Algorithm for computing Kronecker basis

Yoshiaki Kakinuma\textsuperscript{1}, Kazuyuki Hiraoka\textsuperscript{1}, Hiroki Hashiguchi\textsuperscript{1},
Yutaka Kuwajima\textsuperscript{1} and Takaomi Shigehara\textsuperscript{1}

Graduate School of Science and Engineering, Saitama University, 255 Shimo-Okubo, Sakura-
ku, Saitama City, Saitama 338-8570, Japan\textsuperscript{1}

E-mail sigehara@ics.saitama-u.ac.jp

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Abstract

To make clear geometrical structure of an arbitrarily given pencil, it is crucial to understand
Kronecker structure of the pencil. For this purpose, GUPTRI is the only practical numerical
algorithm at present. However, although GUPTRI determines the Kronecker canonical form
(KCF), it does not give any direct information on Kronecker bases (KB). In this paper, we pro-
pose a numerical algorithm which gives a full of information on Kronecker structure including
KB as well as KCF. The main ingredient of the algorithm is singular value decompositions,
which guarantee the backward stability of the algorithm.

Keywords pencil, Kronecker canonical form, Kronecker basis

Research Activity Group Algorithms for Matrix / Eigenvalue Problems and their Applications

1. Introduction

Let \((f, g)_{V,W}\) be a pencil, namely a pair of linear mappings \(f, g\) between two finite-dimensional linear spaces \(V\) and \(W\) over \(\mathbb{C}\). It is known \cite{1} that for an arbitrary
\((f, g)_{V,W}\), there exists a Kronecker basis (KB), namely a pair of bases \(\{x_j\}, \{y_j\}\) of \(V\) and \(W\) composed of sequences, each of which satisfies one of five diagrams;

\[
\begin{align*}
(R) & \quad 0 \xleftarrow{\mu} x_1 \xrightarrow{g} y_1 \xleftarrow{\mu} x_2 \xrightarrow{g} y_2 \xleftarrow{\mu} \cdots \xleftarrow{\mu} x_l \xrightarrow{g} y_l, \\
(S_1) & \quad 0 \xleftarrow{f} x_1 \xrightarrow{g} y_1 \xleftarrow{f} x_2 \xrightarrow{g} y_2 \xleftarrow{f} \cdots \xleftarrow{f} x_l \xrightarrow{g} 0, \\
(S_2) & \quad 0 \xleftarrow{f} x_1 \xrightarrow{g} y_1 \xleftarrow{f} x_2 \xrightarrow{g} y_2 \xleftarrow{f} \cdots \xleftarrow{f} x_l \xrightarrow{g} y_l, \\
(S_3) & \quad \mu \xrightarrow{f} x_1 \xrightarrow{g} y_1 \xleftarrow{f} x_2 \xrightarrow{g} y_2 \xleftarrow{f} \cdots \xleftarrow{f} x_l \xrightarrow{g} 0, \\
(S_4) & \quad y_1 \leftarrow x_1 \xrightarrow{f} \cdots \xrightarrow{f} x_{l-1} \xrightarrow{g}\cdots \xrightarrow{g} y_{l-1},
\end{align*}
\]

where \(\mu\) is a nonzero constant associated to the \(R\) sequence and \(l \geq 1\) is the length of each sequence. Matrix representations of \(R, S_1, S_2, S_3\) and \(S_4\) sequences of length \(l\) lead to Kronecker blocks \(J_l(\mu), J_{l-1}(\mu), J_l(0), N_l\) and \(L_{l-1}^T\) in the Kronecker canonical form (KCF) for
\((f, g)_{V,W}\) in the standard notation. If \(V = W = g = i\) (identity transformation), a KB is just a Jordan basis (JB) of \(V\), composed of only \(R\) and \(S_2\) sequences. In this special case, the constant \(\mu\) corresponds to a nonzero eigenvalue of \(f\). For a general case, we will show later in this paper that \(\mu\) corresponds to a nonzero eigenvalue associated to a regular linear transformation \(g_{s,1}\circ f_s\) naturally induced from the original pencil \((f, g)_{V,W}\).

At present, the most reliable and the only practical numerical algorithm to compute the KCF for an arbitrary pencil is GUPTRI \cite{2,3}. However, it cannot give any direct information on KBs for the pencil. In this paper, we propose a novel numerical algorithm to compute a KB as well as the KCF for an arbitrary pencil under the premise that the eigenvalues of the linear transformation \(g_{s,1}\circ f_s\) are separately computed. The algorithm is based on a recently found constructive proof for the existence of a KB which reveals a multilayered geometrical structure inherent in the pencils \cite{4}. After outlining theoretical issues, we describe the algorithm in details. Numerical examples to test the numerical accuracy of the algorithm are also reported.

The paper is organized as follows. In Section 2, we illustrate the essentials of \cite{4} through a simple but generic example, which serves to understand the subsequent sections. On the basis of Section 2, the algorithm for computing a KB is presented in Section 3 in a form without relying on matrix representations, thereby it is described in a basis-independent, unique form. After a short discussion on a possible matrix representation in Section 4, numerical examples are shown to confirm the numerical accuracy of the algorithm in Section 5.

2. Sketch of theoretical aspects

Hereafter, we assume that \(V\) and \(W\) are unitary spaces over \(\mathbb{C}\). For a linear mapping \(h\), in general, denote the kernel, the image and the adjoint mapping of \(h\) by \(N(h), R(h)\) and \(h^*\), respectively.

**Definition 1** For a pencil \((f, g)_{V,W}\), define a pencil \((f', g')_{V',W'}\), by

\[
\begin{align*}
V' & \equiv \{ f \cap R(g) \subset W, \\
W' & \equiv \{ f* \cap R(g*) \subset W, \\
f' & \equiv i_{R(g*)W'}^* \circ d_{h*}^{-1} \circ i_{R(g)W'}, \\
g' & \equiv i_{R(f*)W'}^* \circ d_{h}^{-1} \circ i_{R(f)W'},
\end{align*}
\]

where \(d_h : R(h*) \to R(h)\) is the restriction of \(h\) to \(R(h*)\) for each \(h = f, g\), and \(U_{1\to 2}\) is the inclusion from a subspace \(U_2\) of \(U_1\) to \(U_1\) in general.

Note that the operation of the adjoint mapping \(i_{U_{1\to 2}}^*\) on \(U_1\) is the orthogonal projection from \(U_1\) to \(U_2\). The assertion below represents the importance of the pencil \((f', g')_{V',W'}\).


Assertion 2 Every $R$ sequence in a $KB$ for $(f,g)_{V,W}$ is obtained by lifting a one-to-one corresponding $R$ sequence with the same $\mu$ and $l$ in a $KB$ for $(f',g')_{V',W'}$, while every $S_i$ sequence in length $l \geq 2$ in a $KB$ for $(f,g)_{V,W}$ is obtained by lifting a one-to-one corresponding $S_i$ sequence with length $l - 1$ in a $KB$ for $(f',g')_{V',W'}$ ($i = 1, \ldots, 4$). Supposing $S_i$ sequences of length 1 ($i = 1, \ldots, 4$), we can construct a $KB$ for a $KB$ for $(f',g')_{V',W'}$. To confirm this, Theorem 3 plays a crucial role.

Theorem 3 (i) Let $x \in V$ and $p$ be the orthogonal projection from $V$ to $W'$, Then we have $g'(p(x)) = p(x)$ if $f(x) \in V'$. Similarly, $f'(g'(x)) = p(x)$ if $g(x) \in V'$. (ii) (a) and (b) are equivalent for $y_1, y_2 \in V'$; (a) There exists $x \in V$ such that $y_1 = f(x)$ and $y_2 = g(x)$. (b) $f'(y_2) = g'(y_1).

To illustrate Assertion 2, consider a simple but generic example. Suppose that $dim V = 9$, $dim W = 8$, $dim V' = 5$, $dim (N(f) \cap N(g)) = 3$, $dim (N(f) + N(g)) = 5$ and that $(f',g')_{V',W'}$ has a $KB$ composed of three sequences;  

1') $0 \xrightarrow{f'} y_{1,1} \xrightarrow{g'} z_{1,1} \xrightarrow{f} y_{1,2} \xrightarrow{g} z_{1,2} \mu \neq 0,

2') $0 \xrightarrow{f} x_{2,1} \xrightarrow{g} y_{2,1} \xrightarrow{f'} x_{2,2} \xrightarrow{g'} y_{2,2},

3') $0 \xrightarrow{f} x_{3,1} \xrightarrow{g} y_{3,1} \xrightarrow{f'} x_{3,2} \xrightarrow{g'} y_{3,2} \neq 0.

Note that the assumption leads to $dim W' = 4$, $dim (N(f) \cap N(g)) = 1$, $dim R(f) = dim R(g) = 6$ and $dim (R(f) + R(g)) = 7$. In this setting, we can find a $KB$ for $(f,g)_{V,W}$ composed of six sequences;  

1) $0 \xrightarrow{f} x_{1,1} \xrightarrow{g} y_{1,1} \xrightarrow{f'} x_{1,2} \xrightarrow{g'} y_{1,2},

2) $0 \xrightarrow{f} x_{2,1} \xrightarrow{g} y_{2,1} \xrightarrow{f'} x_{2,2} \xrightarrow{g'} y_{2,2},

3) $0 \xrightarrow{f} x_{3,1} \xrightarrow{g} y_{3,1} \xrightarrow{f'} x_{3,2} \xrightarrow{g'} y_{3,2} \xrightarrow{f'} x_{3,3} \xrightarrow{g'} 0,

4) $0 \xrightarrow{f} x_{4,1} \xrightarrow{g} y_{4},

5) $y_{5,0} \xrightarrow{f} x_{5,0} \xrightarrow{g} y_{5,0},

6) $y_{6,0} \in N(f') \cap N(g').

To see this, we need three steps. (I) Existence of $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{3,3} \in V$, $y_{2,2} \in W$ in 1)-3): By Theorem 3 (ii), there exist $x_{2,1}$ in 2) and $x_{3,1}, x_{3,2}, x_{3,3}$ in 3). Since $y_{2,1} \in V'$ in 2), there exists $x_{2,2}$ such that $y_{2,1} = f(x_{2,2})$. We set $y_{2,2} = g(x_{2,2})$. Now we show the existence of $x_{1,1}, x_{1,2}$ in 1). The sequence 1') indicates  

$f'(y_{1,1}) = \mu z_{1,1} = g'(\mu y_{1,1}),

f'(y_{1,2}) = \mu z_{1,2} + z_{1,1} = g'(\mu y_{1,2} + y_{1,1}).

Hence, by Theorem 3 (ii), there exist $x_{1,1}, x_{1,2}$ such that  

$f(x_{1,1}) = \mu y_{1,1},

g(x_{1,2}) = y_{1,1},

f(x_{1,2}) = \mu y_{1,2} + y_{1,1},

g(x_{1,2}) = y_{1,2},

and these vectors satisfy the diagram 1).

(II) Construction of the basis of V: By the construction of $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}$ by Theorem 3 (i), we have  

$p(x_{1,1}) = \mu z_{1,1},

p(x_{1,2}) = \mu z_{1,2} + z_{1,1},

p(x_{2,1}) = z_{2,1},

p(x_{2,2}) = z_{2,2}.

By the assumption, the four vectors on the right-hand side are a basis of $W'$ since $\mu \neq 0$. Since $W'^{\perp} = N(f) + N(g)$, we confirm that $x_{1,1}, x_{1,2}, x_{2,2}, x_{3,2}$ are a basis of a complementary space of $N(f) + N(g)$ in $V$. Thus we can construct a basis of $V$ by appending a basis of $N(f) + N(g)$ to this. The vectors $x_{2,1}$ in 2) and $x_{3,1}$ in 3) belong to $N(f)$ by construction. Furthermore, $y_{2,1} = g(x_{2,1}), y_{3,1} = g(x_{3,1})$ are a basis of $N(f')$. Thus we confirm that $x_{2,1}, x_{3,1}$ are a basis of a complementary space of $N(f) \cap N(g)$ in $V$, since $dim N(f) - dim N(g) = 1$. Similarly, we confirm $x_{3,3} \in N(g) - N(f) \cap N(g)$. By taking $x_{4,1} \in N(f) \cap N(g)$ $(x_{4,1} \neq 0)$, $x_{3,3}, x_{3,1}$ are a basis of a two-dimensional subspace of $N(g)$. Furthermore, the subspace includes $N(f) \cap N(g)$. Hence, since $dim N(g) = 3$, we have a basis $x_{3,3}, x_{4,1}, x_{5,1}$ of $N(g)$ by appending $x_{5,1} \in N(g) - (N(f) \cap N(g))$. Now we have a basis $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{3,3}, x_{4,1}, x_{5,1}$ of $V$ which satisfies 1)-5) and the upper table in Table 1.

(III) Construction of the basis of $W$: Set $y_{5,0} = f(x_{5,1}).$ By construction of a basis of $V$ in (II), the images of the six vectors $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2} \in V - N(f)$ by $f$, namely  

$f(x_{1,1}) = \mu y_{1,1},

f(x_{1,2}) = \mu y_{1,2} + y_{1,1},

f(x_{2,1}) = y_{2,1},

f(x_{3,1}) = y_{3,1},

f(x_{3,2}) = y_{3,2},

f(x_{5,1}) = y_{5,0},

are a basis of $R(f)$. Similarly, the images of the six vectors $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2} \in V - N(g)$ by $g$, namely  

$g(x_{1,1}) = y_{1,1},

g(x_{1,2}) = y_{1,2},

g(x_{2,1}) = y_{2,1},

g(x_{2,2}) = y_{2,2},

g(x_{3,1}) = y_{3,1},

g(x_{3,2}) = y_{3,2},

are a basis of $R(g)$. Recalling that the five vectors $y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, y_{3,1}, y_{3,2}, y_{3,3}$ are a basis of $V' = R(f) \cap R(g)$, we confirm that $y_{5,0} \in R(f) - R(f) \cap R(g), y_{5,2} \in R(g) - R(f) \cap R(g)$ and that the seven vectors $y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, y_{3,1}, y_{3,2}, y_{3,3}$ are a basis of $R(f) + R(g)$. Since $dim V = 8$, we have a basis of $W$ by appending $y_{6,0} \in (R(f) + R(g))^\perp = N(f') \cap N(g')$ to this. Now we have a basis $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, y_{5,0}, y_{6,0}$ of $W$ which satisfies 1)-6) and the lower table in Table 1.

By definition of $(f',g')_{V',W'}$, if and only if both of $f,g$ are bijective, we have $V' = W', W' = V$, leading to $f' = g^{-1}, g' = f^{-1}$. Otherwise, we have either $dim V' < dim W$ or $dim W' < dim V$. Since $V, W$ are finite-dimensional, by iterating the procedure to construct $(f_j, g_j)_{V_j,W_j} \equiv (f_{j-1}, g_{j-1})_{V_{j-1}, W_{j-1}}$ ($j = 1, 2, \ldots$) with the initial pencil $(f_0, g_0)_{V_0,W_0}$ several times (say $s$), we reach a pencil $(f_s, g_s)_{V,W}$, where both of $f_s, g_s$ are bijective. For this pencil, $g_s^{-1} \circ f_s$ is a regular
Hereafter, denote a sequence with the property $S$.

Let $V$, $W$, $I$, $R$, and $H$ be defined by $\Psi$.

We define three real matrices $F$, $G$, and $H$ by $\Sigma$.

The matrix $H$ is diagonal with nonzero singular values of $H$ as diagonal entries, and $I_{V \rightarrow R(H)} \in C^{n \times r}$ is a diagonal matrix with nonzero singular values of $H$ as diagonal entries, and $I_{V \rightarrow R(H)} \in C^{m \times r}$ and $I_{W \rightarrow R(H)} \in C^{m \times r}$ are column-orthogonal matrices.

We define $I_{R(F)^{-1}} \in C^{r \times r}$ and $I_{R(G)^{-1}} \in C^{r \times r}$ by setting

$$I_{R(G)^{-1}}(e) = (e_1, \ldots, e_r) \in C^{r \times r}.$$

Finally, calculate the matrix products

$$F' = I_{R(G)^{-1}} \cdot W' \cdot D_{G}^{-1} \cdot I_{R(G)^{-1}} - V' \in C^{m \times n'},$$

$$G' = I_{R(F)^{-1}} \cdot W' \cdot D_{F}^{-1} \cdot I_{R(F)^{-1}} - V' \in C^{m \times n'}.$$

Note that since

$$I_{W' \rightarrow V'} = I_{W' \rightarrow R(F') \cdot I_{R(F')^{-1}}}$$

by construction in step 2, the column vectors of $I_{W' \rightarrow V'}$ are a basis of $V' \subset W'$ and in particular, we confirm $r' \cong \dim V'$. The computation of $c_1, \ldots, c_r$ is carried out also by SVD.

Let $y_f, y_g \in V'$. The linear system for $x \in V$ so that $H = I_{W \rightarrow R(H)}$.

$$H = I_{W \rightarrow R(H)} D_H R_{V' \rightarrow R(H)}.$$

where $D_H \in C^{n \times r_H}$ is a diagonal matrix with nonzero singular values of $H$ as diagonal entries, and $I_{V \rightarrow R(H)} \in C^{n \times r_H}$ and $I_{W \rightarrow R(H)} \in C^{m \times r_H}$ are column-orthogonal matrices.

Note that the column vectors of $I_{V \rightarrow R(H)}$ and $I_{W \rightarrow R(H)}$ are right and left singular vectors associated to nonzero singular values of $H$, respectively.

2. For each $(F, G, m, W', V') = (F, G, m, W, V')$,

$$(F'G' \cdot n, V, W')$$

we calculate a basis $c_1, \ldots, c_r$ of the kernel of $I_{W' \rightarrow R(F')} D_H R_{V' \rightarrow R(H)}$.

and define $I_{R(G)^{-1}} \in C^{r \times r}$ and $I_{R(G)^{-1}} \in C^{r \times r}$ by setting

$$I_{R(G)^{-1}}(e) = (e_1, \ldots, e_r) \in C^{r \times r}.$$

Finally, calculate the matrix products

$$F' = I_{R(G)^{-1}} \cdot W' \cdot D_{G}^{-1} \cdot I_{R(G)^{-1}} - V' \in C^{m \times n'},$$

$$G' = I_{R(F)^{-1}} \cdot W' \cdot D_{F}^{-1} \cdot I_{R(F)^{-1}} - V' \in C^{m \times n'}.$$
is written as
\[
\begin{pmatrix}
I_{W-R(F)}D_F I_{R-v}^{*} (F^{-1}) \\
I_{W-R(G)}D_G I_{R-v}^{*} (G^{-1})
\end{pmatrix}
x = \begin{pmatrix}
I_{W-v} y^f_i \\
I_{W-v} y^g_j
\end{pmatrix},
\]
where \(y^f_i, y^g_j \in \mathbb{C}^n\) are the coordinate vectors of \(y_i, y_j\) with respect to the basis of \(V^d\) determined by the column vectors of \(I_{W-v}\). Thus the linear system in (2) is equivalent to
\[
\begin{pmatrix}
D_F I_{R-v}^{*} (F^{-1}) \\
D_G I_{R-v}^{*} (G^{-1})
\end{pmatrix}x = \begin{pmatrix}
I_{R(F)}-v y^f_i \\
I_{R(G)}-v y^g_j
\end{pmatrix}.
\]
Within a finite-precision computation, the equation might be overdetermined in general. A possible solver in numerics is the least squares method, where SVD (Moore-Penrose inverse) plays a crucial role.

5. Numerical experiment

Numerical computation is carried out in double-precision arithmetic. As described in the previous section, the main ingredient of KBA is SVD. To keep numerical accuracy, cut-off parameter \(\varepsilon\) is required for removing small singular values of a relative size less than \(\varepsilon\) compared to the maximum singular value. At each stage involved in \((F_j, G_j)_{Y \to W}(j = 1, \ldots, s)\), we introduce two parameters; \(\varepsilon_j, \varepsilon_j^0\) for computing SVD of \(F_j\), \(G_j\) and the kernels, and \(\varepsilon_j, \varepsilon_j^0\) to solve linear systems. For the moment, we use a common cut-off parameter \(\varepsilon_j, \varepsilon_j^0 = 10^{-8}\) \((j = 1, \ldots, s)\). This value is adopted as a default value of the cut-off parameter \(\varepsilon\) for SVD in double-precision GUPTRI routine in LAPACK.

To confirm the numerical accuracy, we examine the maximum relative error involved in the sequences in KB;
\[
E_K \equiv \max_{\varepsilon \in \mathbf{K}} \left\{ \left\| F_{X_\varepsilon} - \mu Y_{Y_{\varepsilon}} - Y_{C_{\varepsilon}} \right\|_{\infty} : \left\| G_{X_{\varepsilon}} - Y_{C_{\varepsilon}} \right\|_{\infty} \right\},
\]
Here \(\mathbf{K} \equiv \mathbb{R}_0 \cup \bigcup_{k=1}^{s(n\setminus d)} \mathbb{S}(0)\) is the output of KBA, namely the set of the sequences giving rise to a KB for input pencil \((F, G)_{V \to W}\). For each \(\mu \in k_\varepsilon(\mu) = (y_0, x_1, \ldots, x_{s-1}, y_j) \in \mathbf{K}\), we set \(X_c = (x_1, \ldots, x_{s-1}), Y_{Y_{\varepsilon}} = (y_0, \ldots, y_{s-1})\) and \(Y_{C_{\varepsilon}} = (y_0, \ldots, y_j)\). Note that \(F_{X_\varepsilon} - \mu Y_{Y_{\varepsilon}} - Y_{C_{\varepsilon}} = G_{X_{\varepsilon}} - Y_{C_{\varepsilon}} = 0\) in infinite-precision computation.

We examine two types of test matrices;

Type-A (generic): \(m \times n\) matrices \(F, G\) with random integers \(m, n \in [100, 100] (m \neq n)\) and random numbers in the range \([-1, 1]\) for the elements.

Type-B (non-generic): \(F - \lambda G = PK(\lambda)Q^{-1}\), where \(P, Q\) are invertible matrices with random numbers in the range \([-1, 1]\) for elements, and
\[
K(\lambda) = \bigoplus_{k_1=1}^{n_1} J_{k_1} (\mu_{k_1}) \bigoplus_{k_2=1}^{n_2} L_{k_2} \bigoplus_{k_3=1}^{n_3} J_{k_3} (0) \bigoplus_{k_4=1}^{n_4} N_{k_4} \bigoplus_{k_5=1}^{n_5} L_{k_5}
\]
is a KCF with random integers \(n_j \in [1, 5]\) \((j = 1, \ldots, 5)\), random integers \(k_j \in [1, 5]\) \((k_j = 1, \ldots, n_j; j = 1, \ldots, 5)\) and random numbers \(\mu_{k_1} \in (0, 10)\) \((k_1 = 1, \ldots, n_1)\).

As known for non-square \(m \times n\) generic pencils \((d \equiv |n-m| \neq 0)\), we have for \(m-n > 0\), \(K(\lambda) = (\bigoplus_{k=1}^{s} L_l) \bigoplus (\bigoplus_{k=1}^{s} L_{l+1})\) with \(l = \lfloor m/d \rfloor, s = m-d, s = d-s\), while for \(m-n < 0\), \(K(\lambda) = (\bigoplus_{k=1}^{s} L_l^T) \bigoplus (\bigoplus_{k=1}^{s} L_{l-1}^T)\) with \(l = \lfloor n/d \rfloor, s = n-d, s = d-s\). Type A is expected to simulate a generic case. Meanwhile, type B has a non-trivial general Kronecker structure by construction. The middle (right) column in Table 3 shows a distribution of \(E_K\) for 1000 samples of Type-A (Type-B) matrix pencils.

Table 3. Distribution of \(E_K\).

<table>
<thead>
<tr>
<th>relative error</th>
<th>Type-A</th>
<th>Type-B</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-8} &lt; E_K)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(10^{-10} &lt; E_K &lt; 10^{-8})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(10^{-12} &lt; E_K &lt; 10^{-10})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(10^{-14} &lt; E_K &lt; 10^{-12})</td>
<td>72</td>
<td>553</td>
</tr>
<tr>
<td>(E_K &lt; 10^{-14})</td>
<td>928</td>
<td>0</td>
</tr>
</tbody>
</table>

For Type-A, we can confirm \(E_K \approx 10^{-12}\) even in the worst case. We also numerically confirmed that the KCF is of generic type in all cases, as expected.

For Type-B, we can confirm \(E_K \leq 10^{-8}\) (the value of the common cut-off parameter) in 977 cases. Though \(E_K > 10^{-8}\) in 23 cases, we confirmed in all cases that \(E_K\) is made less than \(10^{-8}\) if the two cut-off parameters \(\varepsilon_j, \varepsilon_j^0\) are appropriately adjusted in the range \([10^{-15}, 10^{-7}]\) at each iterative step \((j = 1, \ldots, s)\). In addition, we observed that KBA works well even with the eigenvalues numerically computed for \((F_c, G_c)_{V \to W}\), if we use an average for closely-spaced eigenvalues.

As well-known, the determination of Kronecker structure is essentially ill-conditioned problem in general. In particular, round-off error in numerics might reduces non-generic Kronecker structures to generic ones. In the present implementation, we numerically confirmed for Type-B matrix pencils that KBA succeeds in reproducing the original KCF for the 97% of all. An extensive analysis on numerical stability of KBA is one of the main issues in future.

References