PSEUDO-THREE DIMENSIONAL LAGRANGIAN PARTICLE
FINITE DIFFERENCE METHOD FOR MODELING
LONG-TRAVELING SOIL FLOWS

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Lagrangian Particle Finite Difference Method (LPFDM) is extended to handle rapid and
long-traveling flows of soil. LPFDM describes a soil mass as a cluster of Lagrangian material points
that move with the material and carry all necessary information, which for each time step is mapped
back and forth between the points and a Eulerian grid where the equations of motion are solved.

Key words: large deformations of soils, Lagrangian particle, Eulerian grid

1. INTRODUCTION

The January 13, 2001, Off the Coast of El Salvador
Earthquake triggered total 445 landslides in this country,
and they were mostly initiated from volcanoes and/or
thick sediments of volcanic products. Among them, Las
Colinas Landslide was the most tragic¹,². The total amount
of soil of about 200,000 m³ was not surprisingly large as
contrasted with the huge soil mass of a couple of tens
million m³, which was initiated at the top of the Mt.
Huascaran (6,700 m EL), and killed more than 20,000
people (1970 Peru Earthquake). The Las Colinas landslide,
however, was substantially large in terms of damage
because the slid soil mass surged across a residential district
of Las Colinas, and washed many houses and thus more
than 500 lives to death.

As for extremely large soil deformations, it is very
difficult and sometimes impossible to stop them. Thus, we
have to deal with the problems by developing and enforc-
ing land-use building ordinances that regulate construc-
tions in areas susceptible to landslides and debris flows.
For this, it is necessary to predict a possible extent of soil
deformation, distal end that a slipped soil mass can reach,
and the velocity of a flowing soil mass.

For studying large deformations of soils, numerical
methods such as FEM or FDM have been widely used.
For example, the finite difference based FLAC (Fast
Lagrangian Analysis of Continua³) successfully calculates
large strains by using low-order strain elements. However,
when dealing with large strains, highly distorted elements
often account for inaccurate results.

In the field of computational fluid dynamics, where
history-dependent materials are less common, purely Euler-
ian methods are often used. Sulsky et al.⁴ extended it
further to solid mechanics. Their method evolved from a
particle-in-cell (PIC) method is referred to as the Lagran-
gian Particle Method (LPM) or the Material Point Method
(MPM). In LPM, all Lagrangian variables such as mass,
stress, strain, position, strength etc are carried by uncon-
nected Lagrangian material points; the points, as an entire
cluster, describe the mass of the material. A Eulerian mesh
is used to solve the equations of motion in each time step.

2. LPFDM ALGORITHM

LPFDM embodies an explicit time-marching scheme⁵,⁶,
while implicit, matrix-oriented solution schemes are more
common in finite element methods. As mentioned, Lagran-
gian material points, carrying all Lagrangian variables (see
Fig. 1), describe the solid phase of the material as a cluster.
A Eulerian mesh is used to solve the equations of motion
in each time step. Data mapped from the material points in
one cell of the mesh contribute to cell’s nodes, and accelerate them. The material points’ variables are updated with mesh solution.

Slopes that fail by translation on planar failure surface are to be analyzed herein. A soil mass flowing thin over a failure surface is modeled as a two-dimensional soil “mat” ($\xi-\psi$ plane in Fig. 2). Eulerian cells are arranged in such a way that their projections on a horizontal x-y plane are a regular square mesh with sides parallel to x and y axes of the Cartesian coordinate system. Though the real slope is not a perfect plane, each cell is assumed to be small enough for the cell and its neighboring cells to be arranged in one plane. The cells on the $\xi-\psi$ plane are thus parallelograms. The orientation of the $\xi-\psi$ plane is described by $\xi$ and $\psi$ axes; the $\xi$ axis is a horizontal line produced by the intersection of the $\xi-\psi$ plane and a horizontal plane, while the $\psi$ axis describes the slope of the $\xi-\psi$ plane.

For each cell on the $\xi-\psi$ plane, strains are computed by simply invoking the Gauss’ divergence theorem, and the average Eulerian strain tensor increments of $J$-th element are thus obtained as:

$$
(\Delta \varepsilon)_{ij} = \frac{1}{2A} \left[ \begin{array}{ll}
\Delta u_{i[1]}^{[1]} + \Delta u_{i[2]}^{[2]} & \frac{1}{2} (i[4] - i[1]) (j[3] - j[2]) - \frac{1}{2} (i[3] - i[2]) (j[4] - j[1]) \\
\Delta u_{j[2]}^{[2]} + \Delta u_{j[3]}^{[3]} & \frac{1}{2} (j[2] - j[3]) (i[4] - i[1]) - \frac{1}{2} (j[1] - j[2]) (i[3] - i[2]) \\
\Delta u_{j[3]}^{[3]} + \Delta u_{j[4]}^{[4]} & \frac{1}{2} (j[1] - j[2]) (i[1] - i[2]) - \frac{1}{2} (j[3] - j[2]) (i[2] - i[3]) \\
\Delta u_{i[4]}^{[4]} + \Delta u_{i[1]}^{[1]} & \frac{1}{2} (i[4] - i[1]) (j[1] - j[2]) - \frac{1}{2} (i[3] - i[2]) (j[4] - j[1]) \\
\end{array} \right]
$$

(1)

where, $A$ is the area of Cell $J$, $(i \ j)$ is either $(\xi \ \psi)$ or $(\psi \ \xi)$ of the Cartesian coordinate system, $\Delta u_i$ is a displacement increment in $i$ direction, and superscripts $[1], [2], [3]$ and $[4]$ denote local nodal point numbers (Fig. 3). All material points included in Cell $J$ are assumed to experience the same strain increments given by equation (1), and the stresses for a material point $k$ included in cell $J$ are computed by means of a specified constitutive model:

$$
(\sigma_{ij})_{k,updated} = M((\Delta \varepsilon_j)_{ij}, (\sigma_{ij})_k, (s_1)_k, (s_2)_k, \ldots)
$$

(2)

where, $M(\ )$ is the constitutive model, and $(s_i)_k$ are state variables for this material point. For the explicit scheme used in LPFDM, the constitutive law is only consulted once per material point during one time step.

By virtue of the simple algorithm for the square mesh, the location of material point $k$ in the cell $J$ deformed at the previous time step is updated not on the $\xi-\psi$ plane but on the x-y plane as:

$$
x(k) = x(k) + \Delta x(k), \
y(k) = y(k) - \Delta y(k)
$$

(3a)
where,
\[ \Delta x(k) = x(k) - (x\text{-coordinate of node}[1]\text{ of Cell } J) \]
\[ \Delta y(k) = y(k) - (y\text{-coordinate of node}[1]\text{ of Cell } J), \]
and displacement increments in x and y directions, respectively, with superscripts [1], [2], [3] and [4] as local nodal point numbers on the x-y plane.

The updated locations of material points are then mapped back on the \( \xi - \psi \) plane, and the Eulerian mesh is then shifted back to its original position for the following calculation cycle.

The mass of each cell is obtained by adding up all the masses of material points included in Cell J:
\[ \langle M \rangle_J = \sum_i m_k \quad \text{(on } \xi - \psi \text{ plane)} \tag{4} \]
Here, it is noted that a set of material points included in Cell J has been also updated.

Stress is smeared over the cell:
\[ \langle \sigma \rangle_J = \left( \sum_{i,j} \frac{m_k}{\rho_k} (\sigma_{ij})_k \right)/A \quad \text{(on } \xi - \psi \text{ plane)} \tag{5} \]
where, the quotient of material point mass and density \( \frac{m_k}{\rho_k} \) is the volume of the material point \( k \). In order to obtain the nodal forces at node \( [k] \) of Cell J (see Fig. 3), a virtual unit displacement is applied to node \( [k] \). The stress components \( \langle \sigma \rangle_J \) smeared over Cell J must perform the same amount of work done by the nodal forces \( F_i^{[k]}(i=\xi \text{ or } \psi) \), this calls for:
\[ F_i^{[k]}(i=\xi \text{ or } \psi) = \frac{\langle \sigma \rangle_J}{2} \left( f_i^{[k+1]} - f_i^{[k-1]} + \langle \sigma \rangle_J (i^{[k+1]} - i^{[k-1]}) \right) \quad \text{(on } \xi - \psi \text{ plane)} \tag{6} \]
where, \((i \ j)\) is either \((\xi \ \psi)\) or \((\psi \ \xi)\).

At each node, the forces from all surrounding lattice zones (elements) are summed up to give the net nodal force, \( \sum_{[k]\in(k)} F_i^{[k]}(i=\xi \text{ or } \psi, \text{ and } (k) \text{ is the global node number; see Fig. 4}). This vector includes contributions from driving forces \( F_{di}^{[k]} \) and the corresponding base shear forces \( F_{bi}^{[k]} \) due to gravity because the soil mass is put on the inclined \( \xi - \psi \) plane. The driving force is computed from:
\[ F_{di}^{[k]} = g m^{[k]} \sin \theta_{[k]}^{(i)} \quad \text{(on } \xi - \psi \text{ plane)} \tag{7} \]
Here, \( g \) is the gravitational acceleration, \( m^{[k]} \) is the lumped mass at the node \( (k) \), defined as 1/4 of the masses of the cells connected to the node, and \( \theta_{[k]}^{(i)} \) is the inclination of the slope in \( i \) direction. Since the \( \xi \) axis is a horizontal line on the \( \xi - \psi \) plane, \( \theta_{[k]}^{(x)} \) is obviously zero, while \( \theta_{[k]}^{(y)} \) is the inclination of the slope at this node. Due attention should be paid to the corresponding base shear force \( F_{bi}^{[k]}(i) \), which will be a complicated function of the displacement increments \( \Delta u_i^{[k]}, \Delta u_y^{[k]} \) at the node \( (k) \) and other state variables, \( s_{[k]}^{(i)} (i=1, 2, 3, \ldots) \). Determining its appropriate expression will be a matter of further discussion, to be sure, but the same assumption as that used in the Newmark’s Sliding Block Analysis will be the simplest. In the assumption, \( F_{bi}^{[k]} \) cancels the total driving force so that the soil mass is automatically put in equilibrium unless the driving force exceeds the available resisting force, \( F_{ri}^{[k]}(i) \), which is given by:
\[ F_{ri}^{[k]} = g m^{[k]} \cos \theta_{[k]}^{(i)} \tan \phi_{[k]}^{(i)} \quad \text{(on } \xi - \psi \text{ plane)} \tag{8} \]
with \( \phi_{[k]}^{(i)} = \text{friction angle mobilized on the slip surface}. \)

The available resistance forces on the slip surface often decreases drastically as the soil mass slips further down because grain crushing causes the buildup of pore water pressure along the slope surface. To reflect this feature of soil, mobilized friction angle \( \phi_{[k]}^{(i)} \) for a material point \( k \) is assumed to be slip-distance dependent, and given by:
\[ \phi_{[k]}^{(i)} = \phi_{[k]}^{(i)}_{\text{initial}} \exp\left(-\left|u_k^{(i)}\right|/u_{\text{ref}}\right) \quad \text{(9)} \]
where, \( \phi_{[k]}^{(i)}_{\text{initial}} = \text{initial value of the friction angle, and} \ u_{\text{ref}} = \text{the distance that the material point has slipped for } 1/e \text{ of its initial friction angle to be reached. Given } \phi_{[k]} \text{ for the material points included in cells surrounding a nodal point } (k), \phi_{[k]}^{(i)} \text{ is obtained through the same smearing procedure as explained in Equation (5).} \]

If a connected cell does not contain any material point, its contribution to the nodal force is omitted. The nodal force accelerates the lumped mass, and the acceleration is integrated to obtain the nodal displacement increment:
\[ \Delta u_i^{[k]}(t+\Delta t/2) = \Delta u_i^{[k]}(t-\Delta t/2) + \Delta \frac{t^2}{m^{[k]}_{\text{total}}} \sum F_i^{[k]} \quad \text{(on } \xi - \psi \text{ plane)} \tag{10} \]
It is noted in Equation (10) that the nodal displacement increment \( \Delta u_i^{[k]}(t-\Delta t/2) \) is not exactly identical to that evaluated at the previous time step of the calculation cycle, because the increment must be evaluated at the node on the Eulerian lattice shifted back to its original position while the point where the previous increment was evaluated has passed by this node. Compensation for this convection is made by the following procedure:
Table 1. Lagrangian parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's modulus</td>
<td>$5 \times 10^7$ N/m$^2$</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>0.30</td>
</tr>
<tr>
<td>Density</td>
<td>1700 kg/m$^3$</td>
</tr>
<tr>
<td>Internal friction angle</td>
<td>0.5 rad</td>
</tr>
<tr>
<td>Cohesion</td>
<td>9800 N/m$^2$</td>
</tr>
<tr>
<td>Strength reduction:</td>
<td></td>
</tr>
<tr>
<td>Initial friction angle on the slip surface</td>
<td>0.5 rad</td>
</tr>
<tr>
<td>$\alpha$ for local non-viscous damping</td>
<td>0.1 m</td>
</tr>
<tr>
<td>$L$: Cell size on $x$-$y$ plane</td>
<td>1 m</td>
</tr>
</tbody>
</table>

Fig. 5a-5c show the plan of the slope. Contour lines in this figure show that there are two slopes A and B making up the configuration. Diagonal contour lines on the left show that Slope A goes diagonally down to the right, while contour lines to the right describe that Slope B dips 45 degrees leftward. Lagrangian points are initially arranged in square on Slope B. The gravitational acceleration was then given at once to the soil mass, and the mass started sliding the slope under its own weight (Fig. 5a-5c). With no confinement on its edge, the soil mass spread wide as it surged across the horizontal plane, and after reaching Slope A, the direction of the mass flow turned avoiding Slope A.

4. SUMMARY

A possible extension of LPFDM (Lagrangian Particle Finite Difference Method) was presented for analyzing a rapid and long-traveling flow of soil that fails by translation on a planar failure surface. The present method describes a soil mass as a two-dimensional soil "mat" spreading thin over the failure surface. The extended LPFDM has essentially the same scheme of time-marching calculation as that for the original LPFDM, excluding that the square Eulerian mesh on a horizontal $x$-$y$ plane is projected upon the slope. Material points can go across the mesh allowing long-traveling flows to be easily described, while driving forces due to gravity are uniquely determined at all nodes of the stationary Eulerian mesh. Thus, the present method, fully utilizing both merits of Lagrangian points and the Eulerian mesh, is well suited for analyzing rapid and long-traveling flows of soil. The numerical example given herein demonstrated the potential of the method.

It is noted however that the change of slope inclination causes normal components of nodal forces with respect to the failure surface to appear. The present method, thus, needs to be improved in such a way that the effect of a largely curved slope is taken into account, and to be verified through numerical simulations of real flows. To allow for more quantitative discussions, appropriate constitutive models for surface soils and slip surfaces must be provided. This will be discussed in future publications.
Fig. 5. Plan of landslide (projection on $x$-$y$ plane).

(a) $t = 0$ s

(b) $t = 5$ s

(c) $t = 10$ s
APPENDIX I: TRANSFORMATION OF COORDINATES

Two sets of Cartesian axes, $\xi - \psi$ and $x-y$ describe a slope and its projection on a horizontal plane, respectively, and material points are mapped back and forth between these two planes. The orientation of the $\xi - \psi$ plane is described by $\xi$ and $\psi$ axes (Fig. A1). The $\xi - \psi$ is a horizontal line produced by the intersection of the $\xi - \psi$ plane and the horizontal plane. Its azimuth with respect to $x$ axis is denoted by $\phi$. The angle $\theta$ between $\psi$ and the horizontal plane is the inclination of the $\xi - \psi$ plane. These sets of coordinates are assumed to have a common origin at Nodal point [1] of Cell J. The projection of $\psi$ on the $x-y$ plane is denoted by $\psi'$, which is given by;

$$\psi' = \psi \cos \theta$$  

$(\xi - \phi')^T$ is obtained by rotating $(x \ y)^T$ by $\phi$, namely;

$$\left(\begin{array}{c}
\xi \\
\psi
\end{array}\right) = \left[\begin{array}{cc}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{array}\right] \left(\begin{array}{c}
x \\
y
\end{array}\right)$$  

(A2)

From Equations (A1) and (A2), one obtains;

$$\left(\begin{array}{c}
\xi \\
\psi
\end{array}\right) = \left[\begin{array}{cc}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{array}\right] \left(\begin{array}{c}
x \\
y
\end{array}\right)$$  

(A3)

and for the inverse transformation,

$$\left(\begin{array}{c}
x \\
y
\end{array}\right) = \left[\begin{array}{cc}
\cos \phi & -\cos \theta \sin \phi \\
\sin \phi & \cos \theta \cos \phi
\end{array}\right] \left(\begin{array}{c}
\xi \\
\psi
\end{array}\right)$$  

(A4)

REFERENCES


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