A Sliding Mode Control of Semi-Active Suspension Systems with Describing Function Method*

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Abstract
This paper presents a sliding mode controller of semi-active suspension systems. The sliding mode controller is designed by the describing function method so that a switching function is enforced into a desired limit cycle instead of a perfect sliding mode. Although the proposed sliding mode controller cannot generate the limit cycle as desired because of the passive constraint of controllable dampers, restricting the switching function in the vicinity of the origin can suppress the deterioration due to the passive constraint, such as increase of jerk of the sprung mass. Finally, simulation results show the effectiveness of the proposed controller.

Key words: Semi-Active Suspension Systems, Passive Constraint, Sliding Mode Control, Describing Function Method, Limit Cycle

1. Introduction

Semi-active suspension systems with controllable dampers for vehicles, which are superior to active suspension systems in respect of restraining energy consumption and securing failsafe, have been studied. Various control theories have been attempted to apply to semi-active suspension systems(1)-(7). One of those theories is the sliding mode control theory(1), (2), whose high-frequency switching of a relay input enforces the plant into sliding modes. The plant in sliding modes is well-known for the high robustness against modeling errors and disturbances, and is well-known for the simplicity of the controller, which can be designed if the upper bound of modeling error is known. Applied to the controller for semi-active suspension systems, the sliding mode control could be expected to acquire the robustness against change of the number of passengers and frictions of suspension mechanics.

Controllable dampers employed as actuator are passive devices, and the damping force acts as resistance force against expansion and contraction of dampers. In the controller design of semi-active suspension system, it is necessary to pay attention to the passive constraint condition that the direction of the damping force is subject to the sign of the piston velocity of the damper. In many researches, control laws are designed first on the assumption that the plant has an active suspension system. Then, approximate control laws, to which are added the passive constraint condition of controllable dampers, are implemented at the time of applying to the semi-active suspension system. For example, the Karnopp's law(3) used frequently for semi-active suspension systems is an approximate
control law of the sky-hook damper system, which needs fully active control. For the passive constraint, the control input is approximated as a minimum control input. However, the approximation for the passive constraint can cause increase of the jerk of the sprung mass, since the damping force can change suddenly when the sign of the piston velocity changes.

For such a problem, a nonlinear $H_\infty$ controller regarding semi-active suspension systems as bilinear systems is proposed\(^4\). Also, a linear $H_\infty$ controller paying attention to the Lissajous figure of the damping force is proposed\(^5\). Those controllers can acquire the effect to the improvement of jerk. However, any researches of the sliding mode control in consideration of the passive constraint have not been investigated.

This paper presents a sliding mode control law that can overcome the problem mentioned above. With the proposed sliding mode controller designed by the describing function method, a limit cycle of a switching function in the vicinity of the origin is occurred instead of a prefect sliding mode. As a result, the influence of the passive constraint can be improved by avoiding the frequent switching of the control input, while the robustness of the proposed controller can be still held.

2. Problem Formulation

Figure 1 shows a quarter car model of the semi-active suspension system. The sprung mass is supported with a semi-active suspension system consisting of a spring and a controllable damper. \( m_2 \) and \( m_1 \) denote the sprung mass and the unsprung mass. \( k_2 \) and \( k_1 \) denote the stiffness. \( c_2 \) and \( c_1 \) denote the damping coefficient. \( y_2(t) \), \( y_1(t) \), and \( d(t) \) denote the sprung mass displacement, the unsprung mass displacement and the road disturbance. The control input \( u(t) \) denotes the damping force of the controllable damper. Then, the motion equations for the quarter car model are given by

\[
\begin{align*}
    m_2 \ddot{y}_2(t) &= -k_2 \{y_2(t) - y_1(t)\} - c_2 \{\dot{y}_2(t) - \dot{y}_1(t)\} - \{u(t) + w(t)\} \\
    m_1 \ddot{y}_1(t) &= k_1 \{y_2(t) - y_1(t)\} + c_1 \{\dot{y}_2(t) - \dot{y}_1(t)\} + \{u(t) + w(t)\} \\
        &\quad - k_1 \{y_1(t) - d(t)\} - c_1 \{\dot{y}_1(t) - \dot{d}(t)\}
\end{align*}
\]

where \( k_1 \{y_1(t) - d(t)\} \) denotes variation of the road holding force, and \( w(t) \) denotes model uncertainties and disturbances in the suspension system. The passivity of the damper brings about the following constraint

\[
u(t) = \begin{cases} 
    f(t) & \text{if } f(t)\{\dot{y}_2(t) - \dot{y}_1(t)\} > 0 \\
    0 & \text{if } f(t)\{\dot{y}_2(t) - \dot{y}_1(t)\} \leq 0
\end{cases}
\]

where \( f(t) \) denotes an ideal control input derived from a control law designed for fully active control, and \( \dot{y}_2(t) - \dot{y}_1(t) \) denotes the piston velocity of the damper. The passive constraint Eq. (3) means that the ideal control force \( f(t) \) can be generated by the controllable damper if the condition \( f(t)\{\dot{y}_2(t) - \dot{y}_1(t)\} \) is positive, otherwise \( f(t) \) is obliged to be approximated as the minimum control input, which is considered as zero in this paper. The goal of this paper is the improvement of increase in the jerk of the sprung mass accompanied by rapidly change of the damping force, while the robustness against the modeling error is ensured. In the next section, a sliding mode control law to derive the control input \( f(t) \) will be introduced.
3. Integral Sliding Mode Control

For the quarter car model shown in Fig. 1, the state space model of the sprung mass system is described as follows.

\[
\dot{x}(t) = Ax(t) + b[f(t) + w(t)] + gd(t)
\]  

(4)

where the state vector \( x(t) \), each matrixes and vectors are given as follows.

\[
x(t) = [x_1(t) \ x_2(t)]^T = [y_2(t) \ \dot{y}_2(t)]^T
\]

\[
A = \begin{bmatrix}
0 & 1 \\
-k_2/m_2 & -c_2/m_2
\end{bmatrix}, \quad b = \begin{bmatrix}
0 \\
-1/m_2
\end{bmatrix}, \quad g = \begin{bmatrix}
k_2 & c_2 \\
m_2 & m_2
\end{bmatrix}
\]

\[
d(t) = [y_1(t) \ \dot{y}_1(t)]^T
\]

Applied the integral sliding mode control theory\(^{(3)}\) to the state space model Eq. (4), the control input \( f(t) \) is defined as follows.

\[
f(t) = f_s(t) + f_r(t)
\]  

(5)

where \( f_s(t) \) is a state feedback control input to lead an ideal closed-loop system.

\[
f_s(t) = Fx(t)
\]  

(6)

As an example of such a state feedback control input, the Karnopp's law is well known.

\[
f_s(t) = \begin{cases}
c_s x_2(t) & \text{if } x_2(t) [\dot{y}_2(t) - \dot{y}_1(t)] > 0 \\
0 & \text{if } x_2(t) [\dot{y}_2(t) - \dot{y}_1(t)] \leq 0
\end{cases}
\]  

(7)

where \( c_s \) denotes a damping coefficient of the skyhook damper. In Eq. (5), \( f_r(t) \) is a relay input to reject the influence of modeling error as follows.

\[
f_r(t) = -\gamma \cdot \text{sign}(\sigma(t))
\]  

(8)

where the relay gain \( \gamma > 0 \) is determined in order to satisfy an existence condition of a sliding mode. The switching function \( \sigma(t) \) is defined as follows.
\[ \sigma(t) = z(t) - Cx(t) \]  
(9)

where \( C = [\alpha \quad 1] \), \( \alpha \geq 0 \), and \( z(t) \) is an auxiliary variable, whose dynamics can be described as follows.

\[ \dot{z}(t) = C \{ Ax(t) + b_f(t) + gd(t) \} \]  
(10)

The sliding mode caused by the control law Eqs. (5), (6), (8)~(10) can be analyzed with the equivalent control method(8). First, set the time derivative of the switching function \( \sigma(t) \) equal to zero in order to show the equivalent control input \( f_{eq}(t) \) of the relay input Eq. (8).

\[ \dot{\sigma}(t) = \dot{z}(t) - Cx(t) \]
\[ = \dot{z}(t) - C \{ Ax(t) + b_f(t) + f_{eq}(t) + w(t) \} + gd(t) = 0 \]  
(11)

Solving Eq. (11) can be described the equivalent control input \( f_{eq}(t) \) as follows.

\[ f_{eq}(t) = -w(t) \]  
(12)

Substituting the equivalent control input Eq. (12) in Eqs. (4) and (5) can be described the closed loop dynamics of the sliding mode as follows

\[ \dot{x}(t) = Ax(t) + b_f(t) + gd(t) \]  
(13)

From Eqs. (12) and (13), it can be seen that the equivalent control input could eliminate influence of the disturbance \( w(t) \), and that the ideal dynamics derived from the state feedback control \( f_0(t) \) in Eq. (6) could be achieved in the sliding mode.

Then, with the Lyapunov’s stability theorem, the existence condition of the sliding mode can be led as follows. When a function

\[ V(\sigma(t)) = \frac{1}{2} \sigma(t)^2 \]  
(14)

is chosen as a candidate of the Lyapunov’s function, the time derivative of \( V(\sigma(t)) \) along the solution of Eq. (11) is given by

\[ \frac{d}{dt} V(\sigma(t)) = \frac{1}{2} \frac{d}{dt} (\sigma(t)^2) = \sigma(t) \cdot \dot{\sigma}(t) < 0 \]  
(15)

For the existence of the sliding mode in a sense of the Lyapunov’s stability, the relay gain \( \gamma \) should satisfy the following inequality

\[ \gamma > |w(t)| \]  
(16)

4. Redesign of Relay Input with Describing Function Method

Replaced the relay input \( f(t) \) given by Eq. (8), a new control input is introduced as follows.
\[ F_1(s) = \frac{\omega_s^2}{s^2 + 2\zeta\omega_s s + \omega_s^2} F_2(s) \]  
(17)

\[ f_2(t) = -\gamma_1 \text{sign}(\sigma(t)) - \gamma_2 \text{sign}(\dot{\sigma}(t)) \]  
(18)

where \( s \) denotes the Laplace operator. \( F_1(s) \) and \( F_2(s) \) denote the Laplace transform of \( f_1(t) \) and \( f_2(t) \) respectively. \( \gamma_1, \gamma_2, \zeta \) and \( \omega_s \) are design parameters to occur a limit cycle of the switching function \( \sigma(t) \). Figure 2 shows the block diagram of the proposed sliding mode control system. In that control system, the transfer function \( G(s) \) from \( f_2(t) \) to \( \sigma(t) \) is given as follows.

\[ G(s) = \frac{\omega_s^2}{m_2(s^2 + 2\zeta\omega_s s + \omega_s^2)} \]  
(19)

Then, the proposed control system could be transformed into a nonlinear feedback system shown in Fig. 3. With the describing function method \(^{(9)}\), the existence condition of a limit cycle on the switching function \( \sigma(t) \) could be described as follows.

\[ 1 + N(a)G(s) = 0 \]  
(20)

where \( a \) denotes amplitude of the limit cycle. \( N(a) \) denotes the describing function of the nonlinear part in the proposed control system, and could be described as follows \(^{(10)}\).

\[ N(a) = \frac{4}{\pi a}(\gamma_1 + \gamma_2 j) \]  
(21)

Let \( s = j\omega \) in the existence condition Eq. (20) for a limit cycle, then

\[ G(j\omega) = -\frac{1}{N(a)} \]  
(22)

The describing function methods states that if Eq. (22) has a solution \((a, \omega)\), then there might be a periodic solution of the system with frequency and amplitude close to \( \omega \) and \( a \). For the existence condition Eq. (22), the condition can be concretely described as follows

\[ 2\omega_s(-\pi a\gamma_2 s^2 + 2\omega_n \gamma_1) + j\{\pi a\omega_n(\omega_s^2 - \omega^2) + 4a^2 \gamma_2 \} = 0 \]  
(23)

![Fig. 2 Block diagram of proposed control system](image-url)

**Fig. 2** Block diagram of proposed control system

![Fig. 3 Analysis model for existence condition of limit cycle](image-url)

**Fig. 3** Analysis model for existence condition of limit cycle
From Eq. (23), the design parameters of the control input Eqs. (17) and (18) should be
determined as follows when $\zeta$ is set to a constant in the first step.

$$\gamma_1 > m_2 \pi \zeta a \omega / 2$$  \hspace{1cm} (24)

$$\omega_n = \pi a m_2 \zeta \omega^2 / 2 \gamma_1$$  \hspace{1cm} (25)

$$\gamma_2 = \pi a m_2 \omega (\omega^2 - \omega_n^2) / 4 \omega_n^2$$  \hspace{1cm} (26)

5. Numerical Simulation

5.1 Simulation Condition

In order to investigate effectiveness of the proposed sliding mode controller, numerical
simulations were carried out with the numerical analysis software MATLAB/Simulink. The
proposed sliding mode controller (proposed SMC) was compared with the Karnopp’s law
and the sliding mode controller with the relay input (SMC with relay). Table 1 shows the
simulation parameters for the numerical simulations. The natural frequency of the sprung
mass is 7.5 [rad/s], and that of the unsprung mass is 75 [rad/s]. Table 2 shows the design
parameters of the proposed SMC for each desired limit cycle of the switching function $\sigma(t)$. Those parameters were determined to satisfy Eqs. (24)~(26) when $\zeta$ was set to 1.0
in the first step. The road disturbance velocity was a band-limited white Gaussian signal,
and the sampling period for control was 1 [ms].

<table>
<thead>
<tr>
<th>Table 1 Simulation parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>$m_2$ [kg]</td>
</tr>
<tr>
<td>$m_1$ [kg]</td>
</tr>
<tr>
<td>$k_1$ [N/m]</td>
</tr>
<tr>
<td>$k_2$ [N/m]</td>
</tr>
<tr>
<td>$c_1$ [Ns/m]</td>
</tr>
<tr>
<td>$c_2$ [Ns/m]</td>
</tr>
<tr>
<td>$c_3$ [Ns/m]</td>
</tr>
<tr>
<td>$\gamma$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2 Design parameters for each desired limit cycle of switching function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>$\omega$</td>
</tr>
<tr>
<td>$\gamma_1$</td>
</tr>
<tr>
<td>12.3</td>
</tr>
<tr>
<td>25.1</td>
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<td>37.7</td>
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<td>50.2</td>
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<tr>
<td>62.8</td>
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<tr>
<td>75.4</td>
</tr>
<tr>
<td>87.9</td>
</tr>
<tr>
<td>100</td>
</tr>
<tr>
<td>113</td>
</tr>
<tr>
<td>126</td>
</tr>
</tbody>
</table>

5.2 Accuracy of limit cycle of switching function

First, numerical simulations were carried out to investigate the accuracy of the limit
cycle led by the proposed SMC when the passive constraint Eq. (3) was disregarded, that is,
in the case of fully active control. The accuracy of the limit cycle should be verified, since
the describing function method is known as an approximation method for finding a periodic
solution. Table 3(a) shows the accuracy of the limit cycle in the case of fully active control.
As an example, Fig. 4(a) shows the time response of the switching function for $\omega_a$=126
[rads]. From Table 3(a) and Fig. 4(a), it can be seen that the fully active control without the
passive constraint Eq. (3) can approximately occur the desired limit cycle of the switching
function $\sigma(t)$. Although the relay gain $\gamma_1$ was set constant in this paper, higher accuracy
can be obtained by adjusting to the parameter for each desired limit cycle.

Table 3(b) shows the accuracy of the limit cycle led by the proposed SMC with the
passive constraint Eq. (3), that is, in the case of a semi-active control. As an example, Fig.
4(b) shows the time response of the switching function for $\omega = 126$ [rad/s]. From Table 3(b), it can be seen that there are more than 30% of errors for each desired limit cycle, especially the error of the amplitude $a$. However, the time response of the switching function shown in Fig. 4(b) can gently be restrained in the vicinity of the origin as intended. The design parameters can be adjusted so that actual limit cycle can be enforced as desired.

### Table 3 Accuracy of limit cycle of switching function

<table>
<thead>
<tr>
<th>Desired</th>
<th>Actual</th>
<th>Error [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>$a$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>12.3</td>
<td>$1.00 \times 10^{-3}$</td>
<td>14.7</td>
</tr>
<tr>
<td>25.1</td>
<td>$1.00 \times 10^{-3}$</td>
<td>25.8</td>
</tr>
<tr>
<td>37.7</td>
<td>$1.00 \times 10^{-3}$</td>
<td>39.3</td>
</tr>
<tr>
<td>50.2</td>
<td>$1.00 \times 10^{-3}$</td>
<td>51.5</td>
</tr>
<tr>
<td>62.8</td>
<td>$1.00 \times 10^{-3}$</td>
<td>64.1</td>
</tr>
<tr>
<td>75.4</td>
<td>$1.00 \times 10^{-3}$</td>
<td>76.0</td>
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<tr>
<td>87.9</td>
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<td>88.6</td>
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<td>$1.00 \times 10^{-3}$</td>
<td>99.3</td>
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<tr>
<td>113</td>
<td>$1.00 \times 10^{-3}$</td>
<td>111</td>
</tr>
<tr>
<td>126</td>
<td>$1.00 \times 10^{-3}$</td>
<td>124</td>
</tr>
</tbody>
</table>

### 5.3 Improvement of deterioration caused by passive constraint

How such restriction in the vicinity of the origin can improve the deterioration due to the passive constraint was investigated. The simulation parameters were considered as nominal. While the acceleration of the sprung mass could be obtained equivalently in every control laws, jerk of the sprung mass and variation of the road holding force were subject to influence of the passive constraint, especially in the SMC with relay. Table 4 shows effectiveness of the proposed SMC in comparison to the SMC with relay. Although the average value of variation of the road holding force was slightly increasing, the average of jerk of the sprung mass was dramatically decreasing. Figure 5 shows the Lissajous figure of the ideal control input for both sliding mode controllers. It can be seen that the proposed SMC could suppress switching of the control input accompanied the passive constraint, compared to the SMC with relay. As performance indexes of the improvement, the switching frequency of the control input due to the passive constraint and the rate of time, which the control input cannot be applied by the passive constraint, were considered as shown in the right side of Table 4. In case of the SMC with relay, the switching frequency was 54.6 [1/s] and the rate of time was 75.9 [%]. From these results, it can be seen that restricting the switching function in the vicinity of the origin can improve the deterioration due to the passive constraint.
5.4 Verification of robustness against parameter variation

In this section, the robustness of the proposed SMC for the parameter variation problem was illustrated. The parameter variation was assumed that the sprung mass $m_2$ decreased 25% and the stiffness $k_2$ increased 25% compared with those nominal values.

Figure 6 shows the power spectral density (P.S.D.) of the sprung mass acceleration for each control laws. The desired frequency of the limit cycle in the proposed SMC was 25.1 [rad/s]. From Fig. 6(a), it can be seen that the parameter variation raised the P.S.D. around the natural frequency of the sprung mass for the Karnopp's law. From Fig. 6(b), it can be seen that the SMC with relay could obtain the robustness against the parameter variation, since the quasi sliding mode could be occurred. However, the chattering in the high frequency range was arisen by switching the relay input Eq. (8). And, from Fig. 6(c), it can be seen that the proposed SMC could show the robustness against the parameter variation without the influence of the chattering in the high frequency range, since the number of the switching due to the passive constraint was reduced as shown in Fig. 7. Table 5 shows variation of the road holding force under the parameter variation in comparison with the Karnopp's law under nominal condition. From these results, it can be seen that the proposed sliding mode controller could achieve higher robustness than other two controllers.

Table 4 Improvement of deterioration caused by passive constraint
(Effectiveness of proposed SMC in comparison to SMC with relay)

<table>
<thead>
<tr>
<th>Desired limit cycle</th>
<th>Jerk [%]</th>
<th>Variation of road holding force [%]</th>
<th>Switching freq due to passive constraint [%]</th>
<th>Rate of time concerning passive constraint [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>$\alpha$</td>
<td>RMS</td>
<td>Max.</td>
<td>RMS</td>
</tr>
<tr>
<td>12.3</td>
<td>$1.00 \times 10^3$</td>
<td>-69.7</td>
<td>-11.0</td>
<td>-0.1</td>
</tr>
<tr>
<td>25.1</td>
<td>$1.00 \times 10^3$</td>
<td>-66.6</td>
<td>-4.4</td>
<td>0.0</td>
</tr>
<tr>
<td>37.7</td>
<td>$1.00 \times 10^3$</td>
<td>-66.5</td>
<td>-6.6</td>
<td>-0.2</td>
</tr>
<tr>
<td>50.2</td>
<td>$1.00 \times 10^3$</td>
<td>-61.8</td>
<td>-9.4</td>
<td>+0.5</td>
</tr>
<tr>
<td>62.8</td>
<td>$1.00 \times 10^3$</td>
<td>-61.5</td>
<td>-12.2</td>
<td>+1.1</td>
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<td>75.4</td>
<td>$1.00 \times 10^3$</td>
<td>-58.7</td>
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<td>87.9</td>
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<td>-8.6</td>
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<td>-54.0</td>
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<td>+9.1</td>
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<tr>
<td>126</td>
<td>$1.00 \times 10^3$</td>
<td>-52.6</td>
<td>+18.3</td>
<td>+9.2</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td>-60.6</td>
<td>-4.1</td>
<td>+3.4</td>
</tr>
</tbody>
</table>

Fig.5 Lissajous figure of ideal control input
Fig. 6 Power spectrum density of sprung mass acceleration under parameter variation (Comparison with Karnopp's law under nominal condition)

Table 5 Variation of road holding force under parameter variation (Comparison with Karnopp's law under nominal condition)

<table>
<thead>
<tr>
<th>Controller</th>
<th>Variation of road holding force</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RMS(%)</td>
</tr>
<tr>
<td>Karnopp's law</td>
<td>-6.8</td>
</tr>
<tr>
<td>SMC with relay</td>
<td>-2.9</td>
</tr>
<tr>
<td>Proposed SMC</td>
<td>-4.8</td>
</tr>
</tbody>
</table>

Fig. 7 Lissajous figure of ideal control input
6. Conclusions

A sliding mode controller, which is designed by the describing function method in order to enforce a switching function into a desired limit cycle instead of a perfect sliding mode, is proposed. Although the proposed sliding mode controller cannot generate the limit cycle as desired, restricting the switching function in the vicinity of the origin can suppress the deterioration due to the passive constraint, such as increase in jerk of the sprung mass. As future work, the relation between the design parameters of the proposed sliding mode controller and the robustness against disturbance, such as parameter variation, will be analyzed.

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References