Approximate Linearization Control of 2-DOF Underactuated-by-1 Systems Using Higher Order Linearization Coordinate*

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Abstract

This paper deals with an approximate linearization control of 2-DOF underactuated-by-1 nonlinear systems, proposing a novel linearization coordinate which reduces the approximation error over the state space around the operating point. The coordinate is analytically constructed in a systematic way by solving two first order linear partial differential equations and the solution is given in an infinite series of configuration variables. The resulting linearization feedback is highly nonlinear and the basin of attraction of the stabilized system using proposed coordinate is large, comparing with those of a conventional first order or other lower order linearization coordinates. The approximate linearization control based on the proposed coordinate is applied to the stabilization of a rotational inverted pendulum; the advantage is verified in simulations and experiments. Some perspectives on availability of the linearization coordinate are discussed and they are computed also for a mobile inverted pendulum, Acrobot, and for Pendubot as examples.

Key words : Approximate Linearization, Partial Differential Equation, Basin of Attraction, Gudermannian, Pendulum, Acrobot, Pendubot

1. Introduction

Feedback linearization control is widely used for nonlinear mechanical systems since early 80's especially in fully actuated case. The number of actuators being the same as the DOF of motion, global stabilization and shaping of the transient response are possible even if the system has nonlinearity. In contrast, in the case of underactuated systems, there being less actuators than DOF of the motion, feedback linearization is partially possible. As a result, there is no promise on the global stability and controller design strategy depends on the property of each system. Therefore, intending large basin of attraction and improvement of transient response, nonlinear controllers based on approximate linearization or passivation technique are proposed and tested on Acrobot⁽²⁾, inverted pendulum^{(3),(6),(7)}, and on mobile inverted pendulum⁽⁹⁾.

Sugie et al. pointed out that the linearization coordinate used in the feedback linearization affects the characteristics of feedback loop⁽⁸⁾. They use a cubic function of pendulum position as the linearization coordinate and make the nonlinear residual in the linearized system small, which enlarges the basin of attraction^{(3),(7)}. As for inverted pendulum, only the first order approximate linearization is possible^{(1),(4)}; thus the approximation error inevitably involves quadratic terms of state variables. Still by making the residual small, the modeling error reduces and the basin of attraction is possibly extended. However, especially for underactuated systems, there is no guideline for choosing linearization coordinate, and the process requires trial and error.

The authors use a novel linearization coordinate which is an infinite series of pendulum position and successfully obtain larger basin of attraction⁽¹⁰⁾. The coordinate also makes the

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linearization error small and can be determined in a systematic way without trial. This paper deals with derivation of the linearization coordinate for a small class of 2-DOF underactuated-by-1 systems.

2. Derivation of Linearization Coordinate

2.1. Upper actuated system

This paper deals with the approximate linearization of 2-DOF underactuated-by-1 nonlinear mechanical systems. Let two configuration variables of the system be generalized position $[q_1, q_2]^T =: \mathbf{q}$ and the corresponding generalized forces be Q_1 and Q_2 . Since the system is underactuated, either force is equal to zero. If the first joint is actuated, it is called an upper actuated system⁽⁵⁾ and is described in the following matrix form:

$$\begin{bmatrix} m_{11}(\mathbf{q}) & m_{12}(\mathbf{q}) \\ m_{12}(\mathbf{q}) & m_{22}(\mathbf{q}) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} h_1(\mathbf{q}, \dot{\mathbf{q}}) \\ h_2(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix} = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix} \quad (1)$$

where $\{m_{ij}(\mathbf{q})\}$ ($i, j \in \{1, 2\}$) is a positive definite inertia matrix and $\{h_i(\mathbf{q}, \dot{\mathbf{q}})\}$ ($i \in \{1, 2\}$) involves Coriolis force, centripetal force, gravity force, and friction force. The inertia matrix depends only on generalized position and not on generalized velocity.

Applying the collocated linearization technique⁽⁵⁾ via input transformation, the system (1) is partially linearized as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & m_{22}(\mathbf{q}) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ h_2(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix} = \begin{bmatrix} 1 \\ -m_{12}(\mathbf{q}) \end{bmatrix} u. \quad (2)$$

u is regarded as a new input and is the generalized acceleration of the first joint:

$$u := \frac{m_{22}(\mathbf{q}) \{Q_1 - h_1(\mathbf{q}, \dot{\mathbf{q}})\} + m_{12}(\mathbf{q}) h_2(\mathbf{q}, \dot{\mathbf{q}})}{m_{11}(\mathbf{q}) m_{22}(\mathbf{q}) - m_{12}^2(\mathbf{q})} \quad (3)$$

Using the state vector $\mathbf{x} := [q_1, q_2, \dot{q}_1, \dot{q}_2]^T$, underactuated system (2) is rewritten as a 4th order nonlinear affine form:

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ 0 \\ -h_2(\mathbf{q}, \dot{\mathbf{q}})/m_{22}(\mathbf{q}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -m_{12}(\mathbf{q})/m_{22}(\mathbf{q}) \end{bmatrix} u =: \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \quad (4)$$

$$-h_2(\mathbf{q}, \dot{\mathbf{q}})/m_{22}(\mathbf{q}) =: f_4(\mathbf{q}, \dot{\mathbf{q}}), \quad -m_{12}(\mathbf{q})/m_{22}(\mathbf{q}) =: g_4(\mathbf{q}). \quad (5)$$

As seen in Eqs. (4) and (5), the drift vector field $\mathbf{f}(\mathbf{x})$ depends on both the generalized position and velocity, while the input vector field $\mathbf{g}(\mathbf{x})$ depends only on the generalized position. Though Eq. (4) is still nonlinear, further exact linearization is not available. Any linearization effort to Eq. (4) must involve some approximation. A conventional way is to linearly approximate $f_4(\mathbf{q}, \dot{\mathbf{q}})$ and $g_4(\mathbf{q})$ by the first order term of their Taylor expansions around an operating point $\mathbf{x}^\circ$. Thus the second and higher order terms have been neglected.

If Eq. (4) were exactly linearizable via coordinate transformation, there exists a scalar field $\lambda(\mathbf{x})$ which satisfies $L_g \lambda(\mathbf{x}) = L_g L_f \lambda(\mathbf{x}) = L_g L_f^2 \lambda(\mathbf{x}) = 0$. Then the system is transformed into the following form:

$$\frac{d}{dt} \begin{bmatrix} \lambda(\mathbf{x}) \\ L_f \lambda(\mathbf{x}) \\ L_f^2 \lambda(\mathbf{x}) \\ L_f^3 \lambda(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} L_f \lambda(\mathbf{x}) \\ L_f^2 \lambda(\mathbf{x}) \\ L_f^3 \lambda(\mathbf{x}) \\ L_f^4 \lambda(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} L_g \lambda(\mathbf{x}) \\ L_g L_f \lambda(\mathbf{x}) \\ L_g L_f^2 \lambda(\mathbf{x}) \\ L_g L_f^3 \lambda(\mathbf{x}) \end{bmatrix} u = \begin{bmatrix} L_f \lambda(\mathbf{x}) \\ L_f^2 \lambda(\mathbf{x}) \\ L_f^3 \lambda(\mathbf{x}) \\ L_f^4 \lambda(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ L_g L_f^3 \lambda(\mathbf{x}) \end{bmatrix} u, \quad (6)$$

and using a new coordinate $\boldsymbol{\xi} := [\lambda(\mathbf{x}), L_f \lambda(\mathbf{x}), L_f^2 \lambda(\mathbf{x}), L_f^3 \lambda(\mathbf{x})]^T$, the linearized systems is written in Brunovski canonical form with a new input v :

$$\frac{d\boldsymbol{\xi}}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{\xi} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v, \quad v := L_f^4 \lambda(\mathbf{x}) + L_g L_f^3 \lambda(\mathbf{x}) u. \quad (7)$$

Following the linearization procedure above, even for underactuated system (4), the author will use a scalar field $\lambda(\mathbf{x})$, which decreases approximation error in the input vector in Eq. (6), as a linearization coordinate. Since exact linearization is not possible for underactuated systems, the next candidate is a scalar field $\lambda(\mathbf{x})$ which satisfies

$$\begin{cases} L_g \lambda(\mathbf{x}) = 0, \\ L_g L_f \lambda(\mathbf{x}) = 0. \end{cases} \quad (8)$$

If such $\lambda(\mathbf{x})$ is available, approximation error at least in the first two equations in Eq. (6) will vanish.

The first equation of (8) is written as follows:

$$L_g \lambda(\mathbf{x}) = \frac{\partial \lambda(\mathbf{x})}{\partial \dot{q}_1} + g_4(\mathbf{q}) \frac{\partial \lambda(\mathbf{x})}{\partial \dot{q}_2} = 0. \quad (9)$$

Eq. (9) is a first order linear homogeneous partial differential equation (PDE) about λ with four independent variables $q_1, q_2, \dot{q}_1$, and $\dot{q}_2$. As pointed before, $g_4(\mathbf{q})$ does not depend on $\dot{q}_1$ nor $\dot{q}_2$, PDE (9) is classified as solvable⁽¹³⁾. The general solution is an arbitrary function of $\mathbf{q}$ and $\{g_4(\mathbf{q})\dot{q}_1 - \dot{q}_2\}$:

$$\lambda(\mathbf{x}) = \lambda\{\mathbf{q}, g_4(\mathbf{q})\dot{q}_1 - \dot{q}_2\}. \quad (10)$$

Hereafter, $\lambda(\mathbf{x})$ which depends only on $\mathbf{q}$ is considered for simplicity:

$$\lambda(\mathbf{x}) = \lambda(\mathbf{q}). \quad (11)$$

Substituting Eq. (11) into the second equation of (8), further condition on $\lambda(\mathbf{q})$ is given as a PDE:

$$L_g L_f \lambda(\mathbf{q}) = L_g \left(\frac{\partial \lambda(\mathbf{q})}{\partial q_1} \dot{q}_1 + \frac{\partial \lambda(\mathbf{q})}{\partial q_2} \dot{q}_2 \right) = 0, \quad \therefore \frac{\partial \lambda(\mathbf{q})}{\partial q_1} + g_4(\mathbf{q}) \frac{\partial \lambda(\mathbf{q})}{\partial q_2} = 0. \quad (12)$$

Eq. (12) is a first order linear homogeneous PDE with only two independent variables: q_1 and q_2 . Availability of analytical solution of Eq. (12) depends on particular form of $g_4(\mathbf{q})$. Some cases where general solution to Eq. (12) can be obtained are:

- (1) $g_4(\mathbf{q})$ does not depend on q_1 nor q_2 ,
- (2) $g_4(\mathbf{q})$ depends on either q_1 or q_2 , and
- (3) Eq. (12) is of separation of variables with q_1 and q_2 .

All examples in this paper, including the rotational inverted pendulum, fall into case (2).

The linearization coordinate $\lambda(\mathbf{q})$ obtained by solving Eq. (12) is an infinite series of configuration variables in general. It is close to the conventional linear coordinate around the operating point, but it differs significantly as the system state is away from the operating point. This property brings small approximation error in the linearized system and large basin of attraction. These are concretely described by examples in Section 3.

2.2. Lower actuated system

Similarly to the upper actuated system, a lower actuated system described by

$$\begin{bmatrix} m_{11}(\mathbf{q}) & m_{12}(\mathbf{q}) \\ m_{12}(\mathbf{q}) & m_{22}(\mathbf{q}) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} h_1(\mathbf{q}, \dot{\mathbf{q}}) \\ h_2(\mathbf{q}, \dot{\mathbf{q}}) \end{bmatrix} = \begin{bmatrix} 0 \\ Q_2 \end{bmatrix} \quad (13)$$

is considered below. Q_2 is the generalized force of the second joint. Eq. (13) is partially linearized as follows via collocated linearization using input change:

$$\begin{bmatrix} m_{11}(\mathbf{q}) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} h_1(\mathbf{q}, \dot{\mathbf{q}}) \\ 0 \end{bmatrix} = \begin{bmatrix} -m_{12}(\mathbf{q}) \\ 1 \end{bmatrix} u, \quad (14)$$

$$u := \frac{m_{11}(\mathbf{q})\{Q_2 - h_2(\mathbf{q}, \dot{\mathbf{q}})\} + m_{12}(\mathbf{q})h_1(\mathbf{q}, \dot{\mathbf{q}})}{m_{11}(\mathbf{q})m_{22}(\mathbf{q}) - m_{12}^2(\mathbf{q})}. \quad (15)$$

The partially linearized system (14) is also written as a 4th order nonlinear affine system:

$$\frac{dx}{dt} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ -h_1(\mathbf{q}, \dot{\mathbf{q}})/m_{11}(\mathbf{q}) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -m_{12}(\mathbf{q})/m_{11}(\mathbf{q}) \\ 1 \end{bmatrix} u =: \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \quad (16)$$

$$\mathbf{f}_3(\mathbf{q}, \dot{\mathbf{q}}) = -h_1(\mathbf{q}, \dot{\mathbf{q}})/m_{11}(\mathbf{q}), \quad \mathbf{g}_3(\mathbf{q}) = -m_{12}(\mathbf{q})/m_{11}(\mathbf{q}). \quad (17)$$

Similar to the upper actuated case, $\mathbf{f}(\mathbf{x})$ depends on both $\mathbf{q}$ and $\dot{\mathbf{q}}$, and $\mathbf{g}(\mathbf{x})$ depends only on $\mathbf{q}$. The scalar field $\lambda(\mathbf{x})$ which defines the linearization coordinate is determined by solving a set of PDEs (8) for system (16). The first PDE

$$L_g \lambda(\mathbf{x}) = g_3(\mathbf{q}) \frac{\partial \lambda(\mathbf{x})}{\partial \dot{q}_1} + \frac{\partial \lambda(\mathbf{x})}{\partial \dot{q}_2} = 0 \quad (18)$$

gives a general solution

$$\lambda(\mathbf{x}) = \lambda\{\mathbf{q}, \dot{q}_1 - g_3(\mathbf{q})\dot{q}_2\}. \quad (19)$$

If we consider $\lambda(\mathbf{x})$ which depends only on $\mathbf{q}$, further condition on $\lambda(\mathbf{q})$ is simplified as

$$L_g L_f \lambda(\mathbf{q}) = L_g \left(\frac{\partial \lambda(\mathbf{q})}{\partial q_1} \dot{q}_1 + \frac{\partial \lambda(\mathbf{q})}{\partial q_2} \dot{q}_2 \right) = 0, \quad \therefore g_3(\mathbf{q}) \frac{\partial \lambda(\mathbf{q})}{\partial q_1} + \frac{\partial \lambda(\mathbf{q})}{\partial q_2} = 0. \quad (20)$$

Eq. (20) is similar to Eq. (12); if $g_3(\mathbf{q})$ applies to either of three cases for $g_4(\mathbf{q})$ and the characteristic equations are integrable, $\lambda(\mathbf{q})$ for lower actuated system is analytically obtained.

3. An Example: Stabilization of Inverted Pendulum

3.1. Inverted pendulum and its model

To illustrate derivation process of the proposed linearization coordinate, a nonlinear controller based on approximate linearization is described for a rotational inverted pendulum shown in Fig. 1⁽¹⁰⁾. The rotational pendulum is almost free from trajectory limitation and is useful for testing such a nonlinear controller with fast and large motion. The pendulum is a hollow pipe of SUS304. The actuator is a direct drive motor. Variables and parameters are shown in Table 1. Values of dynamic parameters are identified in prior experiments.

The equations of motion are given as follows:

$$\begin{cases} I'_{1zz} \ddot{\theta}_1 + I'_{2xx} (\ddot{\theta}_1 \sin^2 \theta_2 + 2\dot{\theta}_1 \dot{\theta}_2 \sin \theta_2 \cos \theta_2) - ms_{2y} (l_1 \ddot{\theta}_2 \cos \theta_2 - l_1 \dot{\theta}_2^2 \sin \theta_2) \\ \quad + F_{v1} \dot{\theta}_1 + F_{c1} \text{sgn}(\dot{\theta}_1) = \tau, \\ I'_{2zz} \ddot{\theta}_2 - I'_{2xx} \dot{\theta}_1^2 \sin \theta_2 \cos \theta_2 - ms_{2y} (l_1 \ddot{\theta}_1 \cos \theta_2 + g \sin \theta_2) = 0. \end{cases} \quad (21)$$

Fig. 1 Single rotational pendulum

Table 1 Variables and parameters of the pendulum system

θ_1	arm position	—
θ_2	pendulum position	—
τ	motor torque	—
l_1	arm length	4.00×10^{-1} m
I'_{1zz}	moment of inertia of arm	2.45×10^{-2} kgm ²
I'_{2xx}	moment of inertia of pendulum (1)	2.08×10^{-3} kgm ²
I'_{2zz}	moment of inertia of pendulum (2)	3.72×10^{-4} kgm ²
ms_{2y}	1st moment of inertia of pendulum	3.37×10^{-3} kgm
F_{v1}	viscous friction coefficient of motor	1.39×10^0 Nms
F_{c1}	Coulomb friction of motor	2.77×10^0 Nm

A pair of Eqs. (21) is a single input 2-DOF upper actuated system, and is written as a 4th order nonlinear system via collocated partial linearization:

$$\frac{dx}{dt} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ 0 \\ \frac{I'_{2xx}\dot{\theta}_1^2 \cos \theta_2 + ms_{2y}g}{I'_{2zz}} \sin \theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{ms_{2y}l_1 \cos \theta_2}{I'_{2zz}} \end{bmatrix} u, \quad (22)$$

$$u := \frac{I'_{2zz}}{I'_{2zz}(I'_{1zz} + I'_{2xx} \sin^2 \theta_2)} \left\{ \tau - 2I'_{2xx} \sin \theta_2 \cos \theta_2 \dot{\theta}_1 \dot{\theta}_2 + ms_{2y}l_1 \sin \theta_2 \dot{\theta}_2^2 - F_{v1}\dot{\theta}_1 - F_{c1} \operatorname{sgn}(\dot{\theta}_1) \right\}. \quad (23)$$

$x := [\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2]$ is the state vector and u is a new input.

3.2. Linearization coordinate

From Eq. (22), $g_4(q) = ms_{2y}l_1 \cos \theta_2 / I'_{2zz}$, the PDE (12) for $\lambda(x)$ takes form:

$$\frac{\partial \lambda}{\partial \theta_1} + \frac{ms_{2y}l_1}{I'_{2zz}} \cos \theta_2 \frac{\partial \lambda}{\partial \theta_2} = 0. \quad (24)$$

In this case, since $g_4(q)$ depends only on θ_2 , the general solution to Eq. (24) is analytically obtained by solving a set of characteristic equations:

$$\begin{cases} \frac{d\theta_1}{dt} = 1, \\ \frac{d\lambda}{dt} = 0, \\ \frac{d\theta_2}{dt} = -\frac{ms_{2y}l_1}{I'_{2zz}} \{\theta_1 - \theta_1(s)\} + \theta_2(s). \end{cases} \quad (25)$$

λ is an arbitrary function of $(ms_{2y}l_1\theta_1/I'_{2zz} - \operatorname{gd}^{-1} \theta_2)$:

$$\lambda = \lambda \left(\frac{ms_{2y}l_1}{I'_{2zz}} \theta_1 - \operatorname{gd}^{-1} \theta_2 \right), \quad (26)$$

where $\operatorname{gd}(\cdot)$ denotes Gudermannian and is defined as

$$\operatorname{gd} u := 2 \arctan e^u - \frac{\pi}{2}.$$

Inverse Gudermannian is given by

$$\operatorname{gd}^{-1} \theta = \frac{1}{2} \log \left(\frac{1 + \sin \theta}{1 - \sin \theta} \right) = \log \left(\frac{1 + \sin \theta}{\cos \theta} \right) = \log \left\{ \tan \left(\frac{\theta}{2} + \frac{\pi}{4} \right) \right\}, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

Among many possible candidates for a scalar field (26), the simplest $\lambda(x)$:

$$\lambda(x) = \theta_1 - \frac{I'_{2zz}}{ms_{2y}l_1} \operatorname{gd}^{-1} \theta_2. \quad (27)$$

Fig. 2 Inverse Gudermannian and other $\lambda(\mathbf{x})$'s; $\theta_1 = 0$

has been chosen to construct the linearization coordinate for Eq. (22).

Figure 2 shows the difference between proposed $\lambda(\mathbf{x})$: Eq. (27) and two conventional $\lambda(\mathbf{x})$'s. The first conventional $\lambda(\mathbf{x})$ is the 1st order approximation of Eq. (27):

$$\lambda_1(\mathbf{x}) = \theta_1 - \frac{I'_{2zz}}{ms_{2y}l_1}\theta_2, \quad (28)$$

which is derived by the 1st order approximate linearization method⁽⁴⁾ and is known as the center of percussion (COP). The other conventional $\lambda(\mathbf{x})$ is the cubic approximation of Eq. (27):

$$\lambda_3(\mathbf{x}) = \theta_1 - \frac{I'_{2zz}}{ms_{2y}l_1}\theta_2 - \frac{I'_{2zz}}{6ms_{2y}l_1}\theta_2^3, \quad (29)$$

which is introduced in Ref. (7). Thus the proposed linearization coordinate is a higher order extension of conventional ones. As shown in Fig. 2, the proposed $\lambda(\mathbf{x})$ diverges as $|\theta_2| \rightarrow \pi/2$; while the conventional $\lambda(\mathbf{x})$'s are bounded function over $\theta_2 \in (-\pi/2, \pi/2)$.

3.3. Approximation error

Even if the linearization coordinate based on $\lambda(\mathbf{x})$ in Eq. (27) is used, only the 1st order approximate linearization is achieved; that is, approximation error involves the 2nd order term of state variables. The approximation error of input vector field in Eq. (6) is

$$\begin{aligned} L_g\lambda(\mathbf{x}) &= 0, \\ L_gL_f\lambda(\mathbf{x}) &= 0, \\ L_gL_f^2\lambda(\mathbf{x}) &\doteq -(97.2\dot{\theta}_1 + 185\dot{\theta}_2)\theta_2 + (31.1\dot{\theta}_1 - 33.3\dot{\theta}_2)\theta_2^3 - (10.5\dot{\theta}_1 + 29.0\dot{\theta}_2)\theta_2^5 + O^7(x). \end{aligned}$$

However, comparing with that of $\lambda_3(\mathbf{x})$:

$$\begin{aligned} L_g\lambda_3(x) &= 0, \\ L_gL_f\lambda_3(x) &\doteq -1.15 \times 10^{-3}\theta_2^2 + 19.3\theta_2^4 - 4.74\theta_2^6 + O^8(x), \\ L_gL_f^2\lambda_3(x) &\doteq -(97.2\dot{\theta}_1 + 185\dot{\theta}_2)\theta_2 + 4.98 \times 10^{-3}\theta_2^2 + (31.1\dot{\theta}_1 + 121\dot{\theta}_2)\theta_2^3 - 83.6\theta_2^4 \\ &\quad + (9.73\dot{\theta}_1 - 41.5\dot{\theta}_2)\theta_2^5 + 33.4\theta_2^6 + O^7(x), \end{aligned}$$

the approximation error of the second equation vanishes and that of the third equation decreases. Figs. 3(a) and (b) are the absolute value of each entry of the input vector fields, where $\dot{\theta}_1 = 0$ and practical values are chosen as ranges for θ_2 and $\dot{\theta}_2$. As seen in the figures, the error decreases especially for large values of $|\theta_2|$ and $|\dot{\theta}_2|$ when the proposed linearization coordinate is used.

3.4. Closed-loop response

To evaluate the closed-loop response of the proposed linearization based control, a static optimal controller is designed for the approximately linearized system⁽⁷⁾. Fig. 4 is the basin of attraction for the initial value response of the closed-loop system, obtained in numerical simulations. Though the shape depends on the feedback gain, the approximate linearization control

Fig. 3 Approximation error: absolute values of the input vector fields with $\dot{\theta}_1 = 0$

Fig. 4 Basin of attraction; simulation

using $\lambda(\mathbf{x})$ gives the largest area in this example. Fig. 4 shows that maximum values of initial position of pendulum are 1.34 radian (76.8 degree) for inverse Gudermannian ($\lambda(\mathbf{x})$), 0.998 radian for $\lambda_3(\mathbf{x})$, and 0.705 radian for COP ($\lambda_1(\mathbf{x})$). Fig. 5(a) shows the simulation results of the initial value response to the largest value of θ_2 in each case. Fig. 5(b) is another set of simulation results using each $\lambda(\mathbf{x})$ but to the same initial value, where inverse Gudermannian achieves faster convergence to the equilibrium state than convetional $\lambda(\mathbf{x})$'s. Fig. 6 is an experimental result of the initial value response using inverse Gudermannian; the maximum

Fig. 5 Initial value response using each linearization coordinate; numerical simulation

Fig. 6 Initial value response using inverse Gudermannian; experimental result

initial value is 65 degree due to the actuator limitation. These results show the advantage of the approximate linearization control based on the proposed linearization coordinate.

4. Linearization Coordinate for Other 2-DOF Underactuated Systems

4.1. Mobile inverted pendulum

The linearization coordinate can be computed for other systems in a similar way. The equation of motion of the mobile inverted pendulum is written as Eq. (30) using variables and parameters in Table 2.

$$\begin{cases} I'_p \ddot{\theta}_1 + I'_w (\ddot{\theta}_1 + \ddot{\theta}_2) + m s_p R (2\ddot{\theta}_1 + \ddot{\theta}_2) \cos \theta_1 - m s_p (R \dot{\theta}_1^2 + g) \sin \theta_1 = 0, \\ I'_w (\ddot{\theta}_1 + \ddot{\theta}_2) + m s_p R (\ddot{\theta}_1 \cos \theta_1 + \dot{\theta}_1^2 \sin \theta_1) + F_{v2} \dot{\theta}_2 + F_{c2} \operatorname{sgn}(\dot{\theta}_2) = \tau. \end{cases} \quad (30)$$

Table 2 Variables and parameters of mobile inverted pendulum

θ_1	pendulum position
θ_2	wheel position
τ	motor torque
R	radius of wheel
I'_p	moment of inertia of pendulum
I'_w	moment of inertia of wheel
$m s_p$	1st moment of inertia of pendulum
F_{v2}	viscous friction coefficient of motor
F_{c2}	Coulomb friction of motor

The configuration variable θ_1 is the pendulum position and θ_2 is the wheel rotation; since the wheel is driven by a motor, the system is lower actuated.

The input vector field of the partially linearized system gives

$$g_3(q) = -\frac{I'_w + m s_p R \cos \theta_1}{I'_w + I'_p + 2m s_p R \cos \theta_1},$$

which depends on θ_1 alone. Substituting $g_3(q)$ into Eq. (20), a PDE for the linearization coordinate is:

$$-\frac{I'_w + m s_p R \cos \theta_1}{I'_w + I'_p + 2m s_p R \cos \theta_1} \frac{\partial \lambda}{\partial \theta_1} + \frac{\partial \lambda}{\partial \theta_2} = 0. \quad (31)$$

The general solution is analytically obtained as follows:

$$\lambda(x) = \theta_1 + \frac{\theta_2}{2} + \frac{(I'_p - I'_w)}{\sqrt{I_w'^2 - m s_p^2 R^2}} \arctan \left(\sqrt{\frac{I'_w - m s_p R}{I'_w + m s_p R}} \tan \frac{\theta_1}{2} \right). \quad (32)$$

4.2. Acrobot & Pendubot

Acrobot and Pendubot have identical structure to the vertical, planer double pendulum. The base is fixed and either joint is actuated. The equation of motion for the both is written as Eq. (33) using variables and parameters in Table 3.

Table 3 Variables and parameters of Acrobot & Pendubot

θ_1	first arm position
θ_2	second arm position
τ	motor torque (either first or second joint)
l_1	first arm length
I'_{1zz}	moment of inertia of first arm
I'_{2zz}	moment of inertia of second arm
$m s_{1x}$	1st moment of inertia of first arm
$m s_{2x}$	1st moment of inertia of second arm
F_v	viscous friction coefficient of motor
F_c	Coulomb friction of motor

$$\begin{cases} I'_{1zz}\ddot{\theta}_1 + I'_{2zz}(\ddot{\theta}_1 + \ddot{\theta}_2) + ms_{2x}l_1\{(\ddot{\theta}_1 + \ddot{\theta}_2)\cos\theta_2 - \dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2)\sin\theta_2\} \\ - \{ms_{1x}\sin\theta_1 + ms_{2x}\sin(\theta_1 + \theta_2)\}g + F_{v1}\dot{\theta}_1 + F_{c1}\operatorname{sgn}(\dot{\theta}_1) = \tau_1, \\ I'_{2zz}(\ddot{\theta}_1 + \ddot{\theta}_2) + ms_{2x}l_1(\ddot{\theta}_1\cos\theta_2 + \dot{\theta}_1^2\sin\theta_2) - ms_{2x}g\sin(\theta_1 + \theta_2) \\ + F_{v2}\dot{\theta}_2 + F_{c2}\operatorname{sgn}(\dot{\theta}_2) = \tau_2. \end{cases} \quad (33)$$

In the case of Acrobot, $\tau_1 = 0$ since the second joint is actuated; thus Acrobot is a lower actuated system. By substituting

$$g_3(q) = -\frac{I'_{2zz} + ms_{2x}l_1\cos\theta_2}{I'_{1zz} + I'_{2zz} + 2ms_{2x}l_1\cos\theta_2}$$

into Eq. (20), a PDE for linearization coordinate is as follows:

$$-\frac{I'_{2zz} + ms_{2x}l_1\cos\theta_2}{I'_{1zz} + I'_{2zz} + 2ms_{2x}l_1\cos\theta_2}\frac{\partial\lambda}{\partial\theta_1} + \frac{\partial\lambda}{\partial\theta_2} = 0. \quad (34)$$

$\lambda(x)$ is analytically obtained as the general solution:

$$\lambda(x) = \theta_1 + \frac{\theta_2}{2} - \frac{(I'_{1zz} - I'_{2zz})}{\sqrt{(I'_{1zz} + I'_{2zz})^2 - 4ms_{2x}^2l_1^2}} \arctan\left(\sqrt{\frac{I'_{1zz} + I'_{2zz} - 2ms_{2x}l_1}{I'_{1zz} + I'_{2zz} + 2ms_{2x}l_1}} \tan\frac{\theta_2}{2}\right). \quad (35)$$

On the contrary, Pendubot is an upper actuated system and the equation of motion is Eq. (33) subject to $\tau_2 = 0$. Substitution of $g_4(q)$ into (12) gives a PDE for $\lambda(x)$:

$$\frac{\partial\lambda}{\partial\theta_1} - \frac{I'_{2zz} + ms_{2x}l_1\cos\theta_2}{I'_{2zz}}\frac{\partial\lambda}{\partial\theta_2} = 0. \quad (36)$$

The linearization coordinate is

$$\lambda(x) = \frac{\theta_1}{2} + \frac{I'_{2zz}}{\sqrt{I_{2zz}^{'2} - ms_{2x}^2l_1^2}} \arctan\left(\sqrt{\frac{I'_{2zz} - ms_{2x}l_1}{I'_{2zz} + ms_{2x}l_1}} \tan\frac{\theta_2}{2}\right). \quad (37)$$

5. Conclusion

This paper proposes the approximate linearization control of 2-DOF underactuated-by-1 systems using a novel linearization coordinate, which is determined by solving a PDE. The PDE is proved of a first order, linear, homogeneous, and with two configuration variables. For this small class of systems the general solution is expected to be obtained analytically. It depends on the particular form of the input vector field after partial linearization.

Highly nonlinear inverse Gudermannian is found for the inverted pendulum as the novel linearization coordinate. The approximation error in the resulting linearized system is small comparing with conventional ones. The basin of attraction is also enlarged, which achieves fast transient response and convergence for large initial value, in both simulations and experiments.

Nonlinear linearization coordinates are computed for other underactuated systems, but their advantages should be studied in future research.

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