Error analysis for a matrix pencil of Hankel matrices with perturbed complex moments

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Abstract

In this paper, we present perturbation results for eigenvalues of a matrix pencil of Hankel matrices for which the elements are given by complex moments. These results are extended to the case that matrices have a block Hankel structure. The influence of quadrature error on eigenvalues that lie inside a given integral path can be reduced by using Hankel matrices of an appropriate size. These results are useful for discussing the numerical behavior of root finding methods and eigenvalue solvers which make use of contour integrals. Results from some numerical experiments are consistent with the theoretical results.

Keywords perturbation results, eigenvalues, block Hankel matrix, matrix-valued moments

Research Activity Group Algorithms for Matrix / Eigenvalue Problems and their Applications

1. Introduction

We consider the problem of determining poles and respective residues from a sequence of complex moments. This problem appears in methods for finding roots of analytic functions [1–3] and eigenvalue solvers [4–8] using contour integrals. In these methods, the problem of determining zeros or eigenvalues in a given circle is reduced to an eigenvalue problem for a matrix pencil corresponding to Hankel matrices.

In this paper, we present perturbation results for the eigenvalues of a matrix pencil of Hankel matrices associated with complex moments. We extend these results to the case where matrices have a block Hankel structure with matrix valued moments. These results are useful in discussing the numerical behavior of moment-based methods, and they can be used to determine parameters for these methods.

Our results suggest that the use of Hankel matrices of an appropriate size reduces the influence of quadrature error on eigenvalues that lie inside a given integral path. Hankel matrices are known to be very ill-conditioned [9]. Indeed, the condition number for Hankel matrices often increases exponentially. However, our element-wise error analysis shows that eigenvalues inside the unit circle, with the rest located outside, can be obtained accurately. This result can be generalized to an arbitrary circle by a shift and scale transformation.

The rest of this paper is organized as follows. In Section 2, we present perturbation results for a matrix pencil of Hankel matrices. In Section 3, we extend the results to the case where the matrix pencil consists of block Hankel matrices. Some numerical experiments, the results of which are consistent with the theoretical results, are reported on in Section 4.

2. Perturbation results for a matrix pencil of Hankel matrices

Let \( f(z) \) be a rational function with \( n \) simple poles \( \eta_i \in \mathbb{C} \) for \( 1 \leq i \leq n \), and let \( \nu_i \in \mathbb{C} \) for \( 1 \leq i \leq n \) be their residues, where \( \mathbb{C} \) denotes the set of complex numbers. Throughout this paper, we assume that \( \eta_1, \ldots, \eta_n \) are mutually distinct and that \( \nu_i \neq 0 \) for \( 1 \leq i \leq n \).

Define the sequence of complex moments as follows:

\[
\mu_k = \frac{1}{2\pi i} \oint_T z^k f(z)dz, \quad k = 0, 1, \ldots, \tag{1}
\]

where \( T \) is the unit circle. Let \( m \) poles \( \eta_1, \ldots, \eta_m \) be located inside the unit circle, with the rest located outside the unit circle. Then, from the residue theorem, \( \mu_k \) is given by

\[
\mu_k = \sum_{i=1}^{m} \nu_i \eta_i^k, \quad k = 0, 1, \ldots.
\]

Let the Hankel matrix \( H_m \in \mathbb{C}^{m \times m} \) associated with \( \{\mu_k\}_{k=0}^{2m-2} \) and the shifted Hankel matrix \( H_m' \in \mathbb{C}^{m \times m} \) associated with \( \{\mu_k\}_{k=1}^{2m-1} \) be

\[
H_m = [\mu_{i+j-2}]_{i,j=1}^{m}, \quad H_m' = [\mu_{i+j-1}]_{i,j=1}^{m},
\]

respectively. Let \( V_m \in \mathbb{C}^{m \times m} \) be the Vandermonde matrix \( V_m = [\eta_i^{j-1}]_{i,j=1}^{m} \). The eigenvalues and eigenvectors of the matrix pencil \( H_m' - \lambda H_m \) can be expressed as follows:

**Theorem 1** The eigenvalues of the matrix pencil \( H_m' - \lambda H_m \) are given by \( \eta_1, \ldots, \eta_m \). The right eigenvector \( \textbf{x}_i \)
with respect to ηi is given by
\[ \mathbf{x}_i = \left( \frac{1}{\sqrt{\nu_i}} \right) \mathbf{V}^{-1}_m \mathbf{e}_i, \]
and the left eigenvector \( \mathbf{y}_i \) is given by \( \mathbf{y}_i = \mathbf{x}_i \) with \( y_i^* \mathbf{H}_m \mathbf{x}_i = 1 \). Here \( \mathbf{e}_i \) is the i-th unit vector.

**Proof** Let \( \mathbf{u}_m \in \mathbb{C}^m \) be
\[ \mathbf{u}_m = [\nu_1^{\frac{1}{2}}, \ldots, \nu_m^{\frac{1}{2}}]^T, \]
and let \( \Delta_m = \text{diag}(\eta_1, \ldots, \eta_m) \). It can be easily seen that
\[ \mu_k = \mathbf{u}_m^T \Delta_m^k \mathbf{u}_m, \quad k = 0, 1, \ldots. \]
It follows that the Hankel matrices can be factorized as follows:
\[ \mathbf{H}_m = \phi_m^T \phi_m, \quad \mathbf{H}_m^\leq = \phi_m^T \Delta_m \phi_m, \]
where
\[ \phi_m = [\mathbf{u}_m \Delta_m \mathbf{u}_m \cdots \Delta_m^{m-1} \mathbf{u}_m] \in \mathbb{C}^{m \times m}. \]
This implies that
\[ \mathbf{H}_m^\leq - \lambda \mathbf{I}_m = \phi_m^T (\Delta_m - \lambda \mathbf{I}_m) \phi_m, \]
where \( \mathbf{I}_m \) is the \( m \times m \) identity matrix. The matrix \( \phi_m \) is nonsingular, because \( \phi_m \) can be expressed as
\[ \phi_m = \text{diag}(\nu_1^{\frac{1}{2}}, \ldots, \nu_m^{\frac{1}{2}}) \mathbf{V}_m, \]
where \( \eta_1, \ldots, \eta_m \) are all distinct, and \( \nu_1, \ldots, \nu_m \) are not zero. Thus, the eigenvalues of the matrix pencil \( \mathbf{H}_m^\leq - \lambda \mathbf{I}_m \) are given by \( \eta_1, \ldots, \eta_m \).

Since \( \Delta_m \mathbf{e}_i = \eta_i \mathbf{e}_i \), it can be verified that
\[ \mathbf{x}_i = \mathbf{phi}_m^{-1} \mathbf{e}_i = \mathbf{V}_m^{-1} \text{diag}(\nu_1^{\frac{1}{2}}, \ldots, \nu_m^{\frac{1}{2}})^{-1} \mathbf{e}_i = \frac{1}{\sqrt{\nu_i}} \mathbf{V}_m^{-1} \mathbf{e}_i. \]
We can also verify that \( \mathbf{y}_i = \phi_m^{-1} \mathbf{e}_i = \mathbf{x}_i \). From these results, we have
\[ \mathbf{y}_i^* \mathbf{H}_m \mathbf{x}_i = \mathbf{e}_i^T (\phi_m^T)^{-1} \phi_m^T \phi_m \phi_m^{-1} \mathbf{e}_i = 1. \]
This proves the theorem.

(QED)

An error estimation for the eigenvalues of a perturbed matrix pencil, when all the eigenvalues are simple, is given in [2]. Let \( \lambda_1, \ldots, \lambda_n \) be eigenvalues of the matrix pencil \( \mathbf{A} - \lambda \mathbf{B} \), and let \( \mathbf{x}_i \) and \( \mathbf{y}_i \) be right and left eigenvectors with respect to \( \lambda_i \), respectively. Then the eigenvalue \( \hat{\lambda}_i \) of the perturbed matrix pencil \( (\mathbf{A} + \Delta \mathbf{A}) - \lambda (\mathbf{B} + \Delta \mathbf{B}) \), where \( \| \Delta \mathbf{A} \| \leq \delta \) and \( \| \Delta \mathbf{B} \| \leq \delta \) for sufficiently small \( \delta > 0 \), satisfies the following relation:
\[ |\hat{\lambda}_i - \lambda_i| \leq \delta (1 + |\lambda_i|) \frac{\| \mathbf{x}_i \|_2 \cdot \| \mathbf{y}_i \|_2}{\| \mathbf{y}_i^* \mathbf{B} \mathbf{x}_i \|} + O(\delta^2). \]

Define
\[ \tau_i(A, B) = (1 + |\lambda_i|) \frac{\| \mathbf{x}_i \|_2 \cdot \| \mathbf{y}_i \|_2}{\| \mathbf{y}_i^* \mathbf{B} \mathbf{x}_i \|}, \]
then \( \tau_i(A, B) \) expresses the condition for the i-th eigenvalue of the matrix pencil \( \mathbf{A} - \lambda \mathbf{B} \). From Theorem 1, we have the following expression.

**Lemma 2**
\[ \tau_i(H_m^\leq, H_m) = \frac{1 + |\eta_i|}{|\nu_i|} \| \mathbf{V}_m^{-1} \mathbf{e}_i \|_2. \]

Suppose that the contour integral (1) is approximated using the \( N \)-point trapezoidal rule:
\[ \hat{\mu}_k = \frac{1}{N} \sum_{j=0}^{N-1} \theta_j^{k+1} f(\theta_j), \quad k = 0, 1, \ldots, \]
with the equi-distributed points on the unit circle:
\[ \theta_j = e^{\frac{2\pi j}{N} (1 + \varepsilon)}, \quad j = 0, 1, \ldots, N - 1. \]
The approximate moments \( \hat{\mu}_k \) suffer from quadrature error. For error analysis of the trapezoidal rule, we use the following estimation.

**Lemma 3** Let \( \eta \) be a complex number with \( |\eta| \neq 1 \). For any integer \( k \) with \( 0 \leq k < N \), the following holds:
\[ \frac{1}{N} \sum_{j=0}^{N-1} \theta_j^{k+1} = \frac{\eta^k}{1 + \eta^N}. \]

**Proof** If \( |\eta| < 1 \), we have
\[ \frac{1}{N} \sum_{j=0}^{N-1} \theta_j^{k+1} = \frac{1}{N} \sum_{j=0}^{N-1} \theta_j^k = \frac{1}{N} \sum_{p=0}^\infty \eta^p \frac{1}{N} \sum_{j=0}^{N-1} \theta_j^{p-k} = \sum_{q=0}^\infty (-1)^q \eta^{Nq+k}. \]
The last step follows from the fact that
\[ \frac{1}{N} \sum_{j=0}^{N-1} \theta_j^p = \begin{cases} (-1)^q & \text{if } p = qN \text{ for } q \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}, \]
where \( \mathbb{Z} \) denotes the set of integers.

Similarly, for the case in which \( |\eta| > 1 \), we have
\[ \frac{1}{N} \sum_{j=0}^{N-1} \theta_j^{k+1} = \frac{1}{N} \sum_{j=0}^{N-1} \left( \frac{-1}{\eta^j} \right) \frac{\theta_j^{k+1}}{1 - \theta_j^{-1}} = \sum_{p=0}^\infty \left( \frac{-1}{\eta^{p+1}} \right) \left( \frac{1}{N} \sum_{j=0}^{N-1} \theta_j^{p+k+1} \right) = \sum_{q=0}^\infty (-1)^{q+1} \eta^{-Nq+k}. \]
From (5) and (6), we have (4).

(QED)

From this Lemma, we derive the following equation.
\[ \hat{\mu}_k = \sum_{i=1}^n \left( \frac{\nu_i}{1 + \eta_i^N} \right) \eta_i^k, \quad k = 0, 1, \ldots, N - 1. \]
This equation implies that \( \hat{\mu}_k \) is a moment with a new weight \( \nu_i/(1 + \eta_i^N) \) instead of \( \nu_i \). Therefore, we see that the quadrature error affects the weight; however, the poles \( \eta_1, \ldots, \eta_n \) are unchanged, if computations are performed without any numerical error.

For \( \eta_i \) such that \( |\eta_i^N| > 1 \), the weight \( \nu_i/(1 + \eta_i^N) \) is close to zero. Suppose that \( \eta_1, \ldots, \eta_n \) are ordered such that \( |\eta_1| \leq \cdots \leq |\eta_n| \). Let \( m' \) be an integer such that \( \hat{\nu}_i = O(\varepsilon) \) for any \( i \) with \( m' < i \leq n \) for sufficiently small \( \varepsilon > 0 \). Then (7) can be expressed as
\[ \hat{\mu}_k = \sum_{i=1}^{m'} \hat{\nu}_i \eta_i^k + O(\varepsilon). \]
Let \( \tilde{\mu}_k = \sum_{i=1}^{m} \tilde{v}_i \tilde{\eta}_i^k \); then \( \tilde{\mu}_k \) can be regarded as a perturbed moment from \( \tilde{\mu}_k \) which is obtained from \( m' \) poles \( \eta_1, \ldots, \eta_{m'} \) with weights \( \tilde{v}_1, \ldots, \tilde{v}_{m'} \). Let 
\[
  \mathcal{F}_{m'} = \text{diag}((1 + \eta_1)^{-1}, \ldots, (1 + \eta_{m'})^{-1}),
\]
then we have 
\[
  \tilde{\mu}_k = (\mathcal{F}_{m'} \mathbf{u}_m)^T \Delta_{m'} (\mathcal{F}_{m'} \mathbf{u}_m).
\]

Therefore, \( \tilde{H}_{m'} \) and \( \tilde{H}_{m'}^\prec \), the \( m' \times m' \) Hankel matrices associated with \( \{\tilde{\mu}_k\} \), can be factorized as follows:
\[
  \tilde{H}_{m'} = (\mathcal{F}_{m'} \phi_m)^T (\mathcal{F}_{m'} \phi_m), \quad (8)
\]
\[
  \tilde{H}_{m'}^\prec = (\mathcal{F}_{m'} \phi_m)^T \Delta_{m'} (\mathcal{F}_{m'} \phi_m). \quad (9)
\]

The right eigenvector of \( \tilde{H}_{m'} - \lambda \tilde{H}_{m'} \) with respect to \( \eta_i \) is given by
\[
  \tilde{x}_i = \frac{1}{\sqrt{\nu_i}} V_{m'}^{-1} e_i = \frac{1}{\sqrt{\nu_i}} \sqrt{1 + \eta_i^N} V_{m'}^{-1} e_i,
\]
and the left eigenvector is given by \( \tilde{y}_i = \tilde{x}_i \). From these results and Lemma 2, the following relation is obtained.

**Theorem 4** Let \( \tilde{H}_{m'} = \hat{H}_{m'} + O(\varepsilon) \) and \( \tilde{H}_{m'}^\prec = \hat{H}_{m'}^\prec + O(\varepsilon) \) with sufficiently small \( \varepsilon > 0 \). Then
\[
  \tau_i(\hat{H}_{m'}, \tilde{H}_{m'}) = |1 + \eta_i^N| \times \tau_i(\hat{H}_{m'}^\prec, \hat{H}_{m'} + O(\varepsilon)).
\]

This theorem shows that the condition on the \( i \)-th eigenvector of the matrix pencil \( \hat{H}_{m'}^\prec - \lambda \hat{H}_{m'} \), which is constructed from the moments calculated by numerical integration, is magnified by a factor \( |1 + \eta_i^N| > 1 \). However, the influence of the quadrature error on our target eigenvalues that lie inside the unit circle is \( |1 + \eta_i^N| \approx 1 \) if \( \tilde{v}_i = O(1) \). We should take \( m' \) large enough for \( \nu_i/(1 + \eta_i^N) = O(\varepsilon) \). This condition can be assessed by the singularity of \( \hat{H}_{m'} \).

### 3. Extension to block Hankel matrices with matrix-valued moments

Now we extend the results in the previous section to the case of matrix-valued moments. Let \( L \) be a positive integer with \( L \leq n \), and let \( N_i \in \mathbb{C}^{L \times L} \), \( 1 \leq i \leq n \) be given by \( N_i = d_i c_i^T \), \( i = 1, 2, \ldots, n \), with vectors \( c_i, d_i \in \mathbb{C}^L \). Define the matrix-valued moments \( M_k \in \mathbb{C}^{L \times L} \) by
\[
  M_k = \frac{1}{2\pi i} \int_{\gamma} z^k F(z) dz, \quad k = 0, 1, \ldots,
\]
where \( F(z) \in \mathbb{C}^{L \times L} \) is the matrix-valued function defined by \( F(z) = \sum_{n=1}^{N} N_i \). This function appears in the block Sakurai-Sugiura method for both generalized and nonlinear eigenvalue problems [4–6].

It can be verified that
\[
  M_k = \sum_{i=1}^{m} N_i \eta_i^k = D_m^T \Delta_m^k C_m, \quad k = 0, 1, \ldots,
\]
where 
\[
  C_m = [c_1, c_2, \ldots, c_m]^T, \quad D_m = [d_1, d_2, \ldots, d_m]^T.
\]

Here, we assume that the column vectors of \( C_m \) and those of \( D_m \) are linearly independent, respectively.

Let \( K \) be an integer such that \( m \leq KL \leq n \). Define the block Hankel matrices \( \hat{H}_{KL} \), \( \hat{H}_{KL}^\prec \in \mathbb{C}^{KL \times KL} \) with elements \( \{M_k\} \) by \( \hat{H}_{KL} = [M_{i+j-1,K}]_{j=1}^K \), \( \hat{H}_{KL}^\prec = [M_{i+j-1,K}]_{j=1}^K \). Let \( \Phi_m, \hat{\Psi}_m, \hat{\Psi}_m, \in \mathbb{C}^{m \times KL} \) be
\[
  \Phi_m, \hat{\Psi}_m = [C_m \Delta_m C_m \ldots \Delta_{KL} C_m],
\]
\[
  \hat{\Psi}_m, = [D_m \Delta_m D_m \ldots \Delta_{KL} D_m].
\]

We define the \( m \times m \) leading submatrices as follows:
\[
  H_m = H_{KL}(1 : m, 1 : m), \quad H_m^\prec = H_{KL}(1 : m, 1 : m),
\]
and also
\[
  \Phi_m = \Phi_m, KL(1 : m, 1 : m), \quad \hat{\Psi}_m = \hat{\Psi}_m, KL(1 : m, 1 : m).
\]

Then \( H_m \) and \( H_m^\prec \), the \( m \times m \) block Hankel matrices corresponding to \( \{M_k\} \), can be factorized as follows:
\[
  H_m = \Psi_m^T \Phi_m, \quad H_m^\prec = \Psi_m^T \Delta_m \Phi_m.
\]

These relations lead to the following theorem.

**Theorem 5** The eigenvalues of \( H_m^\prec - \lambda H_m \) are given by \( \eta_1, \ldots, \eta_m \). The right and left eigenvectors \( \mathbf{x}_i \) and \( \mathbf{y}_i \) with respect to \( \eta_i \) are given by \( \mathbf{x}_i = \Phi_m^{-1} \mathbf{e}_i \) and \( \mathbf{y}_i = (\Psi_m)^{-1} \mathbf{e}_i \), respectively, and \( \mathbf{y}_i^* H_m \mathbf{x}_i = 1 \).

The approximations for \( \tilde{\mu}_k \) are calculated by
\[
  \tilde{M}_k = \frac{1}{N} \sum_{j=0}^{N-1} \theta_j^{k+1} F(\theta_j), \quad k = 0, 1, \ldots
\]

Similar to the case of \( \tilde{\mu}_k \), we have
\[
  \tilde{M}_k = \sum_{i=1}^{m'} N_i (1 + \eta_i^N)^{k} + O(\varepsilon).
\]

Therefore, we can see that \( \tilde{M}_k \) approximately consists of \( m' \) poles \( \eta_1, \ldots, \eta_{m'} \) with the matrix-valued weights \( N_1, \ldots, N_{m'} \). In this case, the quadrature error is \( O(\varepsilon) \), which is small enough.

Setting \( M_k = \sum_{i=1}^{m'} N_i (1 + \eta_i^N)^{k} \), we have the following theorem.

**Theorem 6** Let \( \hat{H}_{m'} = \hat{H}_{m'} + O(\varepsilon) \) and \( \hat{H}_{m'}^\prec = \hat{H}_{m'}^\prec + O(\varepsilon) \) for sufficiently small \( \varepsilon > 0 \). Then
\[
  \tau_i(\hat{H}_{m'}, \hat{H}_{m'}) = |1 + \eta_i^N| \times \tau_i(\hat{H}_{m'}^\prec, \hat{H}_{m'} + O(\varepsilon)).
\]

Thus, we obtain a similar result to that of the scalar moments case. The influence of the quadrature error on matrix-valued moments depends on the location of each eigenvalue \( \eta_i \). For eigenvalues that lie outside the unit circle, the influence of the quadrature error is magnified by \( |1 + \eta_i^N| > 1 \). However, the perturbation resulting from quadrature error is not large for eigenvalues inside the unit circle.

### 4. Numerical examples

In this section, some numerical examples are considered. The computations are performed in MATLAB in double precision arithmetic. The matrix pencil is solved by the MATLAB function eig, and the system of linear equations is solved by mldivide.

**Example 1** The first example simply verifies the error estimation in (3). Let \( n = m = 5 \). Let \( \eta_1, \ldots, \eta_m \)
Table 1. Results of Example 1. Underlines indicate the incorrect digits.

| i | Real($\eta_i$) | $|\hat{\eta}_i - \eta_i|$ | $v_i$ |
|---|----------------|----------------------------|-----|
| 1 | $-1.02073233465553$ | $1.2 \times 10^{-2}$ | $10^{-14}$ |
| 2 | $0.49999999583260$ | $4.5 \times 10^{-8}$ | $10^{-8}$ |
| 3 | $0.50000000000000$ | $5.5 \times 10^{-16}$ | $1.0$ |
| 4 | $1.00000000000000$ | $8.5 \times 10^{-16}$ | $1.0$ |
| 5 | $1.99999999978350$ | $4.7 \times 10^{-11}$ | $10^{-6}$ |

Table 2. Results for the case of $m' = 12$ in Example 2. Parameters are set as $N = 32$ and $L = 5$.

| i | Real($\eta_i$) | $|\hat{\eta}_i - \eta_i|$ | $\tilde{\tau}_i$ |
|---|----------------|----------------------------|-----------|
| 1 | $0.19999999999951$ | $4.9 \times 10^{-14}$ | $1.2 \times 10^{-13}$ |
| 2 | $0.39999999999975$ | $2.4 \times 10^{-13}$ | $7.1 \times 10^{-13}$ |
| 3 | $0.60000000000001$ | $2.0 \times 10^{-14}$ | $1.1 \times 10^{-12}$ |
| 4 | $0.79999999999916$ | $9.4 \times 10^{-14}$ | $3.6 \times 10^{-12}$ |
| 5 | $1.00000000000000$ | $4.6 \times 10^{-13}$ | $4.6 \times 10^{-12}$ |
| 6 | $1.20000001763180$ | $1.8 \times 10^{-10}$ | $2.8 \times 10^{-9}$ |
| 7 | $1.40000019943835$ | $2.0 \times 10^{-8}$ | $2.0 \times 10^{-7}$ |
| 8 | $1.60000155508878$ | $1.6 \times 10^{-7}$ | $4.3 \times 10^{-6}$ |
| 9 | $1.79995725555516$ | $4.3 \times 10^{-5}$ | $1.8 \times 10^{-4}$ |
| 10 | $2.00001557580597$ | $3.2 \times 10^{-4}$ | $2.7 \times 10^{-3}$ |
| 11 | $2.20768817600809$ | $7.9 \times 10^{-3}$ | $1.9 \times 10^{-1}$ |
| 12 | $2.42759435400872$ | $5.0 \times 10^{-2}$ | $5.0 \times 10^{0}$ |

and $\nu_1, \ldots, \nu_m$ be $\{-1.0, 0.5 + i, 0.5 - i, 1.0, 2.0\}$ and $\{10^{-14}, 10^{-8}, 1.0, 1.0, 10^{-6}\}$, respectively.

The values $\hat{\eta}_1, \ldots, \hat{\eta}_n$ are obtained by solving the generalized eigenvalue problem $H_0^*x = \lambda H_m x$. The moments are calculated by $\mu_k = \sum_{i=1}^{n} \hat{\omega}_i^{k} \eta_i$. In Table 1, we show $\eta_i, |\hat{\eta}_i - \eta_i|$ for each $i$. The condition number of $H_m$ is $\text{cond}(H_m) = 1.9 \times 10^{14}$, however, $\eta_i$ and $\hat{\eta}_i$ are calculated numerically with sufficient accuracy from the matrix pencil. Other poles suffer from numerical error where magnitude of each error is proportional to $1/\nu_i$.

Example 2 Let $n = 20$, $\eta_i = 0.2 \times i$ for $1 \leq i \leq n$. Here we set $m = 5$. The elements of $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$ are set by a random number generator from a uniform distribution over the interval $[0,1]$. $M_k, k=0,1,\ldots$ are calculated by the $N$-point trapezoidal rule (10). The parameters are set as $N = 32$, $L = 5$.

For each $\hat{\eta}_i$ with $1 \leq i \leq m$, the error is evaluated by $\text{max}_{1 \leq i \leq m} |\hat{\eta}_i - \eta_i|$. To estimate the perturbation in the Hankel matrices of size $m'$, we computed $\sigma_{m'}/\sigma_1$ for various $m'$, where $\sigma_1, \ldots, \sigma_m$ are the singular values of $H_m$.

In Table 2, we present the results for the case of $m' = 12$. Instead of calculating $\tau_i(H_m^*, H_m^*)$, we calculated $\tilde{\tau}_i$ by using the eigenvectors of $H_m^* - \lambda H_m$. Note that $\eta_i = 1$ is located on the unit circle; however, it can be obtained because it does not meet any quadrature nodes. The condition number of $H_m$ is $\text{cond}(H_m) = 1.1 \times 10^{17}$.

The results for various $m'$ are shown in Table 3. The maximum error for the eigenvalues in the unit circle decreases as the matrix size $m'$ increases. The ratio of the singular values $\sigma_{m'}/\sigma_1$ gives a good evaluation of the perturbation of the coefficients of $H_m$.

5. Conclusions

Perturbation results for the eigenvalues of a matrix pencil of Hankel matrices associated with complex moments have been given. We extended these results to the case where matrices have a block Hankel structure.

From these results, we ascertain that the use of Hankel matrices of an appropriate size reduces the influence of quadrature error for eigenvalues that lie inside a given integral path. In this case, the Hankel matrices are ill-conditioned, however, element-wise error analysis shows that the target eigenvalues can be obtained accurately. The singular values of the Hankel matrix give good information for quadrature errors, and we can estimate an appropriate size of the Hankel matrix.

Numerical examples are consistent with the theoretical results. More detailed error estimations and applications to practical problems are subjects for future study.

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References