A new compressible fluid model for traffic flow with density-dependent reaction time of drivers

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Abstract
In this paper, we have proposed a new compressible fluid model for the one-dimensional traffic flow taking into account a variation of the reaction time of drivers, which is based on the actual measurements. The model is a generalization of the Payne model by introducing a density-dependent reaction time. The linear stability analysis of this new model shows the instability of homogeneous flow around a critical density of vehicles, which is observed in real traffic flow. Moreover, the condition of the nonlinear saturation of density against small perturbation is theoretically derived from the reduction perturbation method.

Keywords jamology, traffic flow, compressible fluid, stability analysis, reaction time

Research Activity Group Applied Integrable Systems

1. Introduction

Among various kinds of jamming phenomena, a traffic jam of vehicles is a very familiar phenomenon and causes several losses in our daily life such as decreasing efficiency of transportation, waste of energy, serious environmental degradation, etc. In particular, highway traffic dynamics has attracted many researchers and has been investigated as a nonequilibrium system of interacting particles for the last few decades [1]. A lot of mathematical models for one-dimensional traffic flow have ever been proposed [2–8] and these models are classified into microscopic and macroscopic models in terms of the treatments of particles. In microscopic models, e.g. car-following model [2, 3] and cellular automaton model [4, 5], the dynamics of traffic flow is described by the movement of individual vehicles. Whereas in macroscopic models, the dynamics is treated as an effectively one-dimensional compressible fluid by focusing on the collective behavior of vehicles [6–8]. Moreover, it is widely known that some of these mathematical models are related with each other, which is shown by using mathematical method such as ultra-discretization method [9] or Euler-Lagrange transformation [10]. That is, rule-184 elementary cellular automaton model [4] is derived from Burgers equation [6] by ultra-discretization method. The specific case of optimal velocity (OV) model [3] is formally derived from the rule-184 cellular automaton model via the Euler-Lagrange transformation. Quite recently, more noteworthy are that the ultra-discrete versions of OV model are shown by Takahashi et al [11] and Kanai et al [12]. In contrast to practically reasonable microscopic models, only a small number of macroscopic models with reasonable expression have been proposed, even in various traffic models based on the hydrodynamic theory of fluids [6–8]. In previous fluid models, one does not have any choice to introduce the diffusion term into the models such as Kerner and Konhäuser model [8], in order to represent the stabilized density wave, which indicates the formation of traffic jam. However, it emerges as a serious problem that some vehicles move backward even under heavy traffic, since the diffusion term has a spatial isotropy. As mentioned by Daganzo in [13], the most essential difference between traffic and fluids is as follows:

“A fluid particle responds to the stimulus from the front and from behind, however a vehicle is an anisotropic particle that mostly responds to frontal stimulus.”

That is, traffic vehicles exhibit an anisotropic behavior, although the behavior of fluid particles with simple diffusion is isotropic. Therefore, unfortunately we would have to conclude that traffic models which include the diffusion term are not reasonable for the realistic expression of traffic flow. Given these factors, we suppose that traffic jam forms as a result of the plateaued growth of small perturbation by the nonlinear saturation effect.

Now let us return to Payne model [7], which is one of the most fundamental and significant fluid models of traffic flow without diffusion term. Payne model is given
which is a positive time optimal velocity function, which represents the desired vehicle density and the average velocity at position $x$.

Furthermore, the linear stability condition is calculated from dispersion relation (4),

$$
\frac{1}{2\tau} > -\rho_0 \frac{dV_{opt}(\rho)}{d\rho} \bigg|_{\rho = \rho_0}.
$$

If one applies velocity-density relation of (3) to this stability condition, the following linear stability condition is obtained

$$
\frac{\rho_{\text{max}}}{2\tau V_0} > \rho_0^2.
$$

Here, let us define the stability function $S(\rho_0)$ by

$$
S(\rho_0) = \frac{\rho_{\text{max}}}{2\tau V_0} - \rho_0^2.
$$

In this function, the condition $S(\rho_0) > 0$ ($S(\rho_0) < 0$) corresponds to the stable (unstable) state.

Fig. 1 shows the stability plots for several constant values of $\tau$. From this figure, we can observe that the instability region of homogeneous flow occurs beyond a critical density of vehicles.

However, Payne model shows the condensation of vehicles due to the momentum equation (2). That is, as the density increases, the value of optimal velocity in the first term of right-hand-side becomes zero and the value of second term also becomes zero. Thus, the vehicles gather in one place due to the nonlinear effect $\nu_c^2$ and the small perturbation blows up without stabilization. Therefore, Payne model is also incomplete to describe the realistic dynamics of traffic flow.

Thus, in this paper we propose a new compressible fluid model, by improving the Payne model in terms of the reaction time of drivers based on the following actual measurements.

2. New compressible fluid model based on experimental data

We have performed car-following experiment on a highway. The leading vehicle cruises with legal velocity and following vehicle pursues the front one. The time-series data of the velocity and position (latitude and longitude) of each vehicle are recorded every 0.2 seconds (5Hz) by a global positioning system (GPS) receiver onboard with high-precision (< 60 centimeters precision).

By dividing the time-series data into two phases, i.e. free-flow phase and jam phase, based on the velocity, we have obtained the synthetograph of two time-series data as shown in Fig. 2, which shows that drivers obviously react to the front car with a slight delay in both two phases. Here, assuming that the reaction time of drivers is considered as a slight delay of behavior, we calculate the correlation coefficient which is denoted by

$$
\sum_{k} (v_i(t^{(k)}) - \bar{v}_i^l) \cdot (v_{i+1}(t^{(k)} + \tau - \bar{v}_{i+1}^l))^2
\sqrt{\sum_{k} (v_i(t^{(k)}) - \bar{v}_i^l)^2 \cdot \sum_{k} (v_{i+1}(t^{(k)} + \tau - \bar{v}_{i+1}^l))^2}
$$

where $v_i(t)$ shows the velocity of $i$-th car at time $t$. Note that, $i$-th car drives in front of $(i + 1)$-th car. The symbol ($) and bar indicates an ensemble average and time-average, respectively. Finally we have obtained the correlation coefficient for each given $\tau$ which is shown in Fig. 3. From this figure, we have found that the peak of correlation coefficient shifts according to the situation of the road. Here, since the reaction time of a driver is considered as $\tau$, the reaction time of a driver is not con-
constant, but obviously changes according to the situation of the road. That is, if the traffic state is free (jam), the reaction time of a driver is longer (shorter).

As a reasonable assumption based on this result, the reaction time of drivers depend on the density on the road. Under this assumption, we have extended Payne model and proposed a new compressible fluid model as follows,

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0, \tag{11}
\]

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \frac{1}{\tau(\rho)} (V_{opt}(\rho) - v) + \frac{1}{2\tau(\rho)} \frac{dV_{opt}(\rho)}{d\rho} \frac{\partial \rho}{\partial x}. \tag{12}
\]

The difference between this model and Payne model is that the reaction time of drivers is changed from constant value to density-dependent function \(\tau(\rho)\).

2.1 Linear stability analysis

Now let us perform the linear stability analysis for our new dynamical model to investigate the instability of homogeneous flow. The homogeneous flow and small perturbation are given by

\[
\rho = \rho_0 + \varepsilon \rho_1, \quad v = V_{opt}(\rho_0) + \varepsilon v_1. \tag{13}
\]

One obtains the form of dispersion relation as

\[
\omega = -kV_{opt} + \frac{1}{2\tau(\rho_0)} \left[ \sqrt{1 + 4a_0^2 \tau(\rho_0)^2 (2k\rho_0 i - k^2)} \right], \tag{14}
\]

where

\[
a_0^2 = -\frac{1}{2\tau(\rho_0)} \left. \frac{dV_{opt}(\rho)}{d\rho} \right|_{\rho=\rho_0} > 0. \tag{15}
\]

Hence, the stability conditions

\[
\frac{1}{2\tau(\rho_0)} > -\rho_0^2 \left. \frac{dV_{opt}(\rho)}{d\rho} \right|_{\rho=\rho_0}, \tag{16}
\]

are obtained. The difference between (6) and (16) comes from the reaction time, which changes from \(\tau\) to \(\tau(\rho_0)\) which is decided by only the initial density of homogeneous flow. Therefore, this stability condition (16) and the stability condition of Payne model (6) are essentially equivalent, that is, our new model also shows the instability of homogeneous flow. Substituting (3) into (16), the stability condition leads to

\[
\frac{\rho_{\text{max}}}{2\tau(\rho_0)V_0} > \rho_0^2. \tag{17}
\]

The most important point of our new model is that it is possible to stabilize the perturbation by the nonlinear effect which is created by the function \(\tau(\rho)\), though this stabilizing mechanism was failed in the Payne model. In order to show this nonlinear effect, the evolution equation of small perturbation is derived in the next subsection.

2.2 Reductive perturbation analysis

Let us define the slowly-varying variables \(X\) and \(T\) by Galilei transformation as

\[
X = \varepsilon (x - c_gt), \quad T = \varepsilon^2 t, \tag{18}
\]

where \(c_g = d\omega/dk\) is the group velocity. Next, we assume that \(\rho(x,t), v(x,t)\) can be expressed in terms of the power series of \(\varepsilon\), i.e.,

\[
\rho \sim \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \varepsilon^3 \rho_3 + \cdots, \tag{19}
\]

\[
v \sim v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \cdots. \tag{20}
\]

Substituting (19) and (20) into (11) and (12), for each order term in \(\varepsilon\), we have, respectively,

\[
\varepsilon^3: \quad \frac{\partial \rho_1}{\partial T} + \frac{\partial}{\partial X} \left( \rho_2(v_0 - c_g) + \rho_1 v_1 + \rho_0 v_2 \right) = 0, \tag{21}
\]

\[
\varepsilon^4: \quad \frac{\partial \rho_2}{\partial T} + \frac{\partial}{\partial X} \left( \rho_1(v_0 - c_g) + \rho_2 v_1 + \rho_1 v_2 + \rho_0 v_3 \right) = 0, \tag{22}
\]

and

\[
\varepsilon^2: \quad (v_0 - c_g) \frac{\partial v_1}{\partial X} = V''_{opt} \rho_1^2 + 2V''_{opt} \rho_2 - 2v_2 + \frac{V'_{opt}}{2\tau(\rho_0)} \frac{\partial \rho_1}{\partial \rho_0} \frac{\partial \rho_1}{\partial X}, \tag{23}
\]

\[
\varepsilon^3: \quad \frac{\partial v_1}{\partial T} + (v_0 - c_g) \frac{\partial v_2}{\partial X} + v_1 \frac{\partial v_1}{\partial X} = \frac{1}{\tau(\rho_0)} \left[ \frac{V''_{opt} \rho_1^3}{6} + V''_{opt} \rho_1 \rho_2^2 + V''_{opt} \rho_1 \rho_2 \right] - \frac{1}{\tau(\rho_0)} \frac{\partial \rho_1}{\partial \rho_0} \frac{\partial \rho_1}{\partial X} \left( V''_{opt} \rho_1 \rho_2^2 + V''_{opt} \rho_2 \right) - \frac{1}{\tau(\rho_0)} \frac{\partial \rho_1}{\partial \rho_0} \frac{\partial \rho_1}{\partial X} \left( V''_{opt} \rho_1 \rho_2 \right) - \frac{V''_{opt} \rho_1}{\rho_0} \frac{\partial \rho_1}{\partial X} \frac{\partial \rho_1}{\partial \rho_0} \left( \frac{\partial \rho_1}{\partial \rho_0} \frac{\partial \rho_1}{\partial X} \right). \tag{24}
\]

Note that the prime means the abbreviation of each derivation.

Putting \(\phi_1 = \rho_1\) as a first-order perturbation quantity and eliminating the second-order quantities \((\rho_2, v_2)\) in (21) and (23), we have obtained the Burgers equation

\[
\frac{\partial \phi_1}{\partial T} = \left( \frac{2(v_0 - c_g)}{\rho_0} - V''_{opt} \rho_0 \right) \frac{\partial \phi_1}{\partial X} + \left( \frac{v_0 - c_g}{2\rho_0} - \frac{\tau(\rho_0)(v_0 - c_g)^2}{2} \right) \frac{\partial^2 \phi_1}{\partial X^2}, \tag{25}
\]

as a evolution equation of first-order quantity. Moreover, eliminating the third-order quantities \((\rho_3, v_3)\) in (22) and
Saturation
Amplification
Amplification
Amplification
(Linear unstable)
$P < 0$
$Q > 0$
$P - \varepsilon Q \Phi > 0$
Saturation
$P - \varepsilon Q \Phi < 0$
Amplification
$Q < 0$
$P - \varepsilon Q \Phi > 0$
Damping
$P - \varepsilon Q \Phi < 0$
Amplification
(Linear stable)
$P > 0$
$Q > 0$
$P - \varepsilon Q \Phi > 0$
Damping
$P - \varepsilon Q \Phi < 0$
Amplification

Table 1. Classification table based on the coefficient of diffusion term.

(24) and defining the perturbation $\Phi$ included the first- and second-order perturbation as $\Phi = \phi_1 + \varepsilon \phi_2$, the higher-order Burgers equation

$$
\frac{\partial \Phi}{\partial t} = \frac{2(v_0 - c_g)}{\rho_0} \frac{\partial \Phi}{\partial X} + \left[ \frac{v_0 - c_g}{2\rho_0} - (v_0 - c_g)^2 \tau(\rho_0) \right] \frac{\partial^2 \Phi}{\partial X^2} - \varepsilon (v_0 - c_g)^2 \left\{ \frac{2\tau(\rho_0)}{\rho_0} + \tau'(\rho_0) \right\} \frac{\partial \Phi}{\partial X}
+ \left[ \frac{\tau(\rho_0)}{\rho_0} - 2(v_0 - c_g)\tau'(\rho_0) \right] \frac{\partial^3 \Phi}{\partial X^3}.
$$

(26)

is obtained. Note that, in this derivation, we put $V''_{\text{opt}} = V''_{\text{opt}} = 0$ due to the relation (3).

Although the first-order equation (25) of our model is essentially equivalent to that of Payne model, the second-order equation (26) is different from that of Payne model in terms of the coefficient of the third term of right-hand side.

In order to analyze the nonlinear effect of our model, let us consider the coefficient of the diffusion term of second-order equation. Let us put the coefficient of the second term of right-hand side in (26) as

$$
P = \frac{v_0 - c_g}{2\rho_0} - \tau(\rho_0)(v_0 - c_g)^2,
$$

(27)

and also put the coefficient of the third term as

$$
Q = \frac{2(v_0 - c_g)^2 \tau(\rho_0)}{\rho_0} + (v_0 - c_g)^2 \tau'(\rho_0).
$$

(28)

Thus, diffusion term of (26) is given by

$$
(P - \varepsilon Q \Phi) \frac{\partial^2 \Phi}{\partial X^2}.
$$

(29)

Since $P = 0$ corresponds to the neutrally stable condition, we assume the value $P$ is negative, which corresponds to the linear unstable case. In the case of Payne model, $Q$ is always positive because $\tau$ is constant, i.e. $\tau'$ is always zero. Therefore, the diffusion coefficient (29) is always negative under the linear unstable condition of Payne model, which makes the model difficult to treat numerically. However, in the case of our model, $\tau'(\rho)$ is always negative since $\tau(\rho)$ is considered as monotonically decreasing function. If $Q$ is negative, the diffusion coefficient becomes positive as $\Phi$ increases even under the linear unstable condition. In this situation, the small perturbation will be saturated by nonlinear effect created by the density-dependent function of reaction time of drivers. The conditions for nonlinear saturation corresponds to $P < 0$ and $Q < 0$, which are transformed into the following expressions,

$$
\tau(\rho) > \frac{1}{2\rho_0(v_0 - c_g)^2}, \quad \tau'(\rho_0) < -\frac{2\tau(\rho_0)}{\rho_0}.
$$

(30)

All conditions which include the other cases are summarized in Table 1.

3. Conclusion

A new compressible fluid model for one-dimensional traffic flow has been proposed by introducing the density-dependent function $\tau(\rho)$ about reaction time of drivers based on actual measurements. Our new model does not include the diffusion term which exhibits the unrealistic isotropic behavior of vehicles, since vehicles mostly respond to the stimulus from the front one. The linear stability analysis for our new model gives us the existence of instability of homogeneous flow. We have found that the stability condition is essentially equivalent to Payne model. Moreover, the behavior of small perturbation of density is classified according to the diffusion coefficient of the higher-order Burgers equation, which is derived from our new model by using reductive perturbation method. From this classification, we have obtained the special condition where the small perturbation is saturated by nonlinear effect.

References