disturbance is shown in Fig. 16. The control system based on the conventional design method is numerically unstable.

6.2. Response of the 10th-order plant based on the proposed method

In order to realize the robust controller based on the proposed method, the degrees of the controller are increased to the 11th-order from the 10th. Considering eight stable plant zeros and one unstable plant zero, the controller based on the proposed method is as follows.

$$A(\bar{\delta}) = A_{11}\bar{\delta}^{11} + \cdots + A_2\bar{\delta}^2$$  \hspace{1cm} (59)

$$M(\bar{\delta}) = M_{11}\bar{\delta}^{11} + \cdots + M_1\bar{\delta} + M_0$$  \hspace{1cm} (60)

$$L(\bar{\delta}) = N_m(\bar{\delta})A_0(\bar{\delta})/(N_9 N^T(0))$$  \hspace{1cm} (61)

$$A_0(\bar{\delta}) = \left(\bar{\delta} + \frac{1}{T_\delta}\right)^3$$  \hspace{1cm} (62)

Then, a Sylvester matrix $S_\bar{\delta}$ becomes a $22 \times 22$ matrix and the condition number $C_n$ at sampling period 50μs is shown in Fig. 17.

The minimum value of the condition number $C_n$ is about $6.5 \times 10^{15}$ and the optimum $T_\delta$ is 0.34 from Fig. 17.
6.3. The verification of the robustness of the proposed controller

The plant model used in a controller design is generally inaccurate due to modeling errors. Therefore, it is important to verify the robustness of the proposed controller. The plant perturbation or modeling errors \(\omega_j + \Delta \omega_j\), \(b_j + \Delta b_j\), and \(\xi_j + \Delta \xi_j\) are shown in the following.

\[
\Delta \omega = [0.05 \omega_1, 0.01 \omega_2, 0.01 \omega_3, 0.05 \omega_4]
\]
\[
\Delta b = [0, 0.1 b_2, 0, 0.1 b_4, 0, 0.1 b_6, 0, 0.1 b_8]
\]
\[
\Delta \xi = [-0.05 \xi_1, -0.05 \xi_2, -0.05 \xi_3, -0.05 \xi_4]
\]

The plant output \(y\) corresponds to the model output in which there is no modeling error, as shown in Fig. (18) (a). The disturbance \(d=0.07N\) is also applied to the plant at 5ms. The step and sinusoidal responses for the plant with modeling errors described above are shown in Fig. 18 (b) and 19, respectively. The controller based on the proposed method realizes the good performance.

![Response of a plant without modeling errors](image)

(a) Response of a plant without modeling errors

![Response of a plant with modeling errors](image)

(b) Response of a plant with modeling errors

Fig. 18 Step response of a 10th-order plant by using a controller based on the proposed method

![Sinusoidal response of a 10th-order plant](image)

Fig. 19 Sinusoidal response of a 10th-order plant by using a controller based on the proposed method
7. Conclusions

The numerical instability arises in the control law design, when the polynomial method is applied to a high-order system. This has prevented the practical application of the polynomial method. In order to reduce this numerical instability, an improved polynomial method based on the modified delta operator was proposed. For a 10th-order plant, the numerical sensitivity of a Sylvester matrix becomes one millionth of the conventional one obtained on the z-plane. Simulated results also show that the control system is robust. According above, the proposed method can be applied to a high-order plant by using a high-speed digital controller. Especially, since any desire vibration modes $\xi$ and $\omega_n$ can be easily included in the model, the proposed method is practical for vibration control. Furthermore, since the design procedure of a controller was also shown, the proposed method can be widely and immediately applied to the computer controlled plants in industry.

References


