Numerical Inclusion of Optimum Point for Linear Programming

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Abstract

This paper concerns with the following linear programming problem:

$$\text{Maximize } c^t x, \text{ subject to } Ax \leq b \text{ and } x \geq 0,$$

where $A \in \mathbb{F}^{m \times n}$, $b \in \mathbb{F}^m$ and $c, x \in \mathbb{F}^n$. Here, $\mathbb{F}$ is a set of floating point numbers.

The aim of this paper is to propose a numerical method of including an optimum point of this linear programming problem provided that a good approximation of an optimum point is given. The proposed method is base on Kantorovich's theorem and the continuous Newton method. Kantorovich's theorem is used for proving the existence of a solution for complimentarity equation and the continuous Newton method is used to prove feasibility of that solution. Numerical examples show that a computational cost to include optimum point is about 4 times than that for getting an approximate optimum solution.

Keywords numerical verification, continuous Newton method, Kantorovich's theorem

Research Activity Group Quality of Computations

1. Introduction

In this paper, we are concerned with the following linear programming problem:

$$\text{Maximize } c^t x, \text{ subject to } Ax \leq b \text{ and } x \geq 0, \quad (1)$$

where $A \in \mathbb{F}^{m \times n}$, $b \in \mathbb{F}^m$ and $c, x \in \mathbb{F}^n$. Here, $\mathbb{F}$ is a set of floating point numbers. The superscript t indicates the transposition. The aim of this paper is to propose a numerical method of including an optimum point of this linear programming problem provided that a good approximation of an optimum point is given.

Let x_f be a feasible point of the primal problem (1), *i.e.*, x_f be a point satisfying

$$Ax_f \leq b \text{ and } x_f \geq 0. \quad (2)$$

It is clear that $c^t x_f$ becomes a lower bound of the optimum value. A dual problem for (1) is given by

$$\text{Minimize } b^t y, \text{ subject to } A^t y \geq c \text{ and } y \geq 0, \quad (3)$$

where $y \in \mathbb{F}^m$. Let y_f be a feasible point of the dual problem (3), *i.e.*, y_f be a point satisfying

$$A^t y_f \geq c \text{ and } y_f \geq 0. \quad (4)$$

It is clear that $b^t y_f$ becomes an upper bound of the optimum value. Thus, an inclusion of the optimum value v^* is given by

$$v^* \in [c^t x_f, b^t y_f] \quad (5)$$

provided that feasible points x_f and y_f can be found. The duality theorem asserts that in principle the width of the interval $[c^t x_f, b^t y_f]$ can make as small as desired. This argument, which is rather well known (cf. Ref. [1]),

gives a method of numerical inclusion of the optimum value.

This paper is concerned with a problem of including numerically an optimum point of (1). For the purpose, we shall consider the following complimentarity problem

$$f(z) = \begin{pmatrix} x(A^t y - c) \\ y(b - Ax) \end{pmatrix} = 0 \in \mathbb{R}^{n+m} \quad (6)$$

subject to

$$x \geq 0, y \geq 0, b - Ax \geq 0 \text{ and } A^t y - c \geq 0, \quad (7)$$

where $z = (x^t, y^t)^t$. Here, for two vectors u, v with the same dimension, uv denotes a vector of the same dimension with components $u_i v_i$. To solve (6) subject to (7) is equivalent to solve the primal and dual problem (1) and (3). Tanabe [2] has applied Kantorovich's theorem to (6) to estimate error in an approximate solution. However, the solution proved to be included by this approach is not guaranteed to satisfy the feasibility condition (7). This paper resolves this difficulty by introducing a continuous Newton method. Namely, this paper proposes a method, in which Kantorovich's theorem is used for proving the existence of a solution for complimentarity equation and the continuous Newton method is exploited to prove feasibility of that solution. Since Kantorovich's theorem for the Newton method is used, non-degeneracy of the optimum solution is required for our analysis. Degenerate cases will be considered in a separate paper.

2. Verification Method

The center path of (6) (cf. for instance Refs. [3–5]) is defined by

$$f(z) = \begin{pmatrix} x(A^t y - c) \\ y(b - Ax) \end{pmatrix} = \gamma e, \quad (8)$$

where $e \in \mathbb{R}^{m+n}$ with all elements being 1. The constant γ is defined by

$$\gamma = \frac{\|f(z)\|_1}{m+n} = \frac{(b^t y - c^t x)}{m+n} \quad (9)$$

provided that z is a feasible point. Namely, if z is a feasible point, γ is the duality gap of the problem divided by $m+n$. This fact is pointed out in Refs. [4, 5].

The Fréchet derivative $f'(z)$ is given by

$$f'(z) = \begin{pmatrix} [A^t y - c] & [x]A^t \\ -[y]A & [b - Ax] \end{pmatrix}, \quad (10)$$

where for a vector $x = (x_1, x_2, \dots, x_n)^t$, $[x]$ denotes $\text{diag}(x_1, x_2, \dots, x_n)$. At a given approximate optimum point z , the Newton direction d_n and the centered direction d_c are defined by

$$f'(z)d_n = - \begin{pmatrix} x(A^t y - c) \\ y(b - Ax) \end{pmatrix} \quad (11)$$

and

$$f'(z)d_c = - \begin{pmatrix} x(A^t y - c) \\ y(b - Ax) \end{pmatrix} + \gamma e, \quad (12)$$

respectively.

In the method we shall propose the following verification procedures. First, an interior point of (6) is searched for a searching direction, which is a linear combination of d_n and d_c , based on the guiding cone method or the penalized norm method [4, 5]. Here, we assume that we can find an interior point z , which is a good approximation of an optimum point. Then, the second step of our method is to check conditions of the following theorem at the point z :

Theorem 1 *Let $z \in \mathbb{R}^{m+n}$ be an interior point, namely a point satisfying (7) with inequality condition:*

$$x > 0, \quad y > 0, \quad b - Ax > 0 \quad \text{and} \quad A^t y - c > 0. \quad (13)$$

Let further constants α and ω be defined by the inequalities $\alpha \geq \|f'(z)^{-1}\|_\infty \|f(z)\|_\infty$ and $\omega \geq 2 \max(\|A\|_\infty, \|A\|_1) \|f'(z)^{-1}\|_\infty$, respectively. If

$$\alpha\omega \leq \frac{1}{4}, \quad (14)$$

there exists an optimal point $z^ = (x^{*t}, y^{*t})^t \in B(z, \rho) = \{z' \in \mathbb{R}^{m+n} \mid \|z' - z\|_\infty \leq \rho\}$, which is a point satisfying (6) and (7), where*

$$\rho = \frac{1 - \sqrt{1 - 3\alpha\omega}}{\omega}. \quad (15)$$

The optimum point z^ is unique in $B(z, \rho)$.*

We note that the half assertion of Theorem 1 can be derived from the following Kantorovich theorem applied to the nonlinear equation (6):

Theorem 2 (Kantorovich's Theorem for (6)) *Let f be defined by (6). We assume that the Fréchet derivative*

$f'(z)$ is nonsingular and satisfies the inequality

$$\alpha' \geq \|f'(z)^{-1} f(z)\|_\infty \quad (16)$$

for a certain positive α' . Furthermore, we assume that f satisfies

$$\begin{aligned} & \|f'(z)^{-1}(f'(z') - f'(z''))\|_\infty \\ & \leq \omega' \|z' - z''\|_\infty \quad \text{for } \forall z', z'' \in \mathbb{R}^{m+n} \end{aligned} \quad (17)$$

with a certain positive constant ω' . If

$$\alpha' \omega' \leq \frac{1}{2}, \quad (18)$$

and

$$\rho' = \frac{1 - \sqrt{1 - 2\alpha'\omega'}}{\omega'}, \quad (19)$$

there exists a point $z^ = (x^{*t}, y^{*t})^t \in B(z, \rho')$ satisfying (6). The solution z^* of (6) is unique in $B(z, \rho')$.*

Proof of Theorem 1 We assume that the conditions of Theorem 1 is satisfied.

In the first place, we shall show that the conditions of Theorem 2 are satisfied. We note that f is defined on $\mathbb{R}^{m+n}$. If we put $\alpha' = 1.5\alpha$, then

$$\begin{aligned} \|f'(z)^{-1} f(z)\|_\infty & \leq \|f'(z)^{-1}\|_\infty \|f(z)\|_\infty \\ & \leq \alpha < \alpha'. \end{aligned} \quad (20)$$

It is further noted that for any $z', z'' \in \mathbb{R}^{m+n}$

$$\begin{aligned} & f'(z') - f'(z'') \\ & = \begin{pmatrix} [A^t(y' - y'')] & [x' - x'']A^t \\ -[y' - y'']A & [-A(x' - x'')] \end{pmatrix}. \end{aligned} \quad (21)$$

Let $e_k = (1, 1, \dots, 1)^t \in \mathbb{R}^k$ and I_k be the identity matrix in $\mathbb{R}^k$. Then, from (21), we have the following elementwise inequality:

$$\begin{aligned} & |f'(z') - f'(z'')| \\ & \leq \|z' - z''\|_\infty \begin{pmatrix} \|A\|_1^t e_m & [e_n] |A|^t \\ [e_m] |A| & \|A\|_\infty e_n \end{pmatrix} \\ & \leq \|z' - z''\|_\infty \begin{pmatrix} \|A\|_1 I_n & |A|^t \\ |A| & \|A\|_\infty I_m \end{pmatrix} \end{aligned} \quad (22)$$

which implies

$$\begin{aligned} & \|f'(z') - f'(z'')\|_\infty \\ & \leq 2 \max(\|A\|_\infty, \|A\|_1) \|z' - z''\|_\infty. \end{aligned} \quad (23)$$

Here, for $x = (x_1, x_2, \dots, x_k)^t$, $y = (y_1, y_2, \dots, y_k)^t \in \mathbb{R}^k$

$$|x| = (|x_1|, |x_2|, \dots, |x_k|)^t \quad (24)$$

and

$$x \leq y \iff x_i \leq y_i, \quad (i = 1, 2, \dots, k). \quad (25)$$

Hence, we can use ω in Theorem 1 as ω' in Theorem 2. If we put $\alpha' = 1.5\alpha$ and $\omega' = \omega$, we have

$$\alpha' \omega' = 1.5\alpha\omega \leq 3/8 < 1/2. \quad (26)$$

Further, ρ' coincides with ρ . Thus, from Kantorovich's theorem (Theorem 2) it is seen that there exists a solution $z^* = (x^{*t}, y^{*t})^t \in B(z, \rho)$ satisfying (6). Kan-

torovich's theorem states also that z^* is unique solution of (6) in the closed ball $B(z, \rho)$.

Next, we show that z^* is feasible, *i.e.*, it satisfies the inequality conditions (7). Let us consider a solution curve of the following continuous Newton method starting from a given interior point z :

$$\frac{dz(t)}{dt} = -f'(z(t))^{-1}f(z(t)) \text{ with } z(0) = z. \quad (27)$$

The elementary theory for differential equations such as the Picard-Lindelöf theorem (see, for example, [6]) states that the solution curve $z(t)$ exists for $t \in [0, M)$ for a certain positive constant M .

Suppose $T < M$ be the smallest value of T such that $z(T)$ is on the boundary of the closed ball $B(z, \rho)$. Then

$$\|z - z(T)\|_\infty \leq \int_0^T \left\| \frac{dz(t)}{dt} \right\|_\infty dt < k\|f(z)\|_\infty, \quad (28)$$

where k is defined by

$$k = \max_{z' \in B} \|f'(z')^{-1}\|_\infty. \quad (29)$$

This result was used in Refs. [3, 7]. In fact, $z(t)$ satisfies

$$\frac{df(z(t))}{dt} = -f(z(t)) \text{ with } z(0) = z. \quad (30)$$

Thus,

$$f(z(t)) = f(z)e^{-t} \quad (31)$$

holds. Hence, we have

$$\begin{aligned} \left\| \frac{dz(t)}{dt} \right\|_\infty &\leq \|f'(z(t))^{-1}\|_\infty \|f(z(t))\|_\infty \\ &\leq k\|f(z)\|_\infty e^{-t}, \end{aligned} \quad (32)$$

which gives

$$\begin{aligned} \int_0^T \left\| \frac{dz(t)}{dt} \right\|_\infty dt &\leq k\|f(z)\|_\infty (1 - e^{-T}) \\ &< k\|f(z)\|_\infty. \end{aligned} \quad (33)$$

Furthermore, $f(z(t)) = f(z)e^{-t}$ implies $z(t)$ starting with an interior point remains to be an interior point for $t \in [0, M)$.

We note that for $z' \in B(z, \rho)$

$$\|f'(z)^{-1}(f'(z) - f'(z'))\|_\infty \leq \omega\|z - z'\|_\infty \quad (34)$$

holds. We note also that (14) implies

$$\omega\rho < 1. \quad (35)$$

Thus, from (34), it follows that

$$\begin{aligned} k &= \max_{z' \in B} \|f'(z')^{-1}\|_\infty \\ &\leq \max_{z' \in B} \frac{\|f'(z)^{-1}\|_\infty}{1 - \|I - f'(z)^{-1}f'(z')\|_\infty} \\ &= \max_{z' \in B} \frac{\|f'(z)^{-1}\|_\infty}{1 - \|f'(z)^{-1}(f'(z) - f'(z'))\|_\infty} \\ &\leq \max_{z' \in B} \frac{\|f'(z)^{-1}\|_\infty}{1 - \omega\|z - z'\|_\infty} \end{aligned}$$

$$\leq \frac{\|f'(z)^{-1}\|_\infty}{1 - \omega\rho}. \quad (36)$$

Therefore, we have

$$\begin{aligned} k\|f(z)\|_\infty &\leq \frac{\|f'(z)^{-1}\|_\infty \|f(z)\|_\infty}{1 - \omega\rho} \\ &\leq \frac{\alpha}{1 - \omega\rho}. \end{aligned} \quad (37)$$

On the other hand, we can show the following inequality

$$\frac{\alpha}{1 - \omega\rho} \leq \rho. \quad (38)$$

In fact, since $4\alpha\omega \leq 1$, we have the inequality

$$1 - 2\alpha\omega \leq \sqrt{1 - 3\alpha\omega}, \quad (39)$$

which implies the inequality

$$\frac{\alpha}{\sqrt{1 - 3\alpha\omega}} \leq \frac{1 - \sqrt{1 - 3\alpha\omega}}{\omega} \quad (40)$$

which is equivalent to the inequality (38).

The inequalities (28), (37) and (38) imply

$$\|z - z(T)\|_\infty < \rho \quad (41)$$

which contradicts the fact that $z(T)$ is on the boundary of $B(z, \rho)$. Therefore, there exists no such T and the solution curve is contained in the interior of the ball $B(z, \rho)$. There is no singularity of the right hand side of (27) in $B(z, \rho)$. By the elementary theory of differential equation on extending solutions (see, for instance, [8]), the solution can be prolonged to the interval $[0, \infty)$, *i.e.*, $M = \infty$ and it converges to z^* as t tends to ∞ . In fact, let z^{**} be a point in the limit set of the solution curve, which is obviously contained in the closed ball $B(z, \rho)$. Then z^{**} is a solution of (6) by (31). By the uniqueness of the solution of (6) in the closed ball $B(z, \rho)$, it is identical to z^* . Therefore, the solution curve converges to z^* as t tends to ∞ .

Since the solution curve is contained in the feasible set, the limit point z^* is also a feasible point.

(QED)

3. Numerical Examples

In this section, let us present numerical examples. For executing verified computation, we have used MATLAB on Windows XP over a personal computer having Core 2 Duo 1.2GHz Intel processor.

A verification function is programmed based on the rounding mode controlled numerical verification method proposed in Ref. [9].

Example 1 Let us consider the following problem:

$$\text{Maximize } c^t x, \text{ subject to } Ax \leq b \text{ and } x \geq 0, \quad (42)$$

where $c^t = (3, 2)$,

$$A = \begin{pmatrix} -1 & 3 \\ 1 & 1 \\ 2 & -1 \end{pmatrix} \quad (43)$$

and $b^t = (12, 8, 10)$. It is known that the optimal solution

is $x = (6, 2)^t$. In this case, we have an interior point

$$x = \begin{pmatrix} 5.999999999999999 \\ 2.000000000000000 \end{pmatrix},$$

$$y = \begin{pmatrix} 4.166666666666667 \times 10^{-017} \\ 2.333333333333334 \\ 3.333333333333334 \times 10^{-1} \end{pmatrix}.$$

For this point, we have

$$\alpha\omega < 7.93 \times 10^{-14}. \quad (44)$$

Thus, there exists an optimum solution of (42) in the ball centered at $z = (x^t, y^t)^t$ with a radius

$$\rho = 5.14 \times 10^{-15}. \quad (45)$$

Further, the objective value is included in $[22.000000000000000, 22.000000000000001]$. These results are consistent with the exact solution $x = (6, 2)^t$.

Example 2 Next, let us consider the following simple linear programming problem:

$$\text{Maximize } c^t x, \text{ subject to } Ax \leq b \text{ and } x \geq 0, \quad (46)$$

where $c^t = (300, 300, 500)$,

$$A = \begin{pmatrix} 150 & 100 & 100 \\ 1 & 2 & 1 \\ 0 & 0 & 150 \end{pmatrix} \quad (47)$$

and $b^t = (3000, 40, 1200)$. In this case, we have a feasible solution

$$x = \begin{pmatrix} 5.9999999999999973 \\ 13.000000000000004 \\ 8.000000000000000 \end{pmatrix},$$

$$y = \begin{pmatrix} 1.5000000000000000 \\ 75.000000000000000 \\ 1.8333333333333335 \end{pmatrix}. \quad (48)$$

For this feasible point, we have

$$\alpha\omega < 1.64 \times 10^{-11}. \quad (49)$$

Thus, there exists an optimum solution of (46) in the ball centered at $z = (x^t, y^t)^t$ with a radius

$$\rho = 1.45 \times 10^{-13}. \quad (50)$$

Example 3 In this example, we shall consider the following problem with $n = 2m$.

$$\text{Maximize } c^t x, \text{ subject to } Ax \leq b \text{ and } x \geq 0, \quad (51)$$

where

$$c = 10c_r, A = 10E + 5A_r, b = 100b_r \quad (52)$$

Here, $c_r \in \mathbb{F}^n$ is a pseudo-random vector whose elements are distributed uniformly in $[0, 1]$, $E \in \mathbb{F}^{m \times n}$ is a constant matrix whose elements are all one, $A_r \in \mathbb{F}^{m \times n}$ is a pseudo-random matrix whose elements are distributed according to the normal distribution with the mean zero and the variation one, and $b_r \in \mathbb{F}^m$ is a pseudo-random vector whose elements are distributed uniformly in $[0, 1]$.

We examine the cases of $m = 100, 200, 300, 400, 500, 600, 700, 800$ and $m = 900$. We solve the problem on MATLAB. The result is summarized in Table 1. In this

table, t_a is time for computing an approximate optimum solution and t_b that for numerical verification.

Table 1. Results of Verification

$m + n$	$\alpha\omega$	ρ	$t_a[\text{sec}]$	$t_v[\text{sec}]$
300	3.78×10^{-6}	3.4×10^{-12}	0.30	0.29
600	1.69×10^{-5}	5.0×10^{-12}	0.79	2.17
900	4.22×10^{-4}	2.9×10^{-11}	1.9	7.4
1200	0.0049	7.3×10^{-11}	3.9	17.4
1500	0.083	3.8×10^{-10}	7.2	33.4
1800	2.91×10^{-4}	1.8×10^{-11}	13	59
2100	0.189	4.7×10^{-10}	21	91
2400	0.051	2.1×10^{-10}	37	137
2700	0.029	1.4×10^{-10}	54	184
3000	0.011	8.3×10^{-11}	72	270

In this numerical example, a computational cost to include optimum point is about 4 times than that for getting an approximate optimum solution.

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