2D tight framelets with orientation selectivity suggested by vision science

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Abstract

In this paper we will construct compactly supported tight framelets with orientation selectivity and Gaussian derivative like filters. These features are similar to one of simple cells in V1 revealed by recent vision science. In order to see the orientation selectivity, we also give a simple example of image processing of a test image.

Keywords wavelet frame, framelet, visual cortex, simple cell, orientation selectivity

Research Activity Group Wavelet Analysis

1. Introduction

Simple cells in V1 of the brain cortex play important roles in the visual information processing, and some mathematical models of such cells have been studied by using the Gabor function or DOG function. However, R. Young established that Gaussian derivative models are suitable for studying simple cells (see \cite{1}). In this paper we construct compactly supported framelets which have similar graphs as Gaussian derivatives and good orientation selectivity. See \cite{2} for the definition of framelets. In \cite{3}, B. Escalante-Ramírez and J. Silván-Cárdenas constructed a multi-channel model with orientation selectivity by means of Gaussian derivatives. However the Gaussian function is not compactly supported. In the previous paper \cite{4}, we presented new wavelet frames with orientation selectivity and Gaussian derivative like shape. The frames are defined only on product of two finite abelian groups, $\mathbf{Z}/N_1 \mathbf{Z} \times \mathbf{Z}/N_2 \mathbf{Z}$, where $\mathbf{Z}$ is the additive abelian group consisting of all integers, and $N_1$ and $N_2$ are positive integers. However our framelet filters presented in this paper define not only tight frames of $l^2(\mathbf{Z}/N_1 \mathbf{Z} \times \mathbf{Z}/N_2 \mathbf{Z})$, but also one of $l^2(\mathbf{Z} \times \mathbf{Z})$, and of $L^2(\mathbf{R}^2)$.

To describe our construction we mention our terminology. Let $\mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$. For a matrix $A$ we denote by $A^T$ the transpose of $A$. For a $2 \times 2$ matrix $M$, let $\mathcal{N}(M)$ be the set $\{(x_1, x_2) : x_1, x_2 \in [0, 1)\} \cap \mathbf{Z}^2$. In this paper we will be concerned with the following matrices: $M_r = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $M_q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $M_h = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$. These matrices are related to decimation of 2D signals: $M_r$ defines a rectangular decimation, $M_q$ a quincunx decimation, and $M_h$ a hexagonal decimation. Suppose $M = M_r, M_q$ or $M_h$. For a stable filter $h = (h[m_1, m_2])_{m_1, m_2 \in \mathbf{Z}}$, let $H(\omega_1, \omega_2)$ be its frequency response function

$$\sum_{m_1, m_2 = -\infty}^{\infty} h[m_1, m_2] e^{-2\pi i m_1 \omega_1} e^{-2\pi i m_2 \omega_2}. $$

The purpose of this paper is to give a simple construction of a finite number of FIR filters $h_s = (h_s[n])_{n \in \mathbf{Z}^2}$, $s \in S$, having the above mentioned properties and the following conditions: For all $\omega \in \mathbf{R}^2$ and for all $r \in \mathcal{N}(M)$ with $r \neq 0$,

$$\sum_{s \in S} |H_s(\omega)|^2 = |\det M|, \quad (1)$$

$$\sum_{s \in S} H_s(\omega) H_s(\omega + r M^{-1}) = 0, \quad (2)$$

where the bar denotes the complex conjugation. For $u \in \mathbf{Z}^2$, let $h_{s,u}[v] = h_s[v - u M^T]$, $v \in \mathbf{Z}^2$, and $h_{s,u} = (h_{s,u}[v])_{v \in \mathbf{Z}^2}$. Obviously $h_{s,u} \in l^2(\mathbf{Z}^2)$. Since filters $h_s$, $s \in S$, satisfy the conditions (1) and (2), $\{h_{s,u}\}_{s \in S, u \in \mathbf{Z}^2}$ is a tight frame of $l^2(\mathbf{Z}^2)$. Moreover as we will show later, when $M = M_r$, constant multiplication of our filters satisfies the unitary extension principle, and therefore we gain also a tight frame of $L^2(\mathbf{R}^2)$.

We note that several wavelet frames with orientation selectivity had been constructed: for example curvelet \cite{5}, contourlet \cite{6}, complex wavelet \cite{7}, wavelet frames in \cite{4} and so on. However our framelet is completely different from them, and satisfies several properties similar to simple cells.

2. Construction

Suppose $n$ is an integer with $n \geq 2$. Let $r_n = 1$ if $n$ is odd, and let $r_n = 0$ if $n$ is even. Then there is a unique positive integer $r$ such that $n = 2r + r_n$. Let

$$\Lambda_f = \{(0, 0), (0, n), (n, 0), (n, n)\}.$$ 

$$\Lambda_g = \{(k, l) : k=0,n,l=1,2,\ldots,n-1 \cup \{(k, l) : l=0,n,k=1,2,\ldots,n-1 \}.$$
\[ \Lambda_n = \{(k,l)\mid k=1,\ldots, n-1, l=1,2,\ldots, n-1\}. \]

We abbreviate \( c(x) = \cos(\pi x) \) and \( s(x) = \sin(\pi x) \). We denote by \( \binom{\lambda}{k} \) the binomial coefficient. Let \( \alpha_k = \binom{\lambda}{k} \), \( \beta_k = \binom{\lambda}{k} \), and \( c_M = |\det M|^{1/2} \). For \( (k,l) \in \Lambda_n \), let

\[ F_{k,l}(x,y) = c_M^{k+l}e^{-\pi r(x+y)}c(x)^{n-k}s(x)^{l}c(y)^{n-l}s(y)^{l}. \]

For \( (k,l) \in \Lambda_n \), let

\[ G_{k,l}(x,y) = 2^{-2\nu}c_M^{k+l}e^{-\pi r(x+y)}c(x)^{n-k}s(x)^{l}c(y)^{n-l}s(y)^{l}. \]

For \( (k,l) \in \Lambda_n \), let

\[ A_{k,l}^n(x,y) = 2^{-\nu}c_M^{k+l}e^{-\pi r(x+y)}c(x)^{n-k}s(x)^{l}c(y)^{n-l}s(y)^{l}. \]

**Theorem 1**

(i) If \( n \) is odd, \( \mathcal{H} \) satisfies the conditions (1) and (2) for \( M_r, M_g \) and \( M_h \).

(ii) If \( n \) is even and \( n \geq 4 \), \( \mathcal{H} \) satisfies the conditions (1) and (2) for \( M_r, M_g \) but does not satisfy (2) for \( M_h \).

If \( n = 2 \), \( \mathcal{H} \) satisfies the condition (1), but does not satisfy (2) for \( M_r, M_g \) and \( M_h \).

**Sketch of the proof**

For real numbers \( p \) and \( q \), let

\[ \Phi(x,y;p,q) = \sum_{(k,l) \in \Lambda_f} F_{k,l}(x,y)F_{k,l}(x+p,y+q) \]

\[ + \sum_{(k,l) \in \Lambda_g} G_{k,l}(x,y)G_{k,l}(x+p,y+q) \]

\[ + 2 \sum_{\kappa=1}^{\nu} A_{k,l}^\kappa(x,y)A_{k,l}^\kappa(x+p,y+q). \]

Suppose \( n \) is odd. By calculation we have that \( \Phi(x,y;p,q) = 0 \) if \( p = 1/2 + m \) \((m \in \mathbb{Z})\) or \( q = 1/2 + m' \) \((m' \in \mathbb{Z})\), and that \( \Phi(x,y;0,0) = |\det M| \). In particular,

\[ \Phi(x,y;m/2,0) = \Phi(x,y;0,1/2) = \Phi(x,y;1/2,1/2) = 0. \]

This implies (i). If \( n \) is even and \( n \geq 4 \), then we have \( \Phi(x,y;0,0) = |\det M| \), and

\[ \Phi(x,y;1/2,0) = \Phi(x,y;0,1/2) = \Phi(x,y;1/2,1/2) = 0. \]

However \( \Phi(x,y;1/2,1/4) \) and \( \Phi(x,y;1/2,3/4) \) are not identically zero. Suppose \( n = 2 \). Then it is easy to show that \( \Phi(x,y;0,0) = |\det M| \), and that \( \Phi(x,y;1/2,1/2) \) is not identically zero.

**(QED)**

Suppose \( M = M_r \) and \( n \) is a positive integer with \( n \geq 3 \). Let \( B_1(x) \) be the characteristic function of the interval \([-1/2, 1/2] \) on \( \mathbb{R} \), and \( B_{n+1}(x) = B_n * B_1(x) \), \( m = 1,2,\ldots \) where \( * \) is the convolution on \( \mathbb{R} \). We consider the SP framelet of degree \( n \). Let

\[ f_{0,0}(x,y) = B_n(x-1/2)B_n(y-1/2). \]

Then the Fourier transform of \( f_{0,0} \) is as follows:

\[ \widehat{f}_{0,0}(\xi_1,\xi_2) = e^{-\pi i(\xi_1 + \xi_2)} \left( \frac{s(\xi_1) \cdot s(\xi_2)}{\pi \xi_1^2 \pi \xi_2^2} \right)^2. \]

Hence we have

\[ \widehat{f}_{0,0}(2\xi_1,2\xi_2) = c_{M_r}^{-1}f_{0,0}(\xi_1,\xi_2) \widehat{f}_{0,0}(\xi_1,\xi_2). \]

Define

\[ f_{k,l}(\xi_1,\xi_2) = c_{M_r}^{-1}F_{k,l}(\xi_1/2,\xi_2/2)f_{0,0}(\xi_1/2,\xi_2/2), \]

\[ g_{k,l}(\xi_1,\xi_2) = c_{M_r}^{-1}G_{k,l}(\xi_1/2,\xi_2/2)g_{0,0}(\xi_1/2,\xi_2/2), \]

\[ a_{k,l}^\kappa(\xi_1,\xi_2) = c_{M_r}^{-1}A_{k,l}^\kappa(\xi_1/2,\xi_2/2)a_{0,0}^\kappa(\xi_1/2,\xi_2/2). \]

By the unitary extension principle \([8]\) we have that \( \{f_{k,l}\}_{(k,l) \in \Lambda_n \cap \{0,0\}} \cup \{g_{k,l}\}_{(k,l) \in \Lambda_n} \cup \{a_{k,l}^\kappa\}_{(k,l) \in \Lambda_n, \kappa \in \{1,2\}} \) is a tight frame of \( L^2(\mathbb{R}^2) \).
3. Discussion related to vision science and image processing

To apply our framelets to computational experiments for studying vision, we discuss here a maximal overlap version of the generalized multiresolution analysis (MOGMRA) defined on \( \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z} \), where \( N_1 \) and \( N_2 \) are positive even integers. We refer our paper [9] why MOGMRA is suitable for studying mathematical models of visual information processing. See [10] and [11] for maximal overlap multiresolution analysis for wavelets. We begin with describing MOGMRA based on our SP framelet of degree \( n \). Suppose \( M = M_r \). For a positive integer \( N \), let \( Z_N = \{0, 1, \ldots, N-1\} \). For \( y = (y[m])_{m \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z}} \), we denote by \( y^M = (y^M[m])_{m \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z}} \) the upsampling of \( y \) by the sampling matrix \( M \), that is, \( y^M[mM] = y[m] \) for \( m \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z} \), and otherwise \( y^M[m] = 0 \). For \( x = (x[m])_{m \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z}} \), let \( x^\circ[m] = x[m] + x[m+1] \mathbb{Z}/N_1 \mathbb{Z} \mathbb{Z}/N_2 \mathbb{Z} \), and let \( S(x) = (x^\circ)^M \). Let \( S^\mu(x) = x \) and \( S^\mu(x) = S(S^{\mu-1}(x)) \) for \( \mu = 1, 2, \ldots \). For a stable filter \( h \in l^1(\mathbb{Z}^2) \), let \( p(h) \) be its periodization, that is, \( p(h) = (p(h)[m])_{m \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z}} \) where for \( m = (m_1, m_2) \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z} \),

\[
p(h)[m] = \sum_{k_1, k_2 = -\infty}^{\infty} h[m_1 + N_1 k_1, m_2 + N_2 k_2].
\]

Let \( T^j(h) = S^{j-1}(p(h)) \), \( j = 1, 2, \ldots \). For \( x = (x[m])_{m \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z}} \), \( (k_1, k_2) \in \mathbb{Z}^2 \), and \( \mu = (\mu_1, \mu_2) \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z} \), let \( x_{\mu_1, \mu_2}(k_1, k_2) = x[m] \). Then \( x_{\mu_1, \mu_2} \) is identified with a signal defined on \( \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z} \). Let \( x^\mu[m] = x_{\mu_1, \mu_2}[-m_1, -m_2] \). For \( x = (x[m])_{m \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z}} \) and \( y = (y[m])_{m \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z}} \), we denote by \( x * y \) the cyclic convolution, that is, \( x * y[m] = \sum_{k \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z}} x_{\mu_1, \mu_2}(k) y_{\mu_1, \mu_2}(m-k), m \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z} \). For \( x = (x[m])_{m \in \mathbb{Z}/N_1 \mathbb{Z} \times \mathbb{Z}/N_2 \mathbb{Z}} \), the first stage of the decomposition of MOGMRA is defined by \( F_{1 \ell}^1(x) = T^{-1}(f_{1 \ell})^\vee * x \), \( G_{k_1, k_2}^1(x) = T^1(g_{k_1, k_2})^\vee * x \), and \( A_{k_1, k_2}^1(x) = T^1(a_{k_1, k_2}^\vee) \). The second stage is defined by \( F_{2 \ell}^0(x) = T^2(f_{2 \ell})^\vee * F_{1 \ell}^1(x) \), \( G_{k_1, k_2}^0(x) = T^0(g_{k_1, k_2})^\vee * F_{1 \ell}^1(x) \), and \( A_{k_1, k_2}^0(x) = T^0(a_{k_1, k_2}^\vee) \). These are the decomposition phase. Let \( F^J = T^J(f_{0,0}) \). For a positive integer \( J \), the synthesis phase is defined as follows:

\[
\begin{align*}
\tilde{F}_{k_1, k_2}^J(x) &= 4^{-J} F^1 \ast \cdots \ast F^{J-1} * T^J(f_{k_1, k_2}) * F_{k_1, k_2}^1(x), \\
\tilde{F}_{k_1, k_2}^{-1}(x) &= 4^{-J+1} F^1 \ast \cdots \ast F^{J-2} * T^{J-1}(f_{k_1, k_2}) * F_{k_1, k_2}^{-1}(x), \\
&\vdots \\
\tilde{F}_{k_1, k_2}^{-J}(x) &= 4^{-1} F^1 * F_{k_1, k_2}^{-1}(x),
\end{align*}
\]

Fig. 1. Filters of MOGMRA (level 2, \( n = 5 \)).
We call this decomposition MOGMRA decomposition of \( \tilde{\kappa} \), \( J \)

\[
G_{k,l}^j(x) = 4^{-J} F^1 \ast \cdots \ast F^{J-1} \ast T^J(\tilde{\kappa},l) \ast G_{k,l}^j(x),
\]

and \( \tilde{\kappa}_{k,l}^j \) are defined by the similar way. Then we obtain

\[
x = \bar{F}_{0,0}^j(x) + \sum_{j=1}^J \left\{ \sum_{(k,l) \in \Lambda_j} \tilde{F}_{k,l}^j(x) \right\}
\]

\[
+ \sum_{(k,l) \in \Lambda_0} \tilde{G}_{k,l}^j(x) + \sum_{(k,l) \in \Lambda_0} \sum_{\kappa=1,2} \tilde{A}_{k,l}^\kappa(x) \right\}.
\]

We call this decomposition MOGMRA decomposition of \( x \) at level \( J \). By the same way as in [4], we can define MOGMRA when \( N_1 \) and \( N_2 \) are not even.

Let \( \delta \) be the 2D unit impulse supported at \( (N_1/2 + 1, N_2/2 + 1) \), and let \( \delta' = p(\delta) \). Suppose \( n = 5 \). Fig. 1 depicts the plots of the outputs of \( \delta' \) by \( F_{k,l}^2, G_{k,l}^2 \), and \( A_{k,l}^2 \) arranged by the following rule:

\[
F_{5,0}^2, G_{4,0}^2, G_{2,0}^2, G_{1,0}^2, F_{0,0}^2, G_{2,1}^2, A_{1,0}^2, A_{2,0}^2, A_{1,1}^2, G_{1,1}^2, A_{2,1}^2, A_{1,2}^2, A_{2,2}^2, A_{1,3}^2, A_{2,3}^2, A_{1,4}^2, A_{2,4}^2, G_{1,5}^2, G_{3,5}^2, G_{5,5}^2, F_{0,5}^2,
\]

Next we consider a test image (Fig. 2). Fig. 3 is the MOGMRA decomposition of the test image at level 2. From this result of image processing we can conclude that our framelet has good orientation selectivity.

\[\text{References}\]